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# Some New Results for Reverse Graph Energies and Graph Operations

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**Abstract.** In this paper, we establish the notions of reverse first Zagreb energy, reverse second Zagreb energy, and reverse degree square sum energy. We show how these energies behave across various graph families. Furthermore, we analyze the splitting graph and shadow graph energies, i.e. their context in the complete graph family. Our research adds to the overall knowledge about graph energies and their use within mathematical graph theory.

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**Key Words:** Graph energy, Shadow graph, Complete graph.

## 1. Introduction

Since graph energy was introduced, graph theory, a subfield of discrete mathematics, has gained a lot of popularity. Graphs are used extensively for solving problems since they provide an intuitive approach prior to formal definitions. Graph theory is applied in a broad spectrum of fields such as mathematics, artificial intelligence, biology, genetics, physics, social sciences, data structures, pattern recognition, and cybernetics.

Graph energies and spectra have had an outstanding application in science and engineering fields like satellite communication [1, 3], facial recognition and air transport. They are increasingly used to analyze the complexity of realworld systems, such as modular and distributed architectures found in fractionated spacecraft [20]. While traditional approaches focus on structural or functional dependencies, spectral techniques including graph energy and its variants offer a powerful perspective by quantifying complexity through eigenvalue-based measures. The eigenvalues of an adjacency matrix were once studied through the application of matrix theory and linear algebra. However, these days, algebraic methods can also successfully deal with symmetric and regular graphs [15].

The application of characteristic polynomials to chemical graph theory commenced in 1930 when in his molecular orbital theory for Π-electron networks of conjugated hydrocarbons, E. Huckel demonstrated the chemical uses of graph theory [10]. Horn in [11] utilized matrix analysis in examining the energy of graphs. Bapat illustrated that a graph's energy can never be an odd number [2], while Parizada revealed that the square root of an odd number can never be the energy of a graph [19]. Different results for bounds of energies of graphs are presented in [18]. Meenakshi computed different energies of regular, non-regular, circulant, and random graphs [16]. Das established upper and lower bounds for Zagreb energy [6], and maximal eccentricity energy of complete bipartite and star graphs was computed in [17]. Zheng studied different splitting graph energies as a multiple of the original graph energy [5].

Additional results for various energies of shadow graphs in [22] and the adjacency energy of m-splitting and m-shadow graphs as multiples of the input graph are given in [23].

Kousar and Nazeer have given results for several graph energies of subdivision graphs in [14]. Moreover, reverse Laplacian energy for some classes of graphs was computed in [13], and reverse maximum degree energy of complete graphs was established in [4, 12]. Lastly, statistical information regarding the studies of graph energies and their uses is presented in [8]. Structural transformations have long been central to graph theory, particularly in the context of connectivity and flow optimization [7]. These operations also influence the spectral characteristics of graphs, motivating the study of reverse graph energies as a tool to understand how such structural changes affect graph complexity. In this paper, we propose and investigate three new graph energy measures: the reverse first Zagreb energy, reverse second Zagreb energy, and reverse degree square sum energy.

Our results show that these new energy measures offer interesting information about the structural properties of different families of graphs.

In particular, we calculate these energies for some standard graph families and study their behaviors in splitting and shadow graphs, with special attention to complete graphs. The findings point to the success of these reverse energy measures in being able to capture the special properties and nuances of various graph structures, thus extending the application of graph energy concepts to theoretical graph analysis.

## 2. Preliminaries

The idea of graph energy was given by [24]. Let $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_\varrho$ be the eigenvalues of a graph $H$. The graph's energy can therefore be defined as the sum of the absolute values of the eigenvalues and is denoted by [9]

$$E(H) = \sum_{i=1}^{\varrho} |\gamma_i|.$$

Graph energy is a significant measure in the study of spectral graph theory, providing insights into various structural properties of graphs. This concept has applications in chemistry, particularly in the study of molecular orbitals where the graph's eigenvalues correspond to the energies of electron states in a molecule.

Let $H$ be a simple graph having $\varrho$ vertices and $\psi$ edges. Let $\Delta(H)$ be the largest degree among the vertices of $H$. The reverse vertex degree of a vertex $\omega_i$ in $H$ is given as [12]

$$\Upsilon_{\omega_i} = \Delta(H) - d_{\omega_i} + 1,$$

where $d_{\omega_i}$ is the vertex degree of vertex $\omega_i$. This definition scales the vertex degree to take the maximum degree within the graph into consideration, delivering a normalized measurement that may find application in comparison studies across multiple graphs.

In [21], based on the first and second Zagreb topological indices, Gutman presented the first and second Zagreb energies. These were initially formulated for describing molecular structure in chemistry but have now seen applications in more areas of graph theory.

A connected simple graph $H$'s first Zagreb energy $EZ_1$ is determined by adding the absolute values of its eigenvalues of the first Zagreb matrix $Z^{(1)}(H)$ of $H$, where $Z^{(1)}(H) = [z_{ij}^{(1)}]$ and

$$z_{ij}^{(1)} = \begin{cases} d_{\omega_i} + d_{\omega_j}, & \text{if } \omega_i\omega_j \in \Psi(H), \\ 0, & \text{otherwise.} \end{cases}$$

This is a measure that considers the degrees of the neighboring vertices and thus a means of expressing the structural complexity of the graph.

A connected simple graph $H$'s second Zagreb energy $EZ_2$ is determined by adding the absolute values of its eigenvalues of the second Zagreb matrix $Z^{(2)}(H)$ of $H$, where $Z^{(2)}(H) = [z_{ij}^{(2)}]$ and

$$z_{ij}^{(2)} = \begin{cases} d_{\omega_i} \cdot d_{\omega_j}, & \text{if } \omega_i\omega_j \in \Psi(H), \\ 0, & \text{otherwise.} \end{cases}$$

This is included as the product of the adjacent vertices' degrees, adding additional stress on connected nodes' interaction.

A connected simple graph $H$'s degree square sum energy $EDSS(H)$ is determined by adding the absolute values of its eigenvalues of degree square sum matrix $DSS(H) = [dss_{ij}]$,

$$dss_{ij} = \begin{cases} d_{\omega_i}^2 + d_{\omega_j}^2, & \text{if } \omega_i\omega_j \in \Psi(H), \\ 0, & \text{otherwise.} \end{cases}$$

Based on the ideas of first Zagreb energy, second Zagreb energy, and reverse vertex degree, we introduce the reverse first Zagreb energy, reverse second Zagreb energy, and reverse degree square sum energy. The reverse first Zagreb matrix is given as

$$z_{(1)_R} = \begin{cases} \Upsilon_{\omega_i} + \Upsilon_{\omega_j}, & \text{if } \omega_i\omega_j \in \Psi(H), \\ 0, & \text{otherwise.} \end{cases}$$

The reverse first Zagreb energy is the sum of the absolute eigenvalues of the reverse first Zagreb matrix and is represented as $EZ_{1_R}(H)$.

In the same manner, the reverse second Zagreb matrix is defined as

$$z_{(2)_R} = \begin{cases} \Upsilon_{\omega_i} \cdot \Upsilon_{\omega_j}, & \text{if } \omega_i\omega_j \in \Psi(H), \\ 0, & \text{otherwise.} \end{cases}$$

The reverse second Zagreb energy is the sum of the absolute eigenvalues of the reverse second Zagreb matrix and is represented as $EZ_{2_R}(H)$.

This is the representation of the reverse degree square sum matrix

$$dss_R = \begin{cases} \Upsilon_{\omega_i}^2 + \Upsilon_{\omega_j}^2, & \text{if } \omega_i\omega_j \in \Psi(H), \\ 0, & \text{otherwise.} \end{cases}$$

The sum of the absolute eigenvalues of the reverse degree square sum matrix is known as the reverse degree square sum energy and is represented as $EDSS_R(H)$.

The current study examines the relationship between different graph energies and energies of extended graphs derived via some graph operations like the splitting and shadow graphs. Adding a new vertex $w'$ to each vertex $w$ with the property that $w'$ is connected with every vertex with which $w$ is connected in $H$ creates the splitting graph $S'(H)$ of a graph $H$.

By taking two copies of $H$, represented by $H_1$ and $H_2$, and connecting each vertex $w$ in $H_1$ to the neighbours of the corresponding vertex $w$ in $H_2$, the shadow graph $D_2(H)$ of a connected graph $H$ is created.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of order $a \times \psi$ and $b \times \varrho$, respectively. Their tensor product, $A \otimes B$, is the matrix formed by replacing each entry $a_{ij}$ by the block $a_{ij}B$ and is of order ab times $\varrho\psi$.

**Proposition 2.1.** [11] *Let $A \in M^q$ and $B \in M^p$. Also, let $\alpha$ be an eigenvalue of the matrix $A$ with corresponding eigenvector $y$ and $\beta$ be an eigenvalue of the matrix $B$ with corresponding eigenvector $z$. Then, $\alpha\beta$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $yz$.*

## 3. Main Results and Discussions

3.1. **Reverse First Zagreb Energy.** In this section, the $EZ_{1_R}$ for star graph, complete graph and complete bipartite graph has been shown. The $EZ_{1_R}$ offers a quantitative measure of the structural complexity of the graph in terms of the adjusted degrees of the vertices.

3.1.1. *Reverse First Zagreb Energy of Star Graph.* The eigenvalues of a matrix connected to a graph are used to calculate the $EZ_{1_R}$. It is determined specifically for the star graph $S_\varrho$.

**Theorem 3.2.** *For the star graph $S_\varrho$, the reverse first Zagreb energy is given by*

$$EZ_{1_R}(S_\varrho) = 2(\varrho + 1)\sqrt{\varrho}.$$

*Proof.* Consider the star graph $S_\varrho$ with vertices $\omega_1, \omega_2, \ldots, \omega_\varrho$. Then,

$$Z_{1_R}(S_\varrho) = \begin{bmatrix} 0 & \varrho+1 & \varrho+1 & \cdots & \varrho+1 \\ \varrho+1 & 0 & 0 & \cdots & 0 \\ \varrho+1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varrho+1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

This matrix's characteristic polynomial is

$$\begin{vmatrix} -\gamma & \varrho+1 & \varrho+1 & \cdots & \varrho+1 \\ \varrho+1 & -\gamma & 0 & \cdots & 0 \\ \varrho+1 & 0 & -\gamma & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varrho+1 & 0 & 0 & \cdots & -\gamma \end{vmatrix}.$$

The spectrum of the $Z_{1_R}(S_\varrho)$

$$\text{Spec}(Z_{1_R}(S_\varrho)) = \begin{pmatrix} 0 & -(\varrho+1)\sqrt{\varrho} & (\varrho+1)\sqrt{\varrho} \\ \varrho-1 & 1 & 1 \end{pmatrix}.$$

Thus, the reverse first Zagreb energy is

$$EZ_{1_R}(S_\varrho) = (\varrho-1)|0| + |-(\varrho+1)\sqrt{\varrho}| + |(\varrho+1)\sqrt{\varrho}| = 2(\varrho+1)\sqrt{\varrho}.$$

□

3.2.1. *Reverse First Zagreb Energy of Complete Graph.* For the complete graph $K_\varrho$, the $EZ_{1_R}$ can also be calculated.

**Theorem 3.3.** *For the complete graph $K_\varrho$, the reverse first Zagreb energy is given by*

$$EZ_{1_R}(K_\varrho) = 4(\varrho-1).$$

*Proof.* Let $K_\varrho$ be a complete graph with vertices $\omega_1, \omega_2, \ldots, \omega_\varrho$. Then,

$$Z_{1_R}(K_\varrho) = \begin{bmatrix} 0 & 2 & 2 & \cdots & 2 \\ 2 & 0 & 2 & \cdots & 2 \\ 2 & 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & 0 \end{bmatrix}.$$

This matrix's characteristic polynomial is

$$\begin{vmatrix} -\gamma & 2 & 2 & \cdots & 2 \\ 2 & -\gamma & 2 & \cdots & 2 \\ 2 & 2 & -\gamma & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & -\gamma \end{vmatrix}.$$

The spectrum of the $Z_{1_R}(K_\varrho)$ is

$$\text{Spec}(Z_{1_R}(K_\varrho)) = \begin{pmatrix} -2 & 2\varrho-2 \\ \varrho-1 & 1 \end{pmatrix}.$$

Thus, the reverse first Zagreb energy is

$$EZ_{1_R}(K_\varrho) = (\varrho-1)|-2| + |2\varrho-2| = 4(\varrho-1).$$

□

3.3.1. *Reverse First Zagreb Energy of Complete Bipartite Graph.* For the complete bipartite graph $K_{\varrho,\varrho}$, the $EZ_{1_R}$ can also be calculated.

**Theorem 3.4.** *For the complete bipartite graph $K_{\varrho,\varrho}$, the reverse first Zagreb energy is given by*

$$EZ_{1_R}(K_{\varrho,\varrho}) = 4\varrho.$$

*Proof.* Let $K_{\varrho,\varrho}$ be a complete bipartite graph with vertices $\omega_1, \omega_2, \ldots, \omega_\varrho$. Then,

$$Z_{1_R}(K_{\varrho,\varrho}) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 0 & \cdots & 0 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 0 & \cdots & 0 & 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 2 & \cdots & 2 & 0 & 0 & \cdots & 0 \\ 2 & 2 & 2 & \cdots & 2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & 2 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

This matrix's characteristic polynomial is

$$\begin{vmatrix} -\gamma & 0 & 0 & \cdots & 0 & 2 & 2 & \cdots & 2 \\ 0 & -\gamma & 0 & \cdots & 0 & 2 & 2 & \cdots & 2 \\ 0 & 0 & -\gamma & \cdots & 0 & 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\gamma & 2 & 2 & \cdots & 2 \\ 2 & 2 & 2 & \cdots & 2 & -\gamma & 0 & \cdots & 0 \\ 2 & 2 & 2 & \cdots & 2 & 0 & -\gamma & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & 2 & 0 & 0 & \cdots & -\gamma \end{vmatrix}.$$

The spectrum of the $Z_{1_R}(K_{\varrho,\varrho})$ is

$$\text{Spec}(Z_{1_R}(K_{\varrho,\varrho})) = \begin{pmatrix} 0 & 2\varrho & -2\varrho \\ 2\varrho - 2 & 1 & 1 \end{pmatrix}.$$

Thus, the reverse first Zagreb energy is

$$EZ_{1_R}(K_{\varrho,\varrho}) = (2\varrho - 2)|0| + |-2\varrho| + |2\varrho| = 4\varrho.$$

□

3.5. **Reverse Second Zagreb Energy.** In this section the $EZ_{2_R}$ for stat graph, complete graph and complete bipartite graph is calculated. The $EZ_{2_R}$ provides a numerical assessment of the graph's structural complexity based on the vertices' adjusted degrees.

3.5.1. *Reverse Second Zagreb Energy of Star Graph.* The $EZ_{2_R}$ is a measure derived from the eigenvalues of a matrix associated with a graph. Specifically, it is computed for the star graph $S_\varrho$.

**Theorem 3.6.** *The star graph $S_\varrho$'s reverse second Zagreb energy is provided by*

$$EZ_{2_R}(S_\varrho) = 2\varrho\sqrt{\varrho}.$$

*Proof.* Consider the star graph $S_\varrho$ with vertices $\omega_1, \omega_2, \ldots, \omega_\varrho$. Then,

$$Z_{2_R}(S_\varrho) = \begin{bmatrix} 0 & \varrho & \varrho & \cdots & \varrho \\ \varrho & 0 & 0 & \cdots & 0 \\ \varrho & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varrho & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The characteristic polynomial of $Z_{2_R}(S_\varrho)$ is:

$$\begin{vmatrix} -\gamma & \varrho & \varrho & \cdots & \varrho \\ \varrho & -\gamma & 0 & \cdots & 0 \\ \varrho & 0 & -\gamma & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varrho & 0 & 0 & \cdots & -\gamma \end{vmatrix}.$$

The spectrum of $Z_{2_R}(S_\varrho)$ is:

$$\text{Spec}(Z_{2_R}(S_\varrho)) = \begin{pmatrix} 0 & -\varrho\sqrt{\varrho} & \varrho\sqrt{\varrho} \\ \varrho - 1 & 1 & 1 \end{pmatrix}.$$

The reverse second Zagreb energy is given by:

$$EZ_{2_R}(S_\varrho) = (\varrho - 1)\left|0\right| + \left|-\varrho\sqrt{\varrho}\right| + \left|\varrho\sqrt{\varrho}\right| = 2\varrho\sqrt{\varrho}.$$

□

3.6.1. *Reverse Second Zagreb Energy of Complete Graph.* The $EZ_{2_R}$ can also be computed for the complete graph $K_\varrho$.

**Theorem 3.7.** *The complete graph $K_\varrho$'s reverse second Zagreb energy is provided by*

$$EZ_{2_R}(K_\varrho) = 2(\varrho - 1).$$

*Proof.* Let $K_\varrho$ be a complete graph with vertices $\omega_1, \omega_2, \ldots, \omega_\varrho$. Then,

$$Z_{2_R}(K_\varrho) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}.$$

This matrix's characteristic polynomial is

$$\begin{vmatrix} -\gamma & 1 & 1 & \cdots & 1 \\ 1 & -\gamma & 1 & \cdots & 1 \\ 1 & 1 & -\gamma & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & -\gamma \end{vmatrix}.$$

$Z_{2_R}(K_\varrho)$'s spectrum is

$$\text{Spec}(Z_{2_R}(K_\varrho)) = \begin{pmatrix} -1 & \varrho - 1 \\ \varrho - 1 & 1 \end{pmatrix}.$$

Now, the reverse second Zagreb energy

$$EZ_{2_R}(K_\varrho) = (\varrho - 1)\left|-1\right| + \left|\varrho - 1\right| = 2(\varrho - 1).$$

□

3.7.1. *Reverse Second Zagreb Energy of Complete Bipartite Graph.* The $EZ_{2_R}$ can also be computed for the complete bipartite graph $K_{\varrho,\varrho}$.

**Theorem 3.8.** *The complete bipartite graph $K_{\varrho,\varrho}$'s reverse second Zagreb energy is provided by*

$$EZ_{2_R}(K_{\varrho,\varrho}) = 2\varrho.$$

*Proof.* Suppose that $K_{\varrho,\varrho}$ is a complete bipartite graph with vertices $\omega_1, \omega_2, \ldots, \omega_\varrho$. Then,

$$Z_{2_R}(K_{\varrho,\varrho}) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The characteristic polynomial of $Z_{2_R}(K_{\varrho,\varrho})$ is:

$$\begin{vmatrix} -\gamma & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & -\gamma & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 0 & -\gamma & \cdots & 0 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\gamma & 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & -\gamma & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 1 & 0 & -\gamma & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 1 & 0 & 0 & -\gamma & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\gamma \end{vmatrix}.$$

The spectrum of $Z_{2_R}(K_{\varrho,\varrho})$ is:

$$\mathrm{Spec}(Z_{2_R}(K_{\varrho,\varrho})) = \begin{pmatrix} 0 & \varrho & -\varrho \\ 2\varrho - 2 & 1 & 1 \end{pmatrix}.$$

The reverse second Zagreb energy is given as

$$EZ_{2_R}(K_{\varrho,\varrho}) = (2\varrho - 2)\left|0\right| + \left|-\varrho\right| + \left|\varrho\right| = 2\varrho.$$

□

3.9. **Reverse Degree Square Sum Energy of Star Graph.** The $EDSS_R$ is another spectral measure derived from the eigenvalues of a matrix associated with a graph. Specifically, it is computed for the star graph $S_\varrho$.

**Theorem 3.10.** *The $EDSS_R$ of the star graph $S_\varrho$ is given by*

$$EDSS_R(S_\varrho) = 2(\varrho^2 + 1)\sqrt{\varrho}.$$

*Proof.* Consider the star graph $S_\varrho$ with vertices $\omega_1, \omega_2, \ldots, \omega_\varrho$. Then,

$$DSS_R(S_\varrho) = \begin{bmatrix} 0 & \varrho^2 & \varrho^2 & \cdots & \varrho^2 \\ \varrho^2 & 0 & 0 & \cdots & 0 \\ \varrho^2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varrho^2 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The characteristic polynomial of $DSS_R(S_\varrho)$ is:

$$\begin{vmatrix} -\gamma & \varrho^2 & \varrho^2 & \cdots & \varrho^2 \\ \varrho^2 & -\gamma & 0 & \cdots & 0 \\ \varrho^2 & 0 & -\gamma & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varrho^2 & 0 & 0 & \cdots & -\gamma \end{vmatrix}.$$

The spectrum of $DSS_R(S_\varrho)$ is:

$$\mathrm{Spec}(DSS_R(S_\varrho)) = \begin{pmatrix} 0 & -(\varrho^2 + 1)\sqrt{\varrho} & (\varrho^2 + 1)\sqrt{\varrho} \\ \varrho - 1 & 1 & 1 \end{pmatrix}.$$

The reverse degree square sum energy is given by:

$$EDSS_R(S_\varrho) = (\varrho - 1)\left|0\right| + \left|-(\varrho^2+1)\sqrt{\varrho}\right| + \left|(\varrho^2+1)\sqrt{\varrho}\right| = 2(\varrho^2+1)\sqrt{\varrho}.$$

□

## 4. Reverse Energies of Splitting Graph of Complete Graph

In this section, $EZ_{1_R}$, $EZ_{2_R}$ and $EDSS_R$ of the splitting graph of complete graph have been shown.

**4.1. Reverse First Zagreb Energy of Splitting Graph of Complete graph.** This section contains the expression for the $EZ_{1_R}$ of the splitting graph of the complete graph.

**Theorem 4.2.** *The reverse first Zagreb energy of splitting graph for a complete graph $K_\varrho$ is as follows*

$$EZ_{1_R}(S'(K_\varrho)) = \sqrt{26}EZ_{1_R}(K_\varrho).$$

*Proof.* A complete graph's $K_\varrho$ with vertices $\omega_1, \omega_2, \omega_3, ..., \omega_\varrho$ reverse first Zagreb matrix is define as

$$Z_{1_R}(K_\varrho) = \begin{array}{c} \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_\varrho \end{array} \begin{array}{c} \begin{array}{ccccc} \omega_1 & \omega_2 & \omega_3 & \cdots & \omega_\varrho \end{array} \\ \begin{pmatrix} 0 & \Upsilon_1+\Upsilon_2 & \Upsilon_1+\Upsilon_3 & ... & \Upsilon_1+\Upsilon_\varrho \\ \Upsilon_2+\Upsilon_1 & 0 & \Upsilon_2+\Upsilon_3 & ... & \Upsilon_2+\Upsilon_\varrho \\ \Upsilon_3+\Upsilon_1 & \Upsilon_3+d_2 & 0 & ... & \Upsilon_3+\Upsilon_\varrho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon_\varrho+\Upsilon_1 & \Upsilon_\varrho+\Upsilon_2 & \Upsilon_\varrho+\Upsilon_3 & ... & 0 \end{pmatrix} \end{array}.$$

where, for $j = 1, 2, 3, ..., \varrho$, $\Upsilon_j$ is the reverse degree of vertex $\omega_j$. To obtain $S'(K_\varrho)$ such that $N(\omega_i) = N(\omega'_i)$, let $\omega'_1, \omega'_2, \omega'_3, ..., \omega'_\varrho$ be vertices inserted in $K_\varrho$ corresponding to $\omega_1, \omega_2, \omega_3, ..., \omega_\varrho$.Then, $S'(K_\varrho)$'s reverse first Zagreb matrix can be expressed as a block matrix in the manner shown below.:

$$\begin{array}{c} \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_\varrho \\ \omega'_1 \\ \omega'_2 \\ \omega'_3 \\ \vdots \\ \omega'_\varrho \end{array} \begin{array}{c} \begin{array}{cccccccccc} \omega_1 & \omega_2 & \omega_3 & \cdots & \omega_\varrho & \omega'_1 & \omega'_2 & \omega'_3 & \cdots & \omega'_\varrho \end{array} \\ \begin{pmatrix} 0 & \Upsilon_1+\Upsilon_2 & \Upsilon_1+\Upsilon_3 & ... & \Upsilon_1+\Upsilon_\varrho & 0 & \Upsilon_1+\Upsilon'_2 & \Upsilon_1+\Upsilon'_3 & ... & \Upsilon_1+\Upsilon'_\varrho \\ \Upsilon_2+\Upsilon_1 & 0 & \Upsilon_2+\Upsilon_3 & ... & \Upsilon_2+\Upsilon_\varrho & \Upsilon_2+\Upsilon'_1 & 0 & d_2+\Upsilon'_3 & ... & \Upsilon_2+\Upsilon'_\varrho \\ \Upsilon_3+\Upsilon_1 & \Upsilon_3+\Upsilon_2 & 0 & ... & \Upsilon_3+\Upsilon_\varrho & \Upsilon_3+\Upsilon'_1 & \Upsilon_3+\Upsilon'_2 & 0 & ... & \Upsilon_3+\Upsilon'_\varrho \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon_\varrho+\Upsilon_1 & \Upsilon_\varrho+\Upsilon_2 & \Upsilon_\varrho+\Upsilon_3 & ... & 0 & \Upsilon_\varrho+\Upsilon'_1 & \Upsilon_\varrho+\Upsilon'_2 & \Upsilon_\varrho+\Upsilon'_3 & ... & 0 \\ 0 & \Upsilon'_1+\Upsilon_2 & \Upsilon'_1+\Upsilon_3 & ... & \Upsilon'_1+\Upsilon_\varrho & 0 & 0 & 0 & ... & 0 \\ \Upsilon'_2+\Upsilon_1 & 0 & \Upsilon'_2+\Upsilon_3 & ... & \Upsilon'_2+\Upsilon_\varrho & 0 & 0 & 0 & ... & 0 \\ \Upsilon'_3+\Upsilon_1 & \Upsilon'_3+\Upsilon_2 & 0 & ... & \Upsilon'_3+\Upsilon_\varrho & 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon'_\varrho+\Upsilon_1 & \Upsilon'_\varrho+\Upsilon_2 & \Upsilon'_\varrho+\Upsilon_3 & ... & 0 & 0 & 0 & 0 & ... & 0 \end{pmatrix} \end{array},$$

where, for $j = 1, 2, 3, ..., \varrho$, $\Upsilon'_j$ is the reverse degree of vertex $\omega'_j$.

$$Z_{1_R}(S'(K_\varrho)) = \begin{bmatrix} Z_{1_R}(K_\varrho) & \frac{5}{2}Z_{1_R}(K_\varrho) \\ \frac{5}{2}Z_{1_R}(K_\varrho) & 0 \end{bmatrix}.$$

Or

$$Z_{1_R}(S'(K_\varrho)) = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 0 \end{bmatrix} \bigotimes Z_{1_R}(K_\varrho).$$

Here, the reverse first Zagreb spectrum of $S'(K_\varrho)$ is

$$\begin{pmatrix} (\frac{1-\sqrt{26}}{2})\gamma_j & (\frac{1+\sqrt{26}}{2})\gamma_j \\ \varrho & \varrho \end{pmatrix},$$

where the eigenvalues of $Z_{1_R}(K_\varrho)$ are $\gamma_j$ for $j = 1, 2, 3, ..., \varrho$ and $(\frac{1\pm\sqrt{26}}{2})$ are the eigenvalues of $\begin{bmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}$.

Hence,

$$\begin{aligned} EZ_{1_R}(S'(K_\varrho)) &= \sum_{j=1}^{j=\varrho} \left| \left( \frac{1 \pm \sqrt{26}}{2} \right) \gamma_j \right|, \\ &= \sum_{j=1}^{j=\varrho} |\gamma_j| \left( \left| \frac{1-\sqrt{26}}{2} \right| + \left| \frac{1+\sqrt{26}}{2} \right| \right), \\ &= \sum_{j=1}^{j=\varrho} |\gamma_j| \left( \frac{\sqrt{26}-1}{2} + \frac{1+\sqrt{26}}{2} \right), \\ &= \sqrt{26} EZ_{1_R}(K_\varrho). \end{aligned}$$

□

4.3. **Reverse Second Zagreb Energy of Splitting Graph of Complete graph.** This section contains the expression for the reverse second Zagreb energy of the splitting graph of the complete graph.

**Theorem 4.4.** *The reverse second Zagreb energy of splitting graph for a complete graph $K_\varrho$ is as follows*

$$EZ_{2_R}(S'(K_\varrho)) = \sqrt{65} EZ_{2_R}(K_\varrho).$$

*Proof.* A complete graph's $K_\varrho$ with vertices $\omega_1, \omega_2, \omega_3, ..., \omega_\varrho$ reverse second Zagreb matrix is define as

$$Z_{2_R}(K_\varrho) = \begin{matrix} & \omega_1 & \omega_2 & \omega_3 & ... & \omega_\varrho \\ \omega_1 & 0 & \Upsilon_1.\Upsilon_2 & \Upsilon_1.\Upsilon_3 & ... & \Upsilon_1.\Upsilon_\varrho \\ \omega_2 & \Upsilon_2.\Upsilon_1 & 0 & \Upsilon_2.\Upsilon_3 & ... & \Upsilon_2.\Upsilon_\varrho \\ \omega_3 & \Upsilon_3.\Upsilon_1 & \Upsilon_3.\Upsilon_2 & 0 & ... & \Upsilon_3.\Upsilon_\varrho \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_\varrho & \Upsilon_\varrho.\Upsilon_1 & \Upsilon_\varrho.\Upsilon_2 & \Upsilon_\varrho.\Upsilon_3 & ... & 0 \end{matrix},$$

where, for $j = 1, 2, 3, ..., \varrho$, $\Upsilon_j$ is the reverse degree of vertex $\omega_j$. To obtain $S'(K_\varrho)$ such that $N(\omega_j) = N(\omega'_j)$, let $\omega'_1, \omega'_2, \omega'_3, ..., \omega'_\varrho$ be vertices inserted in $K_\varrho$ corresponding to $\omega_1, \omega_2, \omega_3, ..., \omega_\varrho$. Next, $Z_{2_R}(S'(K_\varrho))$ is the reverse second Zagreb matrix of $S'(K_\varrho)$, which can be expressed as a square matrix as follows

$$\begin{array}{c} \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_\varrho \\ \omega'_1 \\ \omega'_2 \\ \omega'_3 \\ \vdots \\ \omega'_\varrho \end{array}\begin{array}{c} \begin{array}{cccccccccc} \omega_1 & \omega_2 & \omega_3 & ... & \omega_\varrho & \omega'_1 & \omega'_2 & \omega'_3 & ... & \omega'_\varrho \end{array} \\ \begin{pmatrix} 0 & \Upsilon_1.\Upsilon_2 & \Upsilon_1.\Upsilon_3 & ... & \Upsilon_1.\Upsilon_\varrho & 0 & \Upsilon_1.\Upsilon'_2 & \Upsilon_1.\Upsilon'_3 & ... & \Upsilon_1.\Upsilon'_\varrho \\ \Upsilon_2.\Upsilon_1 & 0 & \Upsilon_2.\Upsilon_3 & ... & \Upsilon_2.\Upsilon_\varrho & \Upsilon_2.\Upsilon'_1 & 0 & \Upsilon_2.\Upsilon'_3 & ... & \Upsilon_2.\Upsilon'_\varrho \\ \Upsilon_3.\Upsilon_1 & \Upsilon_3.\Upsilon_2 & 0 & ... & \Upsilon_3.\Upsilon_\varrho & \Upsilon_3.\Upsilon'_1 & \Upsilon_3.\Upsilon'_2 & 0 & ... & \Upsilon_3.\Upsilon'_\varrho \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon_\varrho.\Upsilon_1 & \Upsilon_\varrho.\Upsilon_2 & \Upsilon_\varrho.\Upsilon_3 & ... & 0 & \Upsilon_\varrho.\Upsilon'_1 & \Upsilon_\varrho.\Upsilon'_2 & \Upsilon_\varrho.\Upsilon'_3 & ... & 0 \\ 0 & \Upsilon'_1.\Upsilon_2 & \Upsilon'_1.\Upsilon_3 & ... & \Upsilon'_1.\Upsilon_\varrho & 0 & 0 & 0 & ... & 0 \\ \Upsilon'_2.\Upsilon_1 & 0 & \Upsilon'_2.\Upsilon_3 & ... & \Upsilon'_2.\Upsilon_\varrho & 0 & 0 & 0 & ... & 0 \\ \Upsilon'_3.\Upsilon_1 & \Upsilon'_3.\Upsilon_2 & 0 & ... & \Upsilon'_3.\Upsilon_\varrho & 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon'_\varrho.\Upsilon_1 & \Upsilon'_\varrho.\Upsilon_2 & \Upsilon'_\varrho.\Upsilon_3 & ... & 0 & 0 & 0 & 0 & ... & 0 \end{pmatrix} \end{array},$$

where, for $j = 1, 2, 3, ..., \varrho$, $\Upsilon'_j$ is the reverse degree of vertex $\omega'_j$.

$$Z_{2_R}(S'(K_\varrho)) = \begin{bmatrix} Z_{2_R}(K_\varrho) & 4Z_{2_R}(K_\varrho) \\ 4Z_{2_R}(K_\varrho) & 0 \end{bmatrix},$$

$$Z_{2_R}(S'(K_\varrho)) = \begin{bmatrix} 1 & 4 \\ 4 & 0 \end{bmatrix} \bigotimes Z_{2_R}(K_\varrho).$$

$S'(K_\varrho)$'s reverse second Zagreb spectrum is

$$\begin{pmatrix} \frac{1+\sqrt{65}}{2}\gamma_j & \frac{1-\sqrt{65}}{2}\gamma_j \\ \varrho & \varrho \end{pmatrix},$$

where the eigenvalues of $Z_{2_R}(K_\varrho)$ are $\gamma_j$ for $j = 1, 2, 3, ..., \varrho$ and $\frac{1\pm\sqrt{65}}{2}$ are the eigenvalues of $\begin{bmatrix} 1 & 4 \\ 4 & 0 \end{bmatrix}$.

Hence,

$$\begin{aligned} EZ_{2_R}(S'(K_\varrho)) &= \sum_{j=1}^{j=\varrho} |\frac{1 \pm \sqrt{65}}{2}\gamma_j|, \\ &= \sum_{j=1}^{j=\varrho} |\gamma_j| \left( \left|\frac{1-\sqrt{65}}{2}\right| + \left|\frac{1+\sqrt{65}}{2}\right| \right), \\ &= \sum_{j=1}^{j=\varrho} |\gamma_j| \left( \frac{\sqrt{65}-1}{2} + \frac{1+\sqrt{65}}{2} \right), \\ &= \sqrt{65} EZ_{2_R}(K_\varrho). \end{aligned}$$

□

4.5. **Reverse Degree Square Sum Energy of Splitting Graph of Complete graph.** This section contains the expression for the splitting graph's $EDSS_R$.

**Theorem 4.6.** *The $EDSS_R$ of splitting graph for a complete graph $K_\varrho$ is as follows*

$$EDSS_R(S'(K_\varrho)) = \sqrt{1 + (\varrho^2 + 1)^2} EDSS_R(K_\varrho).$$

*Proof.* Let $\omega'_1, \omega'_2, \omega'_3, ..., \varrho'_\varrho$ be the vertices added in $K_\varrho$ corresponding to $\omega_1, \omega_2, \omega_3, ..., \omega_\varrho$. This creates the splitting graph $S'(K_\varrho)$ such that for $j = 1, 2, 3, \cdots, \varrho$, $N(\omega_j) = N(\omega'_j)$. Then, $K_\varrho$ and $S'(K_\varrho)$'s $DSS_R$ are provided as follows:

$$DSS_R(K_\varrho) = \begin{array}{c} \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_\varrho \end{array} \begin{array}{c} \begin{array}{ccccc} \omega_1 & \omega_2 & \omega_3 & ... & \omega_\varrho \end{array} \\ \begin{pmatrix} 0 & \Upsilon_1^2+\Upsilon_2^2 & \Upsilon_1^2+\Upsilon_3^2 & ... & \Upsilon_1^2+\Upsilon_\varrho^2 \\ \Upsilon_2^2+\Upsilon_1^2 & 0 & \Upsilon_2^2+\Upsilon_3^2 & ... & \Upsilon_2^2+\Upsilon_\varrho^2 \\ \Upsilon_3^2+\Upsilon_1^2 & \Upsilon_3^2+\Upsilon_2^2 & 0 & ... & \Upsilon_3^2+\Upsilon_\varrho^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon_\varrho^2+\Upsilon_1^2 & \Upsilon_\varrho^2+\Upsilon_2^2 & \Upsilon_\varrho^2+\Upsilon_3^2 & ... & 0 \end{pmatrix} \end{array},$$

and

$$\begin{array}{c} \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_\varrho \\ \omega'_1 \\ \omega'_2 \\ \omega'_3 \\ \vdots \\ \omega'_\varrho \end{array} \begin{array}{c} \begin{array}{cccccccccc} \omega_1 & \omega_2 & \omega_3 & ... & \omega_\varrho & \omega'_1 & \omega'_2 & \omega'_3 & ... & \omega'_\varrho \end{array} \\ \begin{pmatrix} 0 & \Upsilon_1^2+\Upsilon_2^2 & \Upsilon_1^2+\Upsilon_3^2 & ... & \Upsilon_1^2+\Upsilon_\varrho^2 & 0 & \Upsilon_1^2+\Upsilon_2'^2 & \Upsilon_1^2+\Upsilon_3'^2 & ... & \Upsilon_1^2+\Upsilon_\varrho'^2 \\ \Upsilon_2^2+\Upsilon_1^2 & 0 & \Upsilon_2^2+\Upsilon_3^2 & ... & \Upsilon_2^2+\Upsilon_\varrho^2 & \Upsilon_2^2+\Upsilon_1'^2 & 0 & \Upsilon_2^2+\Upsilon_3'^2 & ... & \Upsilon_2^2+\Upsilon_\varrho'^2 \\ \Upsilon_3^2+\Upsilon_1^2 & \Upsilon_3^2+\Upsilon_2^2 & 0 & ... & \Upsilon_3^2+\Upsilon_\varrho^2 & \Upsilon_3^2+\Upsilon_1'^2 & \Upsilon_3^2+\Upsilon_2'^2 & 0 & ... & \Upsilon_3^2+\Upsilon_\varrho'^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon_\varrho^2+\Upsilon_1^2 & \Upsilon_\varrho^2+\Upsilon_2^2 & \Upsilon_\varrho^2+\Upsilon_3^2 & ... & 0 & \Upsilon_\varrho^2+\Upsilon_1'^2 & \Upsilon_\varrho^2+\Upsilon_2'^2 & \Upsilon_\varrho^2+\Upsilon_3'^2 & ... & 0 \\ 0 & \Upsilon_1'^2+\Upsilon_2^2 & \Upsilon_1'^2+\Upsilon_3^2 & ... & \Upsilon_1'^2+\Upsilon_\varrho^2 & 0 & 0 & 0 & ... & 0 \\ \Upsilon_2'^2+\Upsilon_1^2 & 0 & \Upsilon_2'^2+\Upsilon_3^2 & ... & \Upsilon_2'^2+\Upsilon_\varrho^2 & 0 & 0 & 0 & ... & 0 \\ \Upsilon_3'^2+\Upsilon_1^2 & \Upsilon_3'^2+\Upsilon_2^2 & 0 & ... & \Upsilon_3'^2+\Upsilon_\varrho^2 & 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon_\varrho'^2+\Upsilon_1^2 & \Upsilon_\varrho'^2+\Upsilon_2^2 & \Upsilon_\varrho'^2+\Upsilon_3^2 & ... & 0 & 0 & 0 & 0 & ... & 0 \end{pmatrix} \end{array},$$

where for $j = 1, 2, 3, ..., \varrho$, $\Upsilon_j$ is the reverse degree of vertex $\omega_j$ and $\Upsilon'_j$ is the reverse degree of vertex $\omega'_j$.

$$DSS_R(D(K_\varrho)) = \begin{bmatrix} DSS_R(K_\varrho) & (\frac{\varrho^2+1}{2})DSS_R(K_\varrho) \\ (\frac{\varrho^2+1}{2})DSS_R(K_\varrho) & 0 \end{bmatrix},$$

$$DSS_R(D(K_\varrho)) = \begin{bmatrix} 1 & (\frac{\varrho^2+1}{2}) \\ (\frac{\varrho^2+1}{2}) & 0 \end{bmatrix} \bigotimes DSS_R(K_\varrho).$$

$DSS_R(D(K_\varrho))$'s spectrum is

$$\begin{pmatrix} \frac{\sqrt{1+(\varrho^2+1)^2}}{2}\gamma_j & \frac{\sqrt{1+(\varrho^2+1)^2}}{2}\gamma_j \\ \varrho & \varrho \end{pmatrix},$$

where $\gamma_j$ for $j = 1, 2, 3, ..., \varrho$ are the eigenvalues of $DSS_R(K_\varrho)$ and $\frac{1\pm\sqrt{1+(\varrho^2+1)^2}}{2}$ are the eigenvalues of $\begin{bmatrix} 1 & \frac{\varrho^2+1}{2} \\ \frac{\varrho^2+1}{2} & 0 \end{bmatrix}$. Hence,

$$\begin{aligned}
EDSS_R(D(K_\varrho)) &= \sum_{j=1}^{j=\varrho} \left|\frac{1 \pm \sqrt{1+(\varrho^2+1)^2}}{2}\gamma_j\right|, \\
&= \sum_{j=1}^{j=\varrho} |\gamma_j| \left( \left|\frac{1-\sqrt{1+(\varrho^2+1)^2}}{2}\right| + \left|\frac{1+\sqrt{1+(\varrho^2+1)^2}}{2}\right| \right), \\
&= \sum_{j=1}^{j=\varrho} |\gamma_j| \left( \frac{\sqrt{1+(\varrho^2+1)^2}-1}{2} + \frac{1+\sqrt{1+(\varrho^2+1)^2}}{2} \right), \\
&= \sqrt{1+(\varrho^2+1)^2}\, EDSS_R(K_\varrho).
\end{aligned}$$

□

## 5. Reverse Energies of Shadow Graph of Complete Graph

In this section, the reverse first Zagreb energy, reverse second Zagreb energy and reverse degree square sum energy of the shadow graph of complete graph have been shown.

### 5.1. **Reverse First Zagreb Energy of Shadow Graph of Complete graph.**

This section contains the expression for the shadow graph of reverse first Zagreb energy.

**Theorem 5.2.** *The reverse first Zagreb energy of a shadow graph for a complete graph $K_\varrho$ is as follows:*

$$EZ_{1_R}(D(K_\varrho)) = \sqrt{34}\, EZ_{1_R}(K_\varrho).$$

*Proof.* Take a complete graph $K_\varrho$ with vertices $\omega_1, \omega_2, \omega_3, ..., \omega_\varrho$. Following that, $Z^{(1)}(K_\varrho)$ is the reverse first Zagreb matrix of $K_\varrho$ and is defined as

$$Z_{1_R}(K_\varrho) = \begin{array}{c} \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_\varrho \end{array} \begin{array}{c} \begin{array}{ccccc} \omega_1 & \omega_2 & \omega_3 & \cdots & \omega_\varrho \end{array} \\ \begin{pmatrix} 0 & \Upsilon_1+\Upsilon_2 & \Upsilon_1+\Upsilon_3 & ... & \Upsilon_1+\Upsilon_\varrho \\ \Upsilon_2+\Upsilon_1 & 0 & \Upsilon_2+\Upsilon_3 & ... & \Upsilon_2+\Upsilon_\varrho \\ \Upsilon_3+\Upsilon_1 & \Upsilon_3+d_2 & 0 & ... & \Upsilon_3+\Upsilon_\varrho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon_\varrho+\Upsilon_1 & \Upsilon_\varrho+\Upsilon_2 & \Upsilon_\varrho+\Upsilon_3 & ... & 0 \end{pmatrix} \end{array},$$

where, for $j = 1, 2, 3, ..., \varrho$, $\Upsilon_j$ is the reverse degree of vertex $\omega_j$. To obtain $D(K_\varrho)$ such that $N(\omega_i) = N(\omega_i')$, let $\omega_1', \omega_2', \omega_3', ..., \omega_\varrho'$ be vertices inserted in $K_\varrho$ corresponding to $\omega_1, \omega_2, \omega_3, ..., \omega_\varrho'$. Then, the reverse first Zagreb matrix of $D(K_\varrho)$ can then be expressed as a block matrix in the manner described below

$$\begin{array}{c} \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_\varrho \\ \omega'_1 \\ \omega'_2 \\ \omega'_3 \\ \vdots \\ \omega'_\varrho \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} \omega_1 & \omega_2 & \omega_3 & \dots & \omega_\varrho & \omega'_1 & \omega'_2 & \omega'_3 & \dots & \omega'_\varrho \end{array} \\
\begin{pmatrix}
0 & \Upsilon_1+\Upsilon_2 & \Upsilon_1+\Upsilon_3 & \dots & \Upsilon_1+\Upsilon_\varrho & 0 & \Upsilon_1+\Upsilon'_2 & \Upsilon_1+\Upsilon'_3 & \dots & \Upsilon_1+\Upsilon'_\varrho \\
\Upsilon_2+\Upsilon_1 & 0 & \Upsilon_2+\Upsilon_3 & \dots & \Upsilon_2+\Upsilon_\varrho & \Upsilon_2+\Upsilon'_1 & 0 & d_2+\Upsilon'_3 & \dots & \Upsilon_2+\Upsilon'_\varrho \\
\Upsilon_3+\Upsilon_1 & \Upsilon_3+\Upsilon_2 & 0 & \dots & \Upsilon_3+\Upsilon_\varrho & \Upsilon_3+\Upsilon'_1 & \Upsilon_3+\Upsilon'_2 & 0 & \dots & \Upsilon_3+\Upsilon'_\varrho \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\Upsilon_\varrho+\Upsilon_1 & \Upsilon_\varrho+\Upsilon_2 & \Upsilon_\varrho+\Upsilon_3 & \dots & 0 & \Upsilon_\varrho+\Upsilon'_1 & \Upsilon_\varrho+\Upsilon'_2 & \Upsilon_\varrho+\Upsilon'_3 & \dots & 0 \\
0 & \Upsilon'_1+\Upsilon_2 & \Upsilon'_1+\Upsilon_3 & \dots & \Upsilon'_1+\Upsilon_\varrho & 0 & \Upsilon'_1+\Upsilon'_2 & \Upsilon'_1+\Upsilon'_3 & \dots & \Upsilon'_1+\Upsilon'_\varrho \\
\Upsilon'_2+\Upsilon_1 & 0 & \Upsilon'_2+\Upsilon_3 & \dots & \Upsilon'_2+\Upsilon_\varrho & \Upsilon'_2+\Upsilon'_1 & 0 & \Upsilon'_2+\Upsilon'_3 & \dots & \Upsilon'_2+\Upsilon'_\varrho \\
\Upsilon'_3+\Upsilon_1 & \Upsilon'_3+\Upsilon_2 & 0 & \dots & \Upsilon'_3+\Upsilon_\varrho & \Upsilon'_3+\Upsilon'_1 & \Upsilon'_3+\Upsilon'_2 & 0 & \dots & \Upsilon'_3+\Upsilon'_\varrho \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\Upsilon'_\varrho+\Upsilon_1 & \Upsilon'_\varrho+\Upsilon_2 & \Upsilon'_\varrho+\Upsilon_3 & \dots & 0 & \Upsilon'_\varrho+\Upsilon'_1 & \Upsilon'_\varrho+\Upsilon'_2 & \Upsilon'_\varrho+\Upsilon'_3 & \dots & 0
\end{pmatrix}
\end{array},$$

where, for $j = 1, 2, 3, ..., \varrho$, $\Upsilon'_j$ is the reverse degree of vertex $\omega'_j$.

$$Z_{1_R}(D(K_\varrho)) = \begin{bmatrix} Z_{1_R}(K_\varrho) & \frac{\varrho+1}{2} Z_{1_R}(K_\varrho) \\ \frac{\varrho+1}{2} Z_{1_R}(K_\varrho) & \varrho Z_{1_R}(K_\varrho) \end{bmatrix}.$$

Or

$$Z_{1_R}(K_\varrho) = \begin{bmatrix} 1 & \frac{\varrho+1}{2} \\ \frac{\varrho+1}{2} & \varrho \end{bmatrix} \bigotimes Z_{1_R}(K_\varrho).$$

Here, $D(K_\varrho)$'s reverse first Zagreb spectrum is

$$\begin{pmatrix} (\frac{(\varrho+1)-\sqrt{2+2\varrho^2}}{2})\gamma_j & (\frac{(\varrho+1)+\sqrt{2+2\varrho^2}}{2})\gamma_j \\ \varrho & \varrho \end{pmatrix},$$

where the eigenvalues of $Z_{1_R}(K_\varrho)$ are $\gamma_j$ for $j = 1, 2, 3, ..., \varrho$ and $(\frac{(\varrho+1)\pm\sqrt{2+2\varrho^2}}{2})$ are the eigenvalues of $\begin{bmatrix} 1 & \frac{\varrho+1}{2} \\ \frac{\varrho+1}{2} & \varrho \end{bmatrix}$. Hence,

$$\begin{aligned}
EZ_{1_R}(D(K_\varrho)) &= \sum_{j=1}^{j=\varrho} \left| \left( \frac{(\varrho+1) \pm \sqrt{2+2\varrho^2}}{2} \right) \gamma_j \right|, \\
&= \sum_{j=1}^{j=\varrho} |\gamma_j| \left( \left| \frac{(\varrho+1) - \sqrt{2+2\varrho^2}}{2} \right| + \left| \frac{(\varrho+1)\sqrt{2+2\varrho^2}}{2} \right| \right), \\
&= \sum_{j=1}^{j=\varrho} |\gamma_j| \left( \frac{-(\varrho+1) + \sqrt{2+2\varrho^2}}{2} + \frac{(\varrho+1) + \sqrt{2+2\varrho^2}}{2} \right), \\
&= \sqrt{2+2\varrho^2} EZ_{1_R}(K_\varrho).
\end{aligned}$$

□

5.3. **Reverse Second Zagreb Energy of Shadow Graph of Complete graph.** This section contains the expression for the shadow graph's reverse second Zagreb energy.

**Theorem 5.4.** *The reverse Second Zagreb energy of a shadow graph for a complete graph $K_\varrho$ is as follows*

$$EZ_{2_R}(D(K_\varrho)) = (\varrho^2 + 1) EZ_{2_R}(K_\varrho).$$

*Proof.* Let $\omega'_1, \omega'_2, \omega'_3, ..., \varrho'_\varrho$ be the vertices added in $K_\varrho$ corresponding to $\omega_1, \omega_2, \omega_3, ..., \omega_\varrho$. This creates the shadow graph $D(K_\varrho)$ such that for $j = 1, 2, 3, \cdots, \varrho$, $N(\omega_j) = N(\omega'_j)$. Then, $K_\varrho$ and $D(K_\varrho)$'s reverse second Zagreb matrix are provided as follows

$$Z_{2_R}(K_\varrho) = \begin{array}{c} \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_\varrho \end{array} \begin{array}{c} \begin{array}{ccccc} \omega_1 & \omega_2 & \omega_3 & ... & \omega_\varrho \end{array} \\ \begin{pmatrix} 0 & \Upsilon_1.\Upsilon_2 & \Upsilon_1.\Upsilon_3 & ... & \Upsilon_1.\Upsilon_\varrho \\ \Upsilon_2.\Upsilon_1 & 0 & \Upsilon_2.\Upsilon_3 & ... & \Upsilon_2.\Upsilon_\varrho \\ \Upsilon_3.\Upsilon_1 & \Upsilon_3.\Upsilon_2 & 0 & ... & \Upsilon_3.\Upsilon_\varrho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon_\varrho.\Upsilon_1 & \Upsilon_\varrho.\Upsilon_2 & \Upsilon_\varrho.\Upsilon_3 & ... & 0 \end{pmatrix} \end{array},$$

and

$$\begin{array}{c} \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_\varrho \\ \omega'_1 \\ \omega'_2 \\ \omega'_3 \\ \vdots \\ \omega'_\varrho \end{array} \begin{array}{c} \begin{array}{cccccccccc} \omega_1 & \omega_2 & \omega_3 & ... & \omega_\varrho & \omega'_1 & \omega'_2 & \omega'_3 & ... & \omega'_\varrho \end{array} \\ \begin{pmatrix} 0 & \Upsilon_1.\Upsilon_2 & \Upsilon_1.\Upsilon_3 & ... & \Upsilon_1.\Upsilon_\varrho & 0 & \Upsilon_1.\Upsilon'_2 & \Upsilon_1.\Upsilon'_3 & ... & \Upsilon_1.\Upsilon'_\varrho \\ \Upsilon_2.\Upsilon_1 & 0 & \Upsilon_2.\Upsilon_3 & ... & \Upsilon_2.\Upsilon_\varrho & \Upsilon_2.\Upsilon'_1 & 0 & \Upsilon_2.\Upsilon'_3 & ... & \Upsilon_2.\Upsilon'_\varrho \\ \Upsilon_3.\Upsilon_1 & \Upsilon_3.\Upsilon_2 & 0 & ... & \Upsilon_3.\Upsilon_\varrho & \Upsilon_3.\Upsilon'_1 & \Upsilon_3.\Upsilon'_2 & 0 & ... & \Upsilon_3.\Upsilon'_\varrho \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon_\varrho.\Upsilon_1 & \Upsilon_\varrho.\Upsilon_2 & \Upsilon_\varrho.\Upsilon_3 & ... & 0 & \Upsilon_\varrho.\Upsilon'_1 & \Upsilon_\varrho.\Upsilon'_2 & \Upsilon_\varrho.\Upsilon'_3 & ... & 0 \\ 0 & \Upsilon'_1.\Upsilon_2 & \Upsilon'_1.\Upsilon_3 & ... & \Upsilon'_1.\Upsilon_\varrho & 0 & \Upsilon'_1.\Upsilon'_2 & \Upsilon'_1.\Upsilon'_3 & ... & \Upsilon'_1.\Upsilon'_\varrho \\ \Upsilon'_2.\Upsilon_1 & 0 & \Upsilon'_2.\Upsilon_3 & ... & \Upsilon'_2.\Upsilon_\varrho & \Upsilon'_2.\Upsilon'_1 & 0 & \Upsilon'_2.\Upsilon'_3 & ... & \Upsilon'_2.\Upsilon'_\varrho \\ \Upsilon'_3.\Upsilon_1 & \Upsilon'_3.\Upsilon_2 & 0 & ... & \Upsilon'_3.\Upsilon_\varrho & \Upsilon'_3.\Upsilon'_1 & \Upsilon'_3.\Upsilon'_2 & 0 & ... & \Upsilon'_3.\Upsilon'_\varrho \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon'_\varrho.\Upsilon_1 & \Upsilon'_\varrho.\Upsilon_2 & \Upsilon'_\varrho.\Upsilon_3 & ... & 0 & \Upsilon'_\varrho.\Upsilon'_1 & \Upsilon'_\varrho.\Upsilon'_2 & \Upsilon'_\varrho.\Upsilon'_3 & ... & 0 \end{pmatrix} \end{array},$$

where, for $j = 1, 2, 3, ..., \varrho$, $\Upsilon'_j$ is the reverse degree of vertex $\omega'_j$.

$$Z_{2_R}(D(K_\varrho)) = \begin{bmatrix} Z_{2_R}(K_\varrho) & \varrho Z_{2_R}(K_\varrho) \\ \varrho Z_{2_R}(K_\varrho) & \varrho^2 Z_{2_R}(K_\varrho \end{bmatrix}$$

$$Z_{2_R}(D(K_\varrho)) = \begin{bmatrix} 1 & n \\ \varrho & \varrho^2 \end{bmatrix} \bigotimes Z_{2_R}(K_\varrho).$$

Here, $D'(K_\varrho)$'s reverse second Zagreb spectrum is

$$\begin{pmatrix} \mathbf{0}\gamma_j & (\varrho^2+1)\gamma_j \\ \varrho & \varrho \end{pmatrix}.$$

where the eigenvalues of $Z_{2_R}(K_\varrho)$ are $\gamma_j$ for $j = 1, 2, 3, ..., \varrho$ and $0$ and $(\varrho^2 + 1)$ are the eigenvalues of $\begin{bmatrix} 1 & n \\ \varrho & \varrho^2+1 \end{bmatrix}$. Hence,

$$\begin{aligned} EZ_{2_R}(D(K_\varrho)) &= \sum_{j=1}^{j=\varrho} |0 + (\varrho^2+1)\gamma_j|, \\ &= \sum_{j=1}^{j=\varrho} |\gamma_j|(\varrho^2+1), \\ &= (\varrho^2+1)EZ_{2_R}(K_\varrho). \end{aligned}$$

□

5.5. **Reverse Degree Square Sum Energy of Shadow Graph of Complete graph.** This section contains the expression for the shadow graph's $EDSS_R$ of the complete graph.

**Theorem 5.6.** *The $EDSS_R$ of the shadow graph for a complete graph $K_\varrho$ is as follows*

$$EDSS_R(D(K_\varrho)) = \sqrt{2\varrho^4+2}EDSS_R(K_\varrho).$$

*Proof.* Let $\omega'_1, \omega'_2, \omega'_3, ..., \omega'_\varrho$ be the vertices added in $K_\varrho$ corresponding to $\omega_1, \omega_2, \omega_3, ..., \omega_\varrho$ to obtain the shadow graph $D(K_\varrho)$ such that for $j = 1, 2, 3, \cdots, \varrho$, $N(\omega_j) = N(\omega'_j)$. Next, $K_\varrho$ and $D(K_\varrho)$'s $DSS_R$ are provided a

$$DSS_R(K_\varrho) = \begin{array}{c} \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_\varrho \end{array} \begin{array}{c} \begin{array}{ccccc} \omega_1 & \omega_2 & \omega_3 & \dots & \omega_\varrho \end{array} \\ \begin{pmatrix} 0 & \Upsilon_1^2+\Upsilon_2^2 & \Upsilon_1^2+\Upsilon_3^2 & \dots & \Upsilon_1^2+\Upsilon_\varrho^2 \\ \Upsilon_2^2+\Upsilon_1^2 & 0 & \Upsilon_2^2+\Upsilon_3^2 & \dots & \Upsilon_2^2+\Upsilon_\varrho^2 \\ \Upsilon_3^2+\Upsilon_1^2 & \Upsilon_3^2+\Upsilon_2^2 & 0 & \dots & \Upsilon_3^2+\Upsilon_\varrho^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon_\varrho^2+\Upsilon_1^2 & \Upsilon_\varrho^2+\Upsilon_2^2 & \Upsilon_\varrho^2+\Upsilon_3^2 & \dots & 0 \end{pmatrix} \end{array},$$

and

$$\begin{array}{c} \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_\varrho \\ \omega'_1 \\ \omega'_2 \\ \omega'_3 \\ \vdots \\ \omega'_\varrho \end{array} \begin{array}{c} \begin{array}{cccccccccc} \omega_1 & \omega_2 & \omega_3 & \dots & \omega_\varrho & \omega'_1 & \omega'_2 & \omega'_3 & \dots & \omega'_\varrho \end{array} \\ \begin{pmatrix} 0 & \Upsilon_1^2+\Upsilon_2^2 & \Upsilon_1^2+\Upsilon_3^2 & \dots & \Upsilon_1^2+\Upsilon_\varrho^2 & 0 & \Upsilon_1^2+\Upsilon_2'^2 & \Upsilon_1^2+\Upsilon_3'^2 & \dots & \Upsilon_1^2+\Upsilon_\varrho'^2 \\ \Upsilon_2^2+\Upsilon_1^2 & 0 & \Upsilon_2^2+\Upsilon_3^2 & \dots & \Upsilon_2^2+\Upsilon_\varrho^2 & \Upsilon_2^2+\Upsilon_1'^2 & 0 & \Upsilon_2^2+\Upsilon_3'^2 & \dots & \Upsilon_2^2+\Upsilon_\varrho'^2 \\ \Upsilon_3^2+\Upsilon_1^2 & \Upsilon_3^2+\Upsilon_2^2 & 0 & \dots & \Upsilon_3^2+\Upsilon_\varrho^2 & \Upsilon_3^2+\Upsilon_1'^2 & \Upsilon_3^2+\Upsilon_2'^2 & 0 & \dots & \Upsilon_3^2+\Upsilon_\varrho'^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon_\varrho^2+\Upsilon_1^2 & \Upsilon_\varrho^2+\Upsilon_2^2 & \Upsilon_\varrho^2+\Upsilon_3^2 & \dots & 0 & \Upsilon_\varrho^2+\Upsilon_1'^2 & \Upsilon_\varrho^2+\Upsilon_2'^2 & \Upsilon_\varrho^2+\Upsilon_3'^2 & \dots & 0 \\ 0 & \Upsilon_1'^2+\Upsilon_2^2 & \Upsilon_1'^2+\Upsilon_3^2 & \dots & \Upsilon_1'^2+\Upsilon_\varrho^2 & 0 & \Upsilon_1'^2+\Upsilon_2'^2 & \Upsilon_1'^2+\Upsilon_3'^2 & \dots & \Upsilon_1'^2+\Upsilon_\varrho'^2 \\ \Upsilon_2'^2+\Upsilon_1^2 & 0 & \Upsilon_2'^2+\Upsilon_3^2 & \dots & \Upsilon_2'^2+\Upsilon_\varrho^2 & \Upsilon_2'^2+\Upsilon_1'^2 & 0 & \Upsilon_2'^2+\Upsilon_3'^2 & \dots & \Upsilon_2'^2+\Upsilon_\varrho'^2 \\ \Upsilon_3'^2+\Upsilon_1^2 & \Upsilon_3'^2+\Upsilon_2^2 & 0 & \dots & \Upsilon_3'^2+\Upsilon_\varrho^2 & \Upsilon_3'^2+\Upsilon_1'^2 & \Upsilon_3'^2+\Upsilon_2'^2 & 0 & \dots & \Upsilon_3'^2+\Upsilon_\varrho'^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Upsilon_\varrho'^2+\Upsilon_1^2 & \Upsilon_\varrho'^2+\Upsilon_2^2 & \Upsilon_\varrho'^2+\Upsilon_3^2 & \dots & 0 & \Upsilon_\varrho'^2+\Upsilon_1'^2 & \Upsilon_\varrho'^2+\Upsilon_2'^2 & \Upsilon_\varrho'^2+\Upsilon_3'^2 & \dots & 0 \end{pmatrix} \end{array},$$

where for $j = 1, 2, 3, ..., \varrho$, $\Upsilon_j$ is the reverse degree of vertex $\omega_j$ and $\Upsilon'_j$ is the reverse degree of vertex $\omega'_j$.

$$DSS_R(D(K_\varrho)) = \begin{bmatrix} DSS_R(K_\varrho) & (\frac{\varrho^2+1}{2})DSS_R(K_\varrho) \\ (\frac{\varrho^2+1}{2})DSS_R(K_\varrho) & \varrho^2 DSS_R(K_\varrho) \end{bmatrix},$$

$$DSS_R(D(K_\varrho)) = \begin{bmatrix} 1 & (\frac{\varrho^2+1}{2}) \\ (\frac{\varrho^2+1}{2}) & \varrho^2 \end{bmatrix} \bigotimes DSS_R(K_\varrho).$$

Here, $DSS_R(D(K_\varrho))$'s spectrum is

$$\begin{pmatrix} \frac{(\varrho^2+1)+\sqrt{2\varrho^4+2}}{2}\gamma_j & \frac{(\varrho^2+1)-\sqrt{2\varrho^4+2}}{2}\gamma_j \\ \varrho & \varrho \end{pmatrix},$$

Where $\gamma_j$ for $j = 1, 2, 3, ..., \varrho$ are the eigenvalues of $DSS_R(K_\varrho)$ and $\frac{(\varrho^2+1)\pm\sqrt{2\varrho^4+2}}{2}$ are the eigenvalues of $\begin{bmatrix} 1 & \frac{\varrho^2+1}{2} \\ \frac{\varrho^2+1}{2} & \varrho^2 \end{bmatrix}$. Hence,

$$\begin{aligned} EDSS_R(D(K_\varrho)) &= \sum_{j=1}^{j=\varrho} \Big|\frac{(\varrho^2+1)\pm\sqrt{2\varrho^4+2}}{2}\gamma_j\Big|, \\ &= \sum_{j=1}^{j=\varrho} |\gamma_j| \left( \left| \frac{(\varrho^2+1)-\sqrt{2\varrho^4+2}}{2} \right| + \left| \frac{(\varrho^2+1)+\sqrt{2\varrho^4+2}}{2} \right| \right), \\ &= \sum_{j=1}^{j=\varrho} |\gamma_j| \left( \frac{\sqrt{2\varrho^4+2}-(\varrho^2+1)}{2} + \frac{(\varrho^2+1)+\sqrt{2\varrho^4+2}}{2} \right), \\ &= \sqrt{2\varrho^4+2}\,EDSS_R(K_\varrho). \end{aligned}$$

□

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## 6. Conclusion

In this research, we have explained the reverse first Zagreb energy, reverse second Zagreb energy, and reverse degree square sum energy, offering a complete explanation of these measures of energy in graph theory. Through a detailed examination of these measures of energy, we have obtained explicit formulas for their values with respect to certain classes of graphs, such as star graphs, complete graphs, and complete bipartite graphs. In addition, we have introduced the proofs describing the behavior of these energy measures under graph operations like splitting and shadowing, specifically for the family of complete graphs. Although this research provides significant insights into the behavior of reverse graph energies, there are a number of avenues that need to be explored in the future. One such direction is exploring the correlation of these reverse energies with other graph parameters, including vertex degrees, graph diameters, or connectivity measures. Further extrapolating these studies to more general classes of graphs than the ones discussed here may provide additional insight into the underlying mechanism controlling graph energy measures. In addition, investigating potential uses of these reverse energies in practical contexts, like network analysis or optimization problems, may reveal practical implications and further elucidate their importance in various fields. In general, this research provides a foundation for future research efforts to demystify the complex properties of graph energy measures and their general implications across different fields.

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