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A class of non-stationary ternary 4-point subdivision schemes based on iterations

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Abstract. This paper presents a novel class of non-stationary ternary 4-point subdivision schemes, capable of generating C^3 continuous limit curves. These schemes are built upon the generalized ternary scheme of order 5 presented in [19] and iterated functions. A distinctive feature is that the shape of the resulting limit curve varies with alterations in the initial parameters. To facilitate lo-calized shaping adjustments, we devise a non-uniform subdivision scheme that extends our non-stationary schemes. This method al-lows for tension paramters to be assigned to each edge of the initial control polygon, offering enhanced flexibility and precision in curve shaping.

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Key Words: Non-stationary ternary scheme; Smoothness; Iteration; Non-uniform.

1. Introduction

Over the last few decades, subdivision algorithms have emerged as the most effi-cient iterative methods for creating curves and surfaces in the domains of Computer Aided Geometric Design and Computer Graphics [4, 5, 9, 10]. Generally speaking, a subdivision scheme is divided into a non-stationary one and a stationary one depending on whether the subdivision rules change during each recursion level. Compared with the stationary schemes [12, 13], a notable benefit of non-stationary schemes lies in its capacity to reproduce exponential polynomials used in biomedical imaging [1]. Beccari et al. [2] constructed a ternary, non-stationary 4-point scheme employing an iterative method. Likewise, Tan et al. [16] introduced a binary, non-stationary 3-point approximating scheme bulid upon a distinct iterative method, capable of generating a diverse range of curves. Based on hyperbolic B-spline basis functions, Siddiqi et al. [15] presented ternary non-stationary three point and four point subdivision schemes. Zhang et al. [17] proposed a generalized cubic exponential B-spline scheme, offering versatility in producing various curve types, among which conics are also included. Since the ternary subdivision scheme yields a limit function exhibiting equal or superior smoothness with significantly reducing its support width (see [3, 11]), this paper presents a new family of non-stationary ternary 4-point schemes, which are derived through iterative modification of the generalized ternary subdivision scheme of order 5. Compared to the non-stationary schemes [2, 16, 17], the proposed non-stationary scheme surpasses them in terms of support width and smoothness. Further, the resulting curves generated by the proposed schemes will be changed when assigning the different initial parameters. To achieve localized management of the curve's shape, we refine the presented non-stationary ternary subdivision method by adopting a non-uniform approach. This is achieved by uniquely assigning a local tension parameter to each side of the initial control polygon.

The article is structured as outlined below. Section 2 provides foundational concepts and definitions related to subdivision schemes. In Section 3, we develop a new class of non-stationary ternary 4-point schemes through an iterative approach, and analyze their convergence and smoothness. Subsequently, Section 4 presents another non-stationary ternary scheme, utilizing a different iteration method. To improve localized management of the generated curve, we introduce a non-uniform scheme. Lastly, Section 5 gives the comprehensive conclusion.

2. Preliminaries

In this section, we introduce the notations for the remainder of the paper and briefly review those tools used in the analysis of the non-stationary subdivision schemes.

Provided a set of initial control points, denoted as $\boldsymbol{P}^0 = \{P_j^0 : j \in \mathbb{Z}\} \in \ell_0(\mathbb{Z})$, where $\ell_0(\mathbb{Z})$ represents the vector space of real-valued sequences that have finite support. For $k \in \mathbb{N}$, $\boldsymbol{P}^{k+1} = \{P_j^{k+1} : j \in \mathbb{Z}\}$ is produced through the non-stationary ternary scheme

$$(\boldsymbol{P}^{k+1})_j = (S_{\boldsymbol{a}^k}\boldsymbol{P}^k)_j := \sum_{i\in\mathbb{Z}} a_{j-3i}^k P_i^k,$$

where $\boldsymbol{a}^k = \{a_j^k : j \in \mathbb{Z}\}$ is the k-level mask with finite length. We use $\{S_{\boldsymbol{a}^k}\}_{k\in\mathbb{N}}$ to represent the non-stationary scheme. The k-level symbol corresponding to $\boldsymbol{a}^k$ is reprsented as $a^k(z) = \sum_{j\in\mathbb{Z}} a_j^k z^j$. If $\boldsymbol{a}^k = \boldsymbol{a}$, i.e., the mask is not influenced by changes in k, the subdivision scheme is classified as stationary and labeled as $S_{\boldsymbol{a}}$.

To analyze the convergence and smoothness of the non-stationary ternary subdvision scheme, we review the relevant definitions and results.

Definition 2.1. [7] *A non-stationary ternary subdivision scheme* $\{S_{\boldsymbol{a}^k}\}_{k\in\mathbb{N}}$ *converges to a continuous function* $f_{\boldsymbol{P}^0} \in C^0$*, if*

$$\lim_{k\to\infty} ||f_{\boldsymbol{P}^0}(j3^{-k}) - P_j^k||_\infty = 0.$$

Under the circumstance, we say that $\{S_{\boldsymbol{a}^k}\}_{k\in\mathbb{N}}$ *is* C^0 *convergent. If* $f_{\boldsymbol{P}^0} \in C^l$, $\{S_{\boldsymbol{a}^k}\}_{k\in\mathbb{N}}$ *is said to be* C^l *convergent.*

Theorem 2.2. [18] *Suppose a stationary ternary subdivision scheme* $S_{\boldsymbol{a}}$ *with* $\boldsymbol{a} = \{a_j\}_{j\in\mathbb{Z}}$ *satisfies* $\sum_{j\in\mathbb{Z}} a_{3j} = \sum_{j\in\mathbb{Z}} a_{3j+1} = \sum_{j\in\mathbb{Z}} a_{3j+2} = 1$. *Denote* S_n *be the nth order difference scheme of* $S_{\boldsymbol{a}}$ *which satisfies*

$$\sum_{j\in\mathbb{Z}} a_{3j}^{(n)} = \sum_{j\in\mathbb{Z}} a_{3j+1}^{(n)} = \sum_{j\in\mathbb{Z}} a_{3j+2}^{(n)} = 1,\ a^{(n)}(z) = \sum_{j\in\mathbb{Z}} a_j^{(n)} z^j = \Big(\frac{3z^2}{1+z+z^2}\Big)^n a(z).$$

If there exists a positive integer L such that $\|(\frac{1}{3}S_{n+1})^L\|_\infty < 1$*, then the scheme* $S_{\boldsymbol{a}}$ *is* C^n *convergent, where*

$$\|(\frac{1}{3}S_{n+1})^L\|_\infty = \Big\{\sum_{j\in\mathbb{Z}} \Big|(\frac{1}{3}a_{3^L j+i}^{(n+1)})^{(L)}\Big| : i = 0, 1, \cdots, 3^L - 1\Big\}.$$

Definition 2.3. [7] *We say that a non-stationary ternary subdivision scheme* $\{S_{\boldsymbol{a}^k}\}_{k\in\mathbb{N}}$ *is asymptotically equivalent a stationary scheme* $S_{\boldsymbol{a}}$*, if*

$$\sum_{k\in\mathbb{Z}} ||S_{a^k} - S_a||_\infty < \infty, \qquad \textit{with} \quad ||S_{\boldsymbol{a}^k}||_\infty = \max\{\sum_{j\in\mathbb{Z}} |a_{i-3j}^k| : i = 0, 1, 2\}.$$

Theorem 2.4. [7] *A non-stationary subdivision scheme* $\{S_{\boldsymbol{a}^k}\}_{k\in\mathbb{N}}$ *is convergent, if there exists a convergent subdivision scheme* $S_{\boldsymbol{a}}$*, which is asymptotically equivalent to* $\{S_{\boldsymbol{a}^k}\}_{k\in\mathbb{N}}$.

Theorem 2.5. [7] *Consider two asymptotically equivalent subdvision schemes with finite support: a non-stationary ternary scheme denoted by* $\{S_{\boldsymbol{a}^k}\}_{k\in\mathbb{N}}$ *and a stationary ternary scheme represented by* $S_{\boldsymbol{a}}$. *Suppose* $S_{\boldsymbol{a}}$ *is* C^n *convergent and*

$$\sum_{k=0}^{\infty} 3^{nk} ||S_{\boldsymbol{a}^k} - S_{\boldsymbol{a}}||_\infty < \infty,$$

then $\{S_{\boldsymbol{a}^k}\}_{k\in\mathbb{N}}$ *is* C^n *convergent.*

Next, we revisit fundamental concepts regarding the generation of exponential polynomials.

Definition 2.6. [14] *Given* $m \in \mathbb{N}$ *and a finite collection of real or imaginary numbers* $\boldsymbol{\alpha} = \{\alpha_0, \alpha_1, \cdots, \alpha_m\}$ *with* $\alpha_m \neq 0$. *The space of m-dimensional exponential polynomials* $V_{m,\boldsymbol{\alpha}}$ *is defined as follows*

$$V_{m,\boldsymbol{\alpha}} := \Big\{f : \mathbb{R} \to \mathbb{C}, f \in C^m(\mathbb{R}) : \sum_{j=0}^{m} \alpha_j D^j f = 0\Big\}.$$

Lemma 2.7. [14] *Let* $\alpha(z)=\sum_{j=0}^{m}\alpha_j z^j$ *and denote by* $\{(\theta_j,\tau_j)\}_{j=0,\cdots,N}$ *the collection of zeros with multiplicity, fulfilling*

$$D^{\beta}\alpha(\theta_j)=0,\quad \beta=0,1,\cdots,\tau_j-1,\ \ j=0,1,\cdots,N.$$

Then $m=\sum_{j=0}^{N}\tau_j$, $V_{m,\alpha}:=span\{x^{\beta}e^{\theta_j x},\beta=0,1,\cdots,\tau_j-1,\ j=0,1,\cdots,N\}$.

Definition 2.8. [8] *Given a sequence of subdivision symbols* $\{a^k(z)\}_{k\in\mathbb{N}}$*, the associated subdivision scheme is termed* $V_{m,\boldsymbol{\alpha}}$*-generating, if it converges and, for any* $g\in V_{m,\boldsymbol{\alpha}}$*, there exists an initial sequence* $\mathbf{g}^{(0)}$ *uniformly selected from* $\tilde{g}\in V_{m,\boldsymbol{\alpha}}$ *fulfilling the property stated below:*

$$\lim_{k\to\infty} S_{\boldsymbol{a}^{n+k}}S_{\boldsymbol{a}^{n+k-1}}\cdots S_{\boldsymbol{a}^{n}}\mathbf{g}^{(0)}=g,\quad n\geq 0.$$

The subsequent results outline the necesessary conditions for $\{S_{\boldsymbol{a}^k}\}_{k\in\mathbb{N}}$ to generate $V_{m,\boldsymbol{\alpha}}$.

Theorem 2.9. [6] *A non-stationary ternary subdivision scheme linked to the symbols* $\{a^k(z)\}_{k\in\mathbb{N}}$ *generates* $V_{m,\boldsymbol{\alpha}}$*, under the fulfillment of the following conditions*

$$D^{\beta}a^k(\mu)=0,\quad \beta=0,1,\cdots,\tau_j-1,$$

for all $\mu\in U_k:=\left\{\varepsilon e^{-\theta_j 3^{-k-1}}:\varepsilon\in\left\{e^{2\pi\iota/3},e^{4\pi\iota/3}\right\},\ \iota^2=-1,\ j=1,2,\cdots,N\right\}$.

3. The non-stationary ternary 4-point subdivision scheme based on an iteration

The goal of this section is to give a class of non-stationary ternary 4-point subdivision schemes, employing an iteration method that builds upon the generalized ternary subdivision scheme of order 5 in [19], and study its smoothness.

3.1. **Construction of non-stationary ternary 4-point subdivision schemes.** The generalized ternary subdivision scheme of order 5 in [19] is characterized by the following refinement rules

$$\begin{cases} P_{3i-1}^{k+1}=\dfrac{4(v^k)^2+6v^k+5}{9(1+2v^k)^2}P_{i-1}^k+\dfrac{28(v^k)^2+24v^k-1}{9(1+2v^k)^2}P_i^k+\dfrac{4(v^k)^2+6v^k+5}{9(1+2v^k)^2}P_{i+1}^k,\\ P_{3i}^{k+1}=\dfrac{2v^k+3}{9(1+2v^k)^2}P_{i-1}^k+\dfrac{24(v^k)^2+20v^k+1}{9(1+2v^k)^2}P_i^k+\dfrac{12(v^k)^2+14v^k+4}{9(1+2v^k)^2}P_{i+1}^k+\dfrac{1}{9(1+2v^k)^2}P_{i+2}^k,\\ P_{3i+1}^{k+1}=\dfrac{1}{9(1+2v^k)^2}P_{i-1}^k+\dfrac{12(v^k)^2+14v^k+4}{9(1+2v^k)^2}P_i^k+\dfrac{24(v^k)^2+20v^k+1}{9(1+2v^k)^2}P_{i+1}^k+\dfrac{2v^k+3}{9(1+2v^k)^2}P_{i+2}^k, \end{cases}\tag{3.1}$$

where

$$v^k=\frac{1}{2}(e^{\frac{t}{3^{k+1}}}+e^{-\frac{t}{3^{k+1}}}),\quad t\in\{0,s,\iota s|s>0\},\ k\in\mathbb{N}.\tag{3.2}$$

From Proposition 2 of [3], we know v^k and v^{k+1} fulfill the subsequent iteration

$$v^{k+1}=\frac{1}{2}Re\Big(\big(v^k+\sqrt{(v^k)^2-1}\big)^{\frac{1}{3}}+\big(v^k+\sqrt{(v^k)^2-1}\big)^{-\frac{1}{3}}\Big),\quad v^0\in(0,+\infty).\tag{3.3}$$

It can be seen from (3.3) that $\{v^k\}_{k\in\mathbb{N}}$ exhibits strictly increasing and converges to 1 as $k\to+\infty$ for $v^0\in(0,1)$, while $\{v^k\}_{k\in\mathbb{N}}$ is strictly decreasing and v^k converges to 1 as $k\to+\infty$ when $v^0\in(1,+\infty)$. Correspondingly, when $v^0=1$, $v^k=1$ for $k\in\mathbb{N}$, which means that the scheme (3.3) simplified to the ternary quartic B-spline. Hence $\lim_{k\to+\infty}v^k=1$.

In view of Theorem 2.5, it is straightforward to deduce that the generalized ternary subdivision scheme of order 5 is C^3 convergent. And according to Theorem 2.9, it has the capacity to generate the function space $E := span\{1, x, x^2, e^{\pm tx}\}$. Consequently, the scheme is capable of generating conic sections.

Actually, the non-stationarity of the subdivision scheme (3.1) results from the iteration (3.3). And the mask of the scheme (3.1) can be regarded as functions of this iteration. Additionally, the utilization of the function $g(x) = \frac{1}{1+2x}$ allows for a reformulation of the generalized ternary subdivision scheme of order 5 in (3.1), into an alternative representation

$$\begin{cases} P_{3i-1}^{k+1} = a_2^k P_{i-1}^k + a_{-1}^k P_i^k + a_{-4}^k P_{i+1}^k, \\ P_{3i}^{k+1} = a_3^k P_{i-1}^k + a_0^k P_i^k + a_{-3}^k P_{i+1}^k + a_{-6}^k P_{i+2}^k, \\ P_{3i+1}^{k+1} = a_4^k P_{i-1}^k + a_1^k P_i^k + a_{-2}^k P_{i+1}^k + a_{-5}^k P_{i+2}^k, \end{cases} \tag{3.4}$$

with the k-level mask

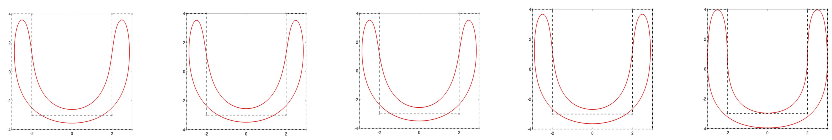
$$\begin{aligned} &a_{-6}^k = a_4^k = \frac{1}{9}g^2(v^k), \quad a_{-5}^k = a_3^k = \frac{1}{9}g(v^k) + \frac{2}{9}g^2(v^k), \quad a_{-4}^k = a_2^k = \frac{1}{9} + \frac{1}{9}g(v^k) + \frac{1}{3}g^2(v^k), \\ &a_{-3}^k = a_1^k = \frac{1}{3} + \frac{1}{9}g(v^k), \quad a_{-2}^k = a_0^k = \frac{2}{3} - \frac{2}{9}g(v^k) - \frac{1}{3}g^2(v^k), \quad a_{-1}^k = \frac{7}{9} - \frac{2}{9}g(v^k) - \frac{2}{3}g^2(v^k). \end{aligned} \tag{3.5}$$

Regarding the function $g(x)$, it is noteworthy $\lim_{x\to 0^+} g(x) = 1, \lim_{x\to 1} g(x) = \frac{1}{3}$. Together with the fact that $\lim_{v^0\to 0^+} g(v^0) = 1, \lim_{k\to\infty} v^k = 1$, we actually have $\lim_{k\to\infty} g(v^k) = \frac{1}{3}$.

Indeed, there exists potential candidates for the function $g(x)$, if using different functions instead of the function $g(x)$ or the iteration (3.3) in (3.4), we can get a different non-stationary ternary subdivision scheme. Hence, we can take a candidate $g_1^k(x) = \frac{1}{2x+x^{\gamma^k}}$ with an iteration $\gamma^{k+1} = \frac{\gamma^k}{3}$ and $\gamma^0 \geq 0$. Note that $g_1^k(x)$ for $\gamma^0 > 0$ exhibits the same characteristic as $g(x)$, namely, $\lim_{x\to 1} g_1^k(x) = \frac{1}{3}$. Besides, when $\gamma^0 = 0$, $g_1^k(x)$ simplifies to $g(x)$. We define the scheme $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$ as the one whose mask is the one in (3.5) with $g(v^k)$ replaced by $g_1^k(v^k)$. In this case, the scheme can be seen as one version of the modified generalized ternary scheme of order 5.

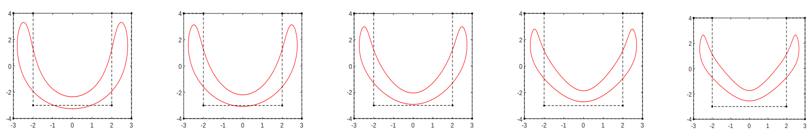
Remark 3.1. *Note that if $\gamma^0 = 0$, then $\gamma^k = 1$ for $k \in \mathbb{N}$, and the scheme $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$ reduces to the generalized ternary scheme of order* 5 *in* [19]*. In particular, when $v^0 = 1$, the scheme transforms into the ternary quartic B-spline scheme.*

Fig 1 illustrates visual comparision of the limit curves generated by the schemes in [**?**, **?**] and the scheme $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$ from the same control polygon. From Fig 1 we can see that the scheme $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$ exhibits superior performance relative to the other two schemes, due to the existence of γ^0. Fig 2 and Fig 3 illustrate the generation of some limit curves by the non-stationary ternary scheme $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$ with different parameters v^0 and γ^0. From Fig 3, it shows that when $v^0 \in (0, 1)$, the limit curves tend to converge towards the initial control polygon as γ^0 decreases. Conversely, when $v^0 \in (1, +\infty)$, the limit curves are close to the initial control polygon with the increasing of the parameter γ^0. When $v^0 = 1$, the limit curve does not change with the variation of the parameter γ^0, because the scheme $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$ turns into a stationary scheme in this case.



(a) (b) (c) (d) (e)

FIGURE 1. Comparison between the limit curves obtained by the generalized ternary subdivision scheme of order 5 in [19](a), the scheme in [15](b-c), the non-stationary ternary subdivision scheme in [15] and the scheme $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$ (d-e). (a) for $v^0 = 10$, (b) for $\alpha = 0.17\pi$, (c) for $\alpha = 0.5\pi$, (d) for $\gamma^0 = 1.5, v^0 = 10$, (e) for $\gamma^0 = 15, v^0 = 10$.



(a) $\gamma^0 = 0$ (b) $\gamma^0 = 4$ (c) $\gamma^0 = 8$ (d) $\gamma^0 = 16$ (e) $\gamma^0 = 64$

FIGURE 2. Limit curves generated by the scheme $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$ with $v^0 = 0.1$ and $\gamma^0 = 0,\ 4, 8, 16, 64$ (from left to right).

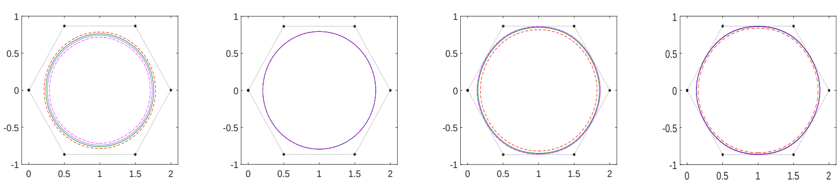


FIGURE 3. Limit curves generated by the scheme $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$ with $v^0 = 0.1, 1, 10, 100$ (from left to right) and $\gamma^0 = 0$(red), 5(green), 25(blue) and 125(magenta), respectively. The black is the initial control polygon.

3.2. **Smoothness of non-stationary ternary 4-point subdivision schemes.**

Theorem 3.2. *For $\gamma^0 \geq 0$ and $v^0 \in (0, +\infty)$, the non-stationary ternary 4-point subdivision scheme $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$ is convergent. And the convergence rate of $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$ is bounded by*

$$\|F_k(t) - S^{\infty}_{a^k} f^0\|_\infty \leq \frac{12C}{7}(\frac{5}{12})^{k-M},$$

where C is a generic constant.

Proof. Note that the symbol about the scheme $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$ can be expressed in the following form:

$$\begin{aligned} a^k(z) = & \Big[\frac{1}{9} + \frac{1}{9}g_1^k(v^k) + \frac{1}{3}(g_1^k(v^k))^2\Big](z^2 + z^{-4}) + \Big[\frac{1}{9}g_1^k(v^k) + \frac{2}{9}(g_1^k(v^k))^2\Big](z^3 + z^{-5}) + \\ & \Big[\frac{2}{3} - \frac{2}{9}g_1^k(v^k) - \frac{1}{3}(g_1^k(v^k))^2\Big](1 + z^{-2}) + \Big[\frac{1}{3} + \frac{1}{9}g_1^k(v^k)\Big](z + z^{-3}) + \frac{1}{9}(g_1^k(v^k))^2 \\ & (z^{-6} + z^4) + \Big[\frac{7}{9} - \frac{2}{9}g_1^k(v^k) - \frac{2}{3}(g_1^k(v^k))^2\Big]z^{-1}. \end{aligned}$$

Let $a^k(z) = (1+z+z^2)b^k(z)$, then

$$\begin{aligned} b^k(z) =& \frac{1}{9}g_1^k(v^k)(z^2+z^{-6}) + [\frac{1}{9}g_1^k(v^k) + \frac{1}{9}(g_1^k(v^k))^2](z+z^{-5}) + [\frac{1}{9}g_1^k(v^k) + \frac{1}{9}(g_1^k(v^k))^2](1+z^{-4}) \\ &+ [\frac{2}{9} - \frac{2}{9}(g_1^k(v^k))^2](z^{-1}+z^{-3}) + [\frac{1}{3} - \frac{2}{9}(g_1^k(v^k))^2 - \frac{2}{9}g_1^k(v^k)]z^{-2}. \end{aligned}$$

From the factorization of $a^k(z)$, we have

$$\begin{aligned} L(\Delta f^{k+1};z) &= \sum_{j\in\mathbb{Z}}(\Delta f^{k+1})_j z^j = \sum_{j\in\mathbb{Z}}(f_j^{k+1} - f_{j-1}^{k+1})z^j \\ &= L(f^{k+1};z) - zL(f^{k+1};z) = (1-z)L(f^{k+1};z) \\ &= (1-z)a^k(z)L(f^k;z^3) = (1-z^2)b^k(z) = L(f^k;z^3) \\ &= b^k(z)L(\Delta f^k;z^3). \end{aligned}$$

which implies

$$S_{b^k}\Delta f^k = \Delta(S_{a^k}f^k). \tag{3.6}$$

Consider the sequence $\{F_k(t)\}_{k\in\mathbb{Z}}$ defined by $F_k(3^k\alpha) = f_\alpha^k, \alpha \in \mathbb{Z}$. To show convergence of S_{a^k}, it is sufficient to show that $\{F_k(t)\}$ is a Cauchy sequence with respect to the sup-norm. By the observation that a piecewise linear function attains its extreme values at its breakpoints

$$\sup_{t\in\mathbb{R}}|F_{k+1}(t) - F_k(t)| = \max\{\sup_{i\in\mathbb{Z}}|f_{3i}^{k+1} - g_{3i}^{k+1}|, \sup_{i\in\mathbb{Z}}|f_{3i+1}^{k+1} - g_{3i+1}^{k+1}|, \sup_{i\in\mathbb{Z}}|f_{3i+2}^{k+1} - g_{3i+2}^{k+1}|\}, \tag{3.7}$$

where

$$g_{3i}^{k+1} = \frac{1}{3}f_{i-1}^k + \frac{2}{3}f_i^k,\ g_{3i+1}^{k+1} = f_i^k,\ g_{3i+2}^{k+1} = \frac{2}{3}f_i^k + \frac{1}{3}f_{i+1}^k. \tag{3.8}$$

In terms of the z-transform, (3.8) can be represented by

$$L(g^{k+1};z) = \frac{(1+z+z^2)^2}{3z}L(f^k;z^3).$$

Thus

$$\begin{aligned} L(f^{k+1};z) - L(g^{k+1};z) &= (a^k(z) - \frac{(1+z+z^2)^2}{3z})L(f^k;z^3) \\ &= (1+z+z^2)(b^k(z) - \frac{1+z+z^2}{3z})L(f^k;z^3) \\ &= (1+z)d^k(z)L(f^k;z^3), \end{aligned}$$

with $d^k(z) = b^k(z) - \frac{1+z+z^2}{3z}$. Since $a^k(1) = 3, d^k(1) = 0$, hence $d^k(z) = (1-z)e^k(z)$, which implies

$$\begin{aligned} e^k(z) =& -\frac{(g_1^k(v^k))^2 z}{9} + \frac{(g_1^k(v^k)^2}{9z^6} + \frac{3 - g_1^k(v^k) - 2(g_1^k(v^k))^2}{9} + \frac{5 - g_1^k(v^k) - 3(g_1^k(v^k))^2}{9z} \\ &+ \frac{6 - (g_1^k(v^k) - g_1^k(v^k))^2}{9z^2} + \frac{g_1^k(v^k + (g_1^k(v^k)^2 + 3}{9z^3} + \frac{1 + g_1^k(v^k + 3(g_1^k(v^k)^2}{9z^4} \\ &+ \frac{g_1^k(v^k + 2(g_1^k(v^k)^2}{9z^5}. \end{aligned}$$

This leads finnally to

$$L(f^{k+1} - g^{k+1};z) = e^k(z)(1-z^3)L(f^k;z^3) = e^k(z)L(\Delta f^k;z^3). \tag{3.9}$$

Recalling that, by (3.7),

$$\|F_{k+1}-F_k\|_\infty=\sup_{j\in\mathbb{Z}}|f_j^{k+1}-g_j^{k+1}|=\|f^{k+1}-g^{k+1}\|_\infty.$$

Combing (3.6) with (3.7), we obtain

$$f^{k+1}-g^{k+1}=S_{e^k}\Delta f^k=S_{e^k}S_{b^1}S_{b^2}\cdots S_{b^k}\Delta f^0.$$

So

$$\|F_{k+1}-F_k\|_\infty=\|f^{k+1}-g^{k+1}\|_\infty\le\|S_{e^k}\|_\infty\|S_{b^1}S_{b^2}\cdots S_{b^k}\Delta f^0\|_\infty.$$

Since $\lim_{k\to\infty}g_1^k(v^k)=\frac{1}{3}$, then there exists $M\in\mathbb{Z}_+$ such that for every $k>M$ we have $g_1^k(v^k)\le\frac{1}{2}\le 1$. Furthermore for every $k>M$,

$$\begin{aligned}\|S_{b^k}\|_\infty=&\max\{\sum_{j\in\mathbb{Z}}|b^k_{i-3j}|\}=\max\{|\frac{3-2(g_1^k(v^k))^2-2g_1^k(v^k)}{9}|,\frac{(g_1^k(v^k))^2}{9}+|\frac{2-2(g_1^k(v^k)^2}{9}|\\&+|\frac{(g_1^k(v^k))^2+1}{9}|+|\frac{g_1^k(v^k)+(g_1^k(v^k))^2}{9}|\le\frac{5}{12},\\ \|S_{e^k}\|_\infty=&\max\{\sum_{j\in\mathbb{Z}}|e^k_{i-3j}|\}=\max\{|\frac{6-(g_1^k(v^k))^2-g_1^k(v^k)}{9}|+|\frac{g_1^k(v^k)+2(g_1^k(v^k))^2}{9}|\\&+\frac{(g_1^k(v^k))^2}{9},|\frac{3-g_1^k(v^k-2(g_1^k(v^k)^2}{9}|+|\frac{g_1^k(v^k)+(g_1^k(v^k))^2+3}{9}|+\frac{(g_1^k(v^k)^2}{9}\\&\frac{5-g_1^k(v^k-3(g_1^k(v^k)^2}{9}+|\frac{1+g_1^k(v^k+3(g_1^k(v^k)^2}{9}|\}\le\frac{7}{9}.\end{aligned}$$

Hence

$$\begin{aligned}\|F_{k+1}-F_k\|_\infty\le&\|S_{e^k}\|_\infty\|S_{b^1}S_{b^2}\cdots S_{b^k}\Delta f^0\|_\infty\\=&\|S_{e^k}\|_\infty\|\|S_{b^1}S_{b^2}\cdots S_{b^M}S_{b^{M+1}}S_{b^{M+2}}\cdots S_{b^k}\Delta f^0\|_\infty\\\le&C(\frac{5}{12})^{k-M},\end{aligned}$$

where C is a generic constant. Thus $\{F_k(t)\}_{k\in\mathbb{Z}}$ is uniformly convergent.

Note that

$$\begin{aligned}\|(F_k-S^\infty_{a^k}f^0)(x)\|_\infty=\lim_{\ell\to\infty}|F_\ell(x)-F_k(x)|\le&\sum_{j=k}^\infty|F_{j+1}(x)-F_j(x)|\\\le&\sum_{j=k}^\infty C(\frac{5}{12})^{j-M}=\frac{12C}{7}(\frac{5}{12})^{k-M}.\end{aligned}$$

□

Theorem 3.3. *For $\gamma^0\ge 0$ and $v^0\in(0,+\infty)$, the non-stationary ternary 4-point subdivision scheme $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$ is C^3 convergent.*

Proof. Let $q^k(z)=(\frac{3z}{1+z+z^2})^3a^k(z)$, where

$$q^k(z)=3\big[g_1^k(v^k)\big]^2(z+z^{-3})+[3g_1^k(v^k)-3(g_1^k(v^k))^2](1+z^{-2})+3[1-2g_1^k(v^k)]z^{-1}.$$

To establish the C^3 convergence of the proposed non-stationary ternary scheme, it suffices to demonstrate that the scheme $\{S_{\boldsymbol{q}^k}\}_{k\in\mathbb{N}}$, which is associated with the function $q^k(z)$, is C^0 convergent.

As k goes to ∞, $q^k(z)$ becomes

$$q(z)=\frac{1}{3}(z+z^{-3})+\frac{2}{3}(1+z^{-2})+z^{-1}.$$

By Theorem 2.2, we have the scheme $S_{\boldsymbol{q}}$ associated with $q(z)$ is C^0. Note that

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}|q_{3j}^k-q_{3j}|\\=&\sum_{j\in\mathbb{Z}}|q_{3j+1}^k-q_{3j+1}|\\=&\Big|3g_1^k(v^k)-3(g_1^k(v^k))^2-\frac{2}{3}\Big|+\Big|3(g_1^k(v^k))^2-\frac{1}{3}\Big|\\\le&\Big|3g_1^k(v^k)-1\Big|+2\Big|3(g_1^k(v^k))^2-\frac{1}{3}\Big|.\end{aligned}$$

Together with the definition of $f_1^k(x)$, it can be seen that there exists a constant c_1 independent of k such that

$$\sum_{j\in\mathbb{Z}}|q_{3j}^k-q_{3j}|=\sum_{j\in\mathbb{Z}}|q_{3j+1}^k-q_{3j+1}|\le c_1|v^k-1|.$$

Similarly, we deduce the existence of another constant c_2, which is not reliant on k, satisfying the following condition:

$$\sum_{j\in\mathbb{Z}}|q_{3j+2}^k-q_{3j+2}|=2|1-3g_1^k(v^k)|\le c_2|v^k-1|.$$

Denote $C=max\{c_1,c_2\}$, to demonstrate the C^0 convergence of the scheme $\{S_{\boldsymbol{q}^k}\}_{k\in\mathbb{N}}$, it suffices to prove the condition that

$$\sum_{k\in\mathbb{Z}}\|S_{q^k}-S_q\|_\infty<\infty,\qquad i.e.,\quad C|v^k-1|<\infty.$$

From (3.3), we derive the existence of a constant $L\in(0,1)$ satisfying

$$|v^k-1|\le L|v^{k-1}-1|.$$

Consequently, we obtain

$$|v^k-1|\le L|v^{k-1}-1|\le L^2|v^{k-2}-1|\le\cdots\le L^k|v^0-1|.$$

Due to $L\in(0,1)$, we get $C|v^{k-1}-1|<\infty$, i.e.,

$$\sum_{k\in\mathbb{Z}}\|S_{q^k}-S_q\|_\infty=\max\{\sum_{j\in\mathbb{Z}}|q_{i+3j}^k-q_{i+3j}|,\ i=0,1,2\}<\infty.$$

Therefore, we can easily verify that the scheme $\{S_{\boldsymbol{q}^k}\}_{k\in\mathbb{N}}$ associated with $q^k(z)$ is C^0 convergent, which completes the proof. □

In comparison to the non-stationary schemes in [3, 16, 17], the proposed scheme $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$ surpasses the others in both the area of support and smoothness. A detailed comparison of these non-stationary schemes is outlined in Table 1.

4. Further discussion

In this section, we extend our exploration by empolying a distinct iteration and a suitable function to derive a similar scheme. Compared to the scheme $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$, the proposed scheme in this section is capable of generating a greater diversity of curves, during to the larger scope of the parameter v^0. Furthermore, we present a locally-controlled, non-uniform ternary 4-point subdivision scheme. This scheme

Table 1. Comparison of the non-stationary schemes.

Schemes	Type	P-ary	Support	Continuity
The scheme in [16]	Approximating	Binary	6	C^3
The scheme in [17]	Interpolating	Binary	4	C^2
The scheme in [3]	Interpolating	Ternary	5	C^2
The scheme$\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$	Approximating	Ternary	5	C^3

provides the capability of assigning tension values to each edge of the initial control polygon, offering greater control.

Specially speaking, we replace the functions $g(x)$ by the function $g_2(x) = \frac{3}{x^2}$ and $v^{k+1} = \sqrt{v^k + 6}$ with $v^0 \in (-6, +\infty)$, respectively. In this way, we can derive a new non-stationary ternary 4-point scheme denoted by $\{S_{\boldsymbol{a}_2^k}\}_{k\in\mathbb{N}}$.

Before focusing on the analysis of the scheme , we present some results about the sequence $\{v^k\}_{k\in\mathbb{N}}$ to be used.

Lemma 4.1. *For the sequence* $v^{k+1} = \sqrt{v^k + 6}$, $v^0 \in (-6, +\infty)$, *it satisfies the following properties:*

(*i*) *if* $v^0 = 3$, *then* $v^k = 3$ *for* $k \in \mathbb{N}$*;*

(*ii*) *if* $v^0 \in (-6, 3)$, *then* $\{v^k\}_{k\in\mathbb{N}}$ *is strictly increasing and* $v^k \in (0, 3)$ *for* $k \in \mathbb{N}$*;*

(*iii*) *if* $v^0 \in (3, +\infty)$, *then* $\{v^k\}_{k\in\mathbb{N}}$ *is strictly decreasing and* $v^k \in (3, +\infty)$ *for* $k \in \mathbb{N}$.

Proposition 4.2. [3] *Given a monotonic sequence* $\{\alpha^k\}_{k\in\mathbb{N}}$, *if it is nondecreasing and possesses an upper bound (nonincreasing with a lower bound), then it converges to the upper (lower) bound.*

Based on the implications derived from Lemma 4.1 and Proposition 4.2, the subsequent conclusion can be drawn.

Proposition 4.3. *Given the initial parameter* $v^0 \in (-6, +\infty)$ *and* $v^{k+1} = \sqrt{v^k + 6}$ *for* $k \in \mathbb{N}$. *Then* $\lim_{k\to\infty} v^k = 3$.

Similar to the proof of Theorem 3.3, together with Proposition 4.3, we have the conclusion about the smoothness of the scheme $\{S_{\boldsymbol{a}_2^k}\}_{k\in\mathbb{N}}$.

Theorem 4.4. *The new scheme* $\{S_{\boldsymbol{a}_2^k}\}_{k\in\mathbb{N}}$ *is* C^3 *convergent, for* $v^0 \in (-6, +\infty)$.

Fig 4 shows some curves generated by the scheme $\{S_{\boldsymbol{a}_2^k}\}_{k\in\mathbb{N}}$ with different values of v^0. Compared to those limit curves generated by the scheme $\{S_{\boldsymbol{a}_1^k}\}_{k\in\mathbb{N}}$, these curves have richer shape, as the wide rage of parameter v^0. Note that, for the scheme $\{S_{\boldsymbol{a}_2^k}\}_{k\in\mathbb{N}}$, altering the initial value of v^0 leads to variations in the shape of the resulting limit curves. To control the limit curves well, we assign an initial local tension parameter for each edge of the initial control polygon, i.e., an initial tension values v_i^0 will be associated with $\overline{P_i^0 P_{i+1}^0}$. After k iterations, a tension v_i^k is

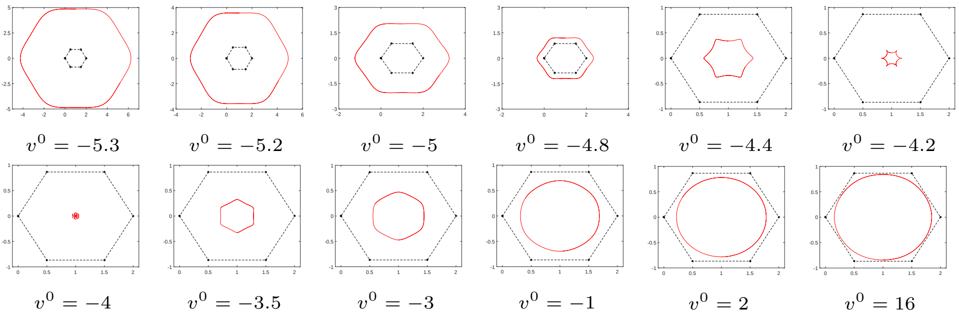


FIGURE 4. The limit curves generated by the scheme $\{S_{\boldsymbol{a}_2^k}\}_{k\in\mathbb{N}}$ with different parameters v^0.

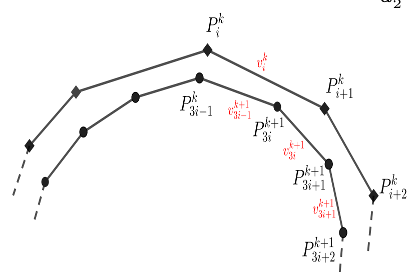


FIGURE 5. Generation of P_{3i}^{k+1} by the scheme (4.1).

assigned to $\overline{P_i^k P_{i+1}^k}$. We make them inherit respectively the tension values

$$v_{3i-1}^{k+1} = v_{3i}^{k+1} = v_{3i+1}^{k+1} = \sqrt{v_i^k + 6}.$$

To align with the specified pattern, the scheme $\{S_{\boldsymbol{a}_2^k}\}_{k\in\mathbb{N}}$ will be consequently described by the following nonuniform non-stationary scheme

$$\begin{cases} P_{3i-1}^{k+1} = \Big[\frac{1}{9} + \frac{1}{9}g_1^k(v_i^k) + \frac{1}{3}(g_1^k(v_i^k))^2\Big]P_{i-1}^k + \Big[\frac{7}{9} - \frac{2}{9}g_1^k(v_i^k) - \frac{2}{3}(g_1^k(v_i^k))^2\Big]P_i^k + \\ \qquad \Big[\frac{1}{9} + \frac{1}{9}g_1^k(v_i^k) + \frac{1}{3}(g_1^k(v_i^k))^2\Big]P_{i+1}^k, \\ P_{3i}^{k+1} = \Big[\frac{1}{9}g_1^k(v_i^k) + \frac{2}{9}(g_1^k(v_i^k))^2\Big]P_{i-1}^k + \Big[\frac{2}{3} - \frac{2}{9}g_1^k(v_i^k) - \frac{1}{3}(g_1^k(v_i^k))^2\Big]P_i^k + \Big[\frac{1}{3} + \\ \qquad \frac{1}{9}g_1^k(v_i^k)\Big]P_{i+1}^k + \frac{1}{9}(g_1^k(v_i^k))^2 P_{i+2}^k, \\ P_{3i+1}^{k+1} = \frac{1}{9}(g_1^k(v_i^k))^2 P_{i-1}^k + \Big[\frac{1}{3} + \frac{1}{9}g_1^k(v_i^k)\Big]P_i^k + \Big[\frac{2}{3} - \frac{2}{9}g_1^k(v_i^k) - \frac{1}{3}(g_1^k(v_i^k))^2\Big]P_{i+1}^k + \\ \qquad \Big[\frac{1}{9}g_1^k(v_i^k) + \frac{2}{9}(g_1^k(v_i^k))^2\Big]P_{i+2}^k. \end{cases} \tag{4.1}$$

The specific process is shown in Fig 5.

Remark 4.5. *The non-uniform non-stationary scheme (4.1) reduces to the uniform non-stationary scheme* $\{S_{\boldsymbol{a}_2^k}\}_{k\in\mathbb{N}}$*, when each initial tension is equal to the same value, i.e.,* $v_i^0 = v^0$*. In particular, the scheme (4.1) becomes the ternary quartic B-spline scheme, when* $v_i^0 \equiv 3$.

Fig 6 and Fig 7 show limit curves generated by the non-uniform scheme (4.1) with different local tension parameters. From Fig 6 and Fig 7, we can see that the

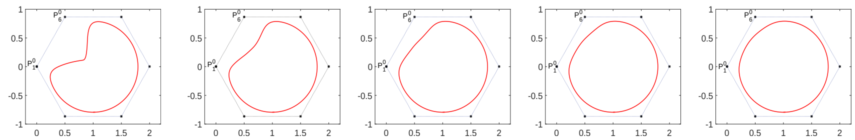


FIGURE 6. Limit curves generated by the scheme (4.1) with $v_i^0 = 3(i \neq 6)$ and $v_6^0 = 1.5, 1.75, 2, 2.25, 2.5$(from left to right).

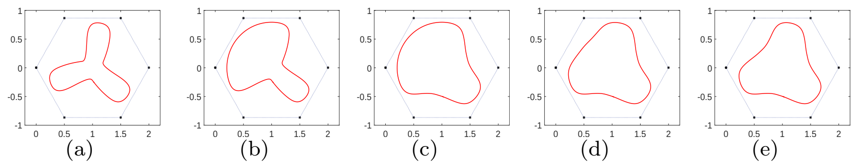


FIGURE 7. Limit curves generated by the scheme (4.1) with the following local tensions v_i^0: (a)[3,1.5,3,1.5,3,1.5], (b)[3,1.5,3,1.5,3,3], (c)[3,1.75,3,1.75,3,3], (d)[3,1.75,3,1.75,3,2], (e)[3,1.75,3,1.75,3,1.75].

resulting curves tend to expand towards the initial polygon with the increasing of the local parameter.

5. Conclusion

This paper presented a family of non-stationary ternary 4-point subdivision schemes utilizing iterative methods. These schemes are capable of generating C^3 limiting curves, offering a diverse range of curve shapes. To achieve localized control over the shape of limit curves, we have devised a non-uniform 4-point scheme that builds upon the non-stationary ternary 4-poit scheme. The non-uniform scheme incorporates an initial local tension parameter for each edge of the initial polygon, enabling precise manipulation of the limit curve's local geometry.

6. Author Contributions

Conceptualization, Z.Z. Zhang; methodology, Z.Z. Zhang and H.C. Zheng; software, Z.Z. Zhang and H.X. Cao; validation, Z.Z. Zhang and H.X. Cao; writing-original draft, Z.Z. Zhang; writing-review and editing, Z.Z. Zhang, H.X. Cao and H.C. Zheng; visualization, Z.Z. Zhang and H.X. Cao; funding acquisition, Z.Z. Zhang and H.X. Cao. All authors have read and agreed to the published version of the manuscript.

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