Punjab University Journal of Mathematics (2025), 57(04), 457-479
https://doi.org/10.52280/pujm.2025.57(04)05

Some Efficient Derivative-Based Semi-Open Quadrature Rules with Error Analysis

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Received
20 January, 2025

Accepted
24 October, 2025

Published Online
13 November, 2025

Abstract. This study suggests four innovative and effective quadrature techniques that combine functional evaluations and their first-order derivatives for data points that are identically spaced, with an emphasis on computational efficiency regarding time and cost utilisation. All the techniques are theoretically derived, and the theorems concerning accuracy, precision, and error terms are also established. The suggested approaches are derivative-based semi-open-type rules. Compared to the conventional rules, the proposed methods are more accurate and possess higher precision degree. Several numerical experiments are conducted to compare the accuracy, truncation errors, rates of convergence, cost evaluation and average execution times of the new approaches compared with the conventional methods. Because of their promisingly lower computational costs, the results of the analysis demonstrate that the developed methods are more efficient than the original methods from both a theoretical and numerical aspects.

AMS (MOS) Subject Classification Codes: 65D3065D32.
Key Words: Semi-open Newton-Cotes formulas, Derivative-based, Precision, Accuracy, Time and cost effectiveness.

1. INTRODUCTION

Numerical integration techniques are essential in computational mathematics, assisting in estimating the integrals in cases where the analytic solution appears impossible or excessively complicated. On the other hand, practitioners have been using the numerical integration routines to better approximate areas, volumes, moment/product of inertia and center of mass in complex cases of integrands and also when discrete data is the only available information [5]. In particular, these methods include Newton-Cotes formulas featuring both open and closed rules which employ evenly spaced weighted points in order to construct the interpolation polynomial. One of the subclasses of these formulas is the semi-open Newton-Cotes which uses only one endpoint in the evaluation of integrals and, therefore, offers a reasonable degree of accuracy with lesser work required. Quadrature approximations are beneficial in several scientific and engineering problems where the need for integration is embedded in the modeling and analysis. For example, the rendering and use of these devices in engineering is well explained by Chapra and Canale [5], which allows solving real engineering problems. In addition, Burden and Faires [4] discussed these techniques in relation to the error analysis that arises from these procedures and the costs associated with their precision. In general, a quadrature formula is described as:

$$\int_{\alpha}^{\beta} g(x)\,dx \approx \sum_{i=0}^{n} w_i g(x_i) \tag{1. 1}$$

Eq. (1. 1) represents Newton-Cotes (NC) technique in general form, where $x_0, x_1, x_2, ...x_{n-1}$ are $n+1$ evenly-spaced nodes in the interval $[\alpha, \beta]$, and the the constant w_i are the weights. In closed Newton-Cotes rules the function evaluation at the endpoints of the interval of integration are included in the formula, while in semi-open rules functional evaluation at one of the end points is excluded from the formula. This paper is focused on the modifications of semi-open quadrature rules where the functional evaluation of one end point of the interval of integration is excluded from the formula. We can rewrite (1. 2) as:

$$\int_{\alpha}^{\beta} g(x)\,dx \approx \int_{x_0}^{x_n} g(x)\,dx \approx \sum_{i=0}^{n} w_i g(x_i) \tag{1. 2}$$

where $x_0, x_1, x_2, ...x_{(n-1)}$ are distinct n integration points and w_i are the weights within the interval $[\alpha, \beta)$ with $x_i = a + ih$, $i = 0, 1, 2, ..., n-1$, and $h = (\beta-\alpha)/(n+1)$. The first few semi-open quadrature rules are: One-point rule:

$$\int_{\alpha}^{\beta} g(x)\,dx = (\beta-\alpha)g(\alpha) + (\beta-\alpha)^2 g'(\epsilon) \tag{1. 3}$$

Two-point rule:

$$\int_{\alpha}^{\beta} g(x)\,dx = (\beta-\alpha)g[(\alpha) + 0 \times \frac{g(\alpha-\beta)}{2}] - \frac{(\beta-\alpha)^3}{12} g^{(2)}(\epsilon) \tag{1. 4}$$

Three-Point rule:

$$\int_{\alpha}^{\beta} g(x)\,dx = \frac{(\beta-\alpha)}{4}[3g(\alpha) + 0 \times g\frac{(2\alpha+\beta)}{3} + g\frac{(\alpha+2\beta)}{3}] - \frac{(\beta-\alpha)^4}{18} g^{(3)}(\epsilon) \tag{1. 5}$$

where $\epsilon \in (\alpha, \beta)$.
This research examines the developments in numerical integration techniques with an emphasis on two-point semi-open Newton-Cotes rules. In the recent past, the literature highlights novel strategies, uses, and improvements in precision and effectiveness of these methods. By tackling problems with error variations as the number of sub-intervals (slits) rises, the work in [2] enhanced an earlier approach for lowering error in combined numerical integration. The novel method found the best quadrature formula combination to stabilise the error and make sure it remained constant and predictable as the number of sub-intervals increased. A simple yet accurate method was introduced in [33] to solve high-order linear Fredholm integro-differential equations. This method was extended to obtain precise and analytically approximate solutions to the high-order nonlinear Volterra-Fredholm-Hammerstein integro-differential equations. The improved method solved nonlinear algebraic equations rather than linear ones. The accuracy of the novel method was higher for nonlinear equations than some of the current methods. A technique for achieving the analytic approximation solution of the nonlinear differential equations arising from heat transport problems, namely through fins, was devised by Turkyilmazoglu in [34]. This approach can be easily employed and articulated without the need for discretisation because of the simplicity of the fundamental functions. In addition to comparing with other methods, Turkyilmazoglu [34]looked at discrepancies. To demonstrate that the proposed technique yielded very accurate results, a few physically interesting fin heat transfer problems were tackled. Numerous scientific and technical domains, such as probability theory [3] and [10], stochastic processes [8], and functional analysis [28], especially the the Hilbert space using self-adjoint operators in [13], highlight extensive uses of integration and quadrature approximations. In recent years, a variety of novel methods have been developed to solve definite integrals by using the derivatives of the function at various statistical means. The derivative at the interval's midpoint was used to suggest a Newton-Cotes rule for algebraic functions in [24]. Similarly, this work was expanded to include the evaluation of functional derivatives measured by using various statistical means at the terminal points of the interval. In [25], the three methods were compared using the three different types of averages. Furthermore, a number of alternative methods for both closed and open Newton-Cotes rules were developed, utilising the contra-harmonic mean [26] to determine the derivatives of the integrand. In [20], Rike and Imran presented two effective derivative-based techniques that employed the arithmetic mean in the mid-point formula.
Mahesar et al. (2023) [16] used a centroidal mean derivative approach to present new quadrature rules for numerical integration. The authors formulated and validated these principles with an emphasis on open Newton-Cotes formulas that did not include interval endpoints, showing increased accuracy and efficiency in integration tasks. Their results point to important developments in numerical techniques that can be used in many other domains. In [35], the efficient open Newton-Cotes quadrature rules were proposed including few or all points of the interval of integration. The methods outperformed and presented efficient results in comparison to existing methods from literature. When it came to error terms and estimated integral values, these new derivative-based schemes outperformed the classic Newton-Cotes formulas.
To increase the accuracy and precision of the usual rules, the previous works adapt to the

point that semi-open Newton-Cotes have received not as much of attention from the standpoint of derivative-based end-point methods. Compared to existing closed approaches, these enhancements may be more effective in handling integrals with an end-point singularity quickly. As a result, more research on the use of semi-open Newton-Cotes possessing the derivatives can be conducted for the numerical solutions of complex line integrals, higher dimensional integrals and Riemann-Stieltjes (RS) integrals. However, in [36], Riemann–Stieltjes coupled integral boundary conditions in Banach spaces were used to develop sufficient requirements for the presence and multiplicity of positive solutions to a class of nonlinear singular fractional differential systems. For every variable, the systems' nonlinear functions may be unique. Using the characteristics of Green's functions, an appropriate cone was built to overcome the difficulties presented by these singularities. The fixed-point theorem on cones was the foundation of the primary method. An example was also provided to illustrate the usefulness of the outcomes.

Moreover, a paper [11] discussed semi-implicit numerical techniques for resolving boundary-value problems (BVPs), with an emphasis on how they might be used to solve difficult or stiff differential equations. A thorough analysis of the various numerical integration techniques for approximating definite integrals, particularly in situations when analytical approaches are impractical, was given in the paper [30]. The Newton-Cotes methods (including Trapezoidal and Simpson's rule), Gaussian quadrature, Romberg integration, Monte Carlo methods, and Adaptive quadrature were among the methods covered. The study also evaluated the accuracy and efficiency of several error analyses techniques, including truncation and round-off errors. It also discussed the limitations and real-world uses of these techniques in a variety of disciplines, including engineering and physics. With an emphasis on error estimates for numerical integration, the comprehensive work in [15] examined symmetric four-point Newton-Cotes-type inequalities. Several groups of functions, such as those with bounded variation, bounded derivatives, Lipschitzian derivatives, convex derivatives, and others, were examined in order to determine how accurate these estimations were. By synthesising and expanding on current knowledge, the study offered a sophisticated understanding of how error bounds relied on the properties of integrated functions. The error estimates of Newton-Cotes quadrature techniques (such as Trapezoidal and Simpson's rules) applied to various function types are examined in the work [18]. It focussed on the variations in error for solitary, oscillatory, and smooth functions. In order to shed light on whether these techniques worked well or needed to be modified, the authors theoretically analysed error bounds and validated them using numerical trials. The sextic B-spline collocation approach was used by Nasir et al. [22] to approximate the solution of the generalised equal width wave equation. The computational outcomes showed that the new findings were superior than those from the earlier studies. Similar to the ordinary Newton-Cotes closed and open quadrature equations, the weighted Newton-Cotes closed and open quadrature rules were obtained [14] for a particular class of weighting functions. A very interesting application of the trigonometrically fitted symplectic approaches based on the closed Newton and Cotes formulae was the integration of a few physical issues [32]. Demir and Şanal [8] derived these inequalities using the perturbed trapezoid inequality. The constraint for the previously described inequality enhanced with the extremely s-convex functions. Lastly, the theorems developed for strongly s-convex functions were reduced to those provided for s-convex functions when the constant from strongly s-convexity disappears. Dehghan et al [6] and [7] worked on numerical improvement of closed and open Newton-Cotes quadrature rules. The new methods possessed higher accuracy and precision than the conventional methods.

The Laplace transform method was used in [29] to numerically solve fractional-order linear delay differential equations. The quadrature approximation of the Bromwich integral served as the basis for the solution, and contour deformation was essential to the approximation procedure. The Gauss–Hermite quadrature rule was suggested as a substitute for the widely used trapezoidal rule. Error analysis and comparisons with other approaches from the literature were used to show how effective this strategy performed. The Ulam–Hyers (UH) stability of the equation and the presence of solutions were also investigated in the work using functional analysis. The work in [1] used mid-point derivative values for integration rules such as Trapezoidal, Simpson's one-third, Simpson's three-eight, and Bool's rule to offer novel, effective ways to improve the efficiency index of mid-point derivative based closed Newton and Cotes quadrature rules. Compared to current methods, the suggested approaches offered greater accuracy and precision. Precision concepts were used to derive error terms. Numerous comparisons demonstrated that these novel techniques performed better in terms of function evaluations and error terms than Zhao and Li's schemes and classical principles. The studies [21] and [9] can be considered as possible venue for the future exploration of applications of numerical integration particularly, and numerical analysis generally. In [21], the third-order stochastic generalized nonlinear Schrodinger equation (SGNLSE) in the Stratonovich sense was studied. Using a modified mapping method, new trigonometric, elliptic, hyperbolic, and rational stochastic solutions were discovered. The generated solutions were utilized to examine a wide range of pertinent physical phenomena because the GNLSE is widely used in nonlinear optical phenomena, optical fiber communication systems, communication, and heat pulse propagation in materials. Lastly it was concluded that the behavior of SGNLSE solutions is affected and stabilized by multiplicative noise. The oscillatory behavior of nonlinear third-order dynamic equations on time scales was examined in [12]. The primary strategy was to convert the semi-canonical form of the problem into a more manageable canonical form. The new oscillation criterion was extracted and simplified the analysis of oscillation behavior. The findings were build upon the previous research in the field, especially for the unique situations where T=R and T=Z. Furthermore, examples were illustrated to show how the defined criterion might be used in practice. As particular families of functions to describe and develop analytical functions of various complex variables, including generalizations of hyper geometric functions, the certain numerical aspects of branched continuing fractions were discussed in [9]. One of the fundamental methods for computing approximations of branched continuous fractions was the backward recurrence algorithm. In addition to creating its own rounding errors, each cycle of the recursive process also carried over the rounding errors made in all of the cycles before it. Also, the work was done on a confluent branched continuing fraction. The key distinction in this case was that the continuing fraction's approximates were the branched continued fraction's so-called figure approximates. When calculating an nth approximation of the branched continuous fraction expansion of Horn's hyper geometric function H4, the backward recurrence approach yielded an estimate of the relative rounding error. In order to create convergence criteria, the derivation made use of techniques from the theory of branched continuing fractions. The backward recurrence algorithm's numerical stability was demonstrated by the numerical examples. In [23] a work offered a numerical technique that combines an iterative approach with Simpson's 3/8 rule to solve nonlinear equations more quickly. This method outperforms traditional methods in terms of accuracy and convergence. A three-point Gaussian quadrature along with a decomposition methodology was used to solve nonlinear equations in the paper by Sana and Noor [31]. The authors compared the results to conventional iterative approaches, the presented approach improved accuracy and processing efficiency. The work in [19] by Mamedov and Isik's (2023) study extends traditional inequality results by examining fractional integral inequalities for p-convex functions. The authors enhanced the study of such functions by establishing additional boundaries and offering applications.

To improve the effectiveness of numerical integration, Mahesar et al. [17] suggested four new quadrature formulae that made use of derivatives. In contrast to more conventional derivative-free approaches like semi-open-Newton-Cotes (SONC) and Gauss-Legendre rules, the study shown that these modified methods (MSONC1, MSONC2, MSONC3 and MSONC4) were both cost and time efficient. The outcomes demonstrated the benefits of the suggested approaches in managing regular, periodic, and oscillatory integrands and exhibited notable improvements in accuracy and reduction in execution times across a range of test integrals. A thorough error analysis and computational cost evaluation were included in the analysis, which validated how well the new methods worked to achieve high precision with lesser resource consumption. Nevertheless, the progress, there were still issues with using SONC rules, especially when it came to managing singularities and enhancing convergence for highly oscillatory integrands.

Since the amount of research on semi-open Newton-Cotes rules has grown dramatically, demonstrating a wide range of creative methods and uses, these advancements demonstrate how these techniques can enhance numerical integration in a variety of domains. To solve current issues and improve the effectiveness of numerical integration methods, additional research is needed. In this work, four novel quadrature rules of semi-open-type are derived for efficient evaluation of a definite integral leading to substantial enhancement in precision and accuracy of the conventional semi-open-type quadrature by Newton and Cotes. The derivation of the proposed formulae have been executed in detail, followed by proved theorems on precision, local and global truncation error terms and extension from basic to composite forms. An exhaustive performance evaluation has been conducted against existing semi-open-type derivative free rules from literature in terms of standard parameters of numerical analysis, like: absolute error drops, observed/computational order of accuracy, computational cost and execution time. The proposed methods exhibit encouraging performance over the existing ones and prove to be efficient way outs for approximating a definite integral numerically.

2. Derivative-Based Quadrature Formulas

Consider the real-valued function $g \in C^{(2n+1)}\,[\alpha, \beta]$. Let $x_0, x_1, ..., x_n$ with $n+1$ nodes and the interval $[\alpha, \beta]$ be partitioned into n sub-intervals. In order to decrease truncation errors exclusive of sacrificing the effectiveness of the current formulas; the modified formulas are founded on the traditional two-point semi-open quadrature rules but include extra derivatives as perturbation terms. In real terms, the new formulas promise to reduce execution time and computing cost while adding up the degree of precision and the order accuracy. With the help of adopted notations, improvements are made in four distinct ways.

2.1. **Two-Point Derivative-Based SONC Rule-1 (DSONC1).** By employing the derivative of order 1 of the function at every point, with the exception of the interval's single end-point $[\alpha, \beta]$, a method posses' higher degree of precision than the two-point conventional semi-open NC rule is proposed, all the while preserving the improvement in accuracy order. The derivation and precision degree of modified DSONC1 are detailed in Theorems 1-2, respectively.

Theorem 1. The two-point DSONC1 method in its basic form is:

$$\int_{\alpha}^{\beta} g(x)\, dx \approx DSONC1 = (\beta-\alpha)g(\alpha) + \frac{(\beta-\alpha)^2}{6}[g'(\alpha) + 2g'(\frac{\alpha-\beta}{2})] \quad (2.\ 6)$$

Proof. DSONC1 uses four evaluations (as evident from (2. 6), so its precision is at most 3, in general $n-1$ for n evaluations. This results in DSONC1 requiring that the integration of monomials of type x^k, where $k = 0, 1, 2, 3$ be accurate. Consequently, a four-by-four system is created. Therefore, we have a 3rd-order polynomial:

$$g(x) = w_0 + w_1 x + w_2 x^2 + w_3 x^3 \quad (2.\ 7)$$

Following the pattern of terms in (2. 7), the following equations are produced for values of k:

$$\int_{\alpha}^{\beta} g(x)dx \approx w_0 g(\alpha) + w_1 g\frac{(\alpha+\beta)}{2} + h[w_2 g'(\alpha) + w_3 g'(\frac{\alpha+\beta}{2})] \quad (2.\ 8)$$

The following equations are produced by this method:

$$g(x) = 1; \int_{\alpha}^{\beta} 1dx = (\beta-\alpha) = w_0 + w_1$$
$$g(x) = x; \int_{\alpha}^{\beta} xdx = \frac{(\beta^2-\alpha^2)}{2} = w_0\alpha + w_1\frac{(\alpha+\beta)}{2} + w_2 + w_3$$
$$g(x) = x^2; \int_{\alpha}^{\beta} x^2dx = \frac{(\beta^3-\alpha^3)}{3} = w_0\alpha^2 + w_1(\frac{\alpha+\beta}{2})^2 + 2\alpha_2 + 2w_3\frac{(\alpha+\beta)}{2}$$
$$g(x) = x^3; \int_{\alpha}^{\beta} x^3dx = \frac{(\beta^4-\alpha^4)}{4} = w_0\alpha^3 + w_1(\frac{\alpha+\beta}{2})^3 + 3\alpha_2 + 3w_3(\frac{\alpha+\beta}{2})^2$$

By solving above system using the method of undetermined coefficients, we were able to find the weight coefficients:$w_0 = (\beta-\alpha), w_1 = 0, w_2 = \frac{(\beta-\alpha)^2}{6} and w_3 = \frac{(\beta-\alpha)^2}{3}$. The following quadrature rule is thus obtained:

$$\int_{\alpha}^{\beta} g(x)\, dx = (\beta-\alpha)g(\alpha) + \frac{(\beta-\alpha)^2}{6}[g'(\alpha) + 2g'\frac{(\alpha-\beta)}{2}] \quad (2.\ 9)$$

The proof of theorem 1 ends here. □

Theorem 2. The precision degree of two-point DSONC1 is three.

Proof. This theorem will be proved by confirming that the new method (2. 9) is accurate for $g(x) = x^n, n = 0, 1, 2, 3$. The exact and approximate integrals using DSONC1 integral for $n = 0, 1, 2, 3$. are given as:

$$\int_{\alpha}^{\beta} g(x)\,dx = \frac{(\beta^{n+1} - \alpha^{n+1})}{n+1} \tag{2. 10}$$

However, the DSONC1 approximation for $g(x) = x^4$ is defined 2. 11 :

$$\int_{\alpha}^{\beta} x^4\,dx = \frac{1}{6}[(\alpha+\beta)^3 + 4\alpha^3][(\alpha-\beta)^2 - \alpha^4(\alpha-\beta)] \tag{2. 11}$$

which is not exact. That is:

$$\int_{\alpha}^{\beta} x^4\,dx \neq \frac{(\beta^5 - \alpha^5)}{5} \tag{2. 12}$$

This demonstrates that the suggested approach DSONC1 has a three degree of precision. The proof of theorem 2 completes here.

2.2. **Two-Point Derivative-Based SONC Rule-2 (DSONC2).** By employing the derivative of order 1 of the function solely at interior points of the interval $[\alpha, \beta]$, a method posseing higher degree of precision than the two-point conventional semi-open NC rule, all the while preserving the improvement in accuracy order is proposed. The derivation and precision degree of modified DSONC2 are detailed in Theorems 3-4, respectively.

Theorem 3. The two-point DSONC2 approach in its basic form is:

$$\int_{\alpha}^{\beta} g(x)\,dx \approx DSONC2 = \frac{(\beta-\alpha)}{3}[g[(\alpha)+g(\frac{\alpha+\beta}{2})]+\frac{(\beta-\alpha)^2}{6}g'(\frac{\alpha+\beta}{2}) \tag{2. 13}$$

Proof. DSONC2 uses three evaluations (as evident from (2. 13), so its precision is at most 2. This results in requiring DSONC2 that the monomials of type x^k, where $k = 0, 1, 2$ be accurate. Consequently, a three-by-three system is created. Therefore, we have a 3rd-order polynomial for n=1:

$$g(x) = w_0 + w_1 x + w_2 x^2 \tag{2. 14}$$

The overall structure of the scheme would be:

$$\int_{\alpha}^{\beta} g(x)dx \approx w_0 g(\alpha) + w_1 g\frac{(\alpha+\beta)}{2} + h[w_2 g'(\frac{\alpha+\beta}{2})] \tag{2. 15}$$

Following the pattern of terms in (2. 14), the following equations are produced for values of k:

$$g(x) = 1; \int_{\alpha}^{\beta} 1dx = (\beta - \alpha) = w_0 + w_1$$

$$g(x) = x; \int_{\alpha}^{\beta} xdx = \frac{(\beta^2 - \alpha^2)}{2} = w_0\alpha + w_1\frac{(\alpha+\beta)}{2} + w_2$$

$$g(x) = x^2; \int_{\alpha}^{\beta} x^2dx = \frac{(\beta^3 - \alpha^3)}{3} = w_0\alpha^2 + w_1(\frac{\alpha+\beta}{2})^2 + 2w_2\frac{(\alpha+\beta)}{2}$$

By solving above system using the method of undetermined coefficients, we were able to find the weight coefficients:$w_0 = \frac{(\beta-\alpha)}{3}, w_1 w_0 = \frac{2}{3}(\beta-\alpha), w_2 = \frac{(\beta-\alpha)}{6}^2$. The following quadrature rule is thus obtained:

$$\int_{\alpha}^{\beta} g(x)\, dx = \frac{(\beta-\alpha)}{3}[g[(\alpha) + \frac{g(\alpha-\beta)}{2}] + \frac{(\beta-\alpha)^2}{6} g'(\frac{\alpha+\beta}{2}) \tag{2. 16}$$

The proof of theorem 3 completes here. □

Theorem 4. The precision degree of two-point DSONC2 is two.

Proof. This theorem is demonstrated by confirming that the new method (2. 16) is accurate for $g(x) = x^n, n = 0, 1, 2$. The exact and approximate integrals results using DSONC2 integral for $n = 0, 1, 2$ is given as:

$$\int_{\alpha}^{\beta} g(x)\, dx = \frac{(\beta^{n+1} - \alpha^{n+1})}{n+1} \tag{2. 17}$$

However, the $DSONC2$ approximation for $g(x) = x^3$ is defined (2. 18).

$$\int_{\alpha}^{\beta} x^3\, dx = [-\frac{\alpha^4}{24} + \frac{7\alpha^3\beta}{6} + \frac{7\alpha^2\beta^2}{4} + \frac{7\alpha\beta^3}{6} - \frac{\beta^4}{24}] \tag{2. 18}$$

which is not exact. That is:

$$\int_{\alpha}^{\beta} x^3\, dx \neq \frac{(\beta^4 - \alpha^4)}{4} \tag{2. 19}$$

This demonstrates that the suggested approach DSONC2 has a two degree of precision. The proof of theorem 4 completes here. □

2.3. **Two-Point Derivative-Based SONC Rule-3 (DSONC3).** By employing derivative of order 1 of the function at at the endpoints of the interval $[\alpha, \beta]$ only, a method posing higher degree of precision than the two-point conventional semi-open NC rule, all the while preserving the improvement in accuracy order is proposed. The derivation and precision degree of modified DSONC3 are detailed in Theorems 5-6, respectively.

Theorem 5. The two-point DSONC3 approach in its basic form is:

$$\int_{\alpha}^{\beta} g(x)\, dx = (\beta-\alpha)[0 \times g(\alpha) + \frac{(\alpha+\beta)}{2}] - \frac{1}{24}(\beta-\alpha)^2[g'(\alpha) - g'(\beta)] \tag{2. 20}$$

Proof. DSONC3 uses four evaluations (as evident from (2. 20), so its precision is at most 3. This results in DSONC3 requiring that the monomials of type x^k where $k = 0, 1, 2, 3$ be accurate. Consequently, a four-by-four system is created. Therefore, we have a 3rd-order polynomial for n=1:

$$g(x) = w_0 + w_1 x + w_2 x^2 + w_3 x^3 \tag{2. 21}$$

The overall structure of the scheme would be:

$$\int_{\alpha}^{\beta} g(x) dx \approx w_0 g(\alpha) + w_1 g\frac{(\alpha+\beta)}{2} + [w_2 g'(\alpha) + w_3 g'(\beta)] \tag{2. 22}$$

Following the pattern of terms in (2. 21), the following equations are produced for values of k:

$$g(x) = 1; \int_{\alpha}^{\beta} 1dx = (\beta - \alpha) = w_0 + w_1$$
$$g(x) = x; \int_{\alpha}^{\beta} xdx = \frac{(\beta^2 - \alpha^2)}{2} = w_0\alpha + w_1\frac{(\alpha + \beta)}{2} + w_2 + w_3$$
$$g(x) = x^2; \int_{\alpha}^{\beta} x^2dx = \frac{(\beta^3 - \alpha^3)}{3} = w_0\alpha^2 + w_1(\frac{\alpha + \beta}{2})^2 + 2\alpha w_2 + w_3\beta$$
$$g(x) = x^3; \int_{\alpha}^{\beta} x^3dx = \frac{(\beta^4 - \alpha^4)}{4} = w_0\alpha^3 + w_1(\frac{\alpha + \beta}{2})^3 + 3\alpha^2 w_2 + 3w_3\beta^2$$
$$g(x) = x^4; \int_{\alpha}^{\beta} x^4dx = \frac{(\beta^5 - \alpha^5)}{5} = w_0\alpha^4 + w_1(\frac{\alpha + \beta}{2})^4 + 4\alpha^3 + w_2 + 4w_3\beta^3$$
$$g(x) = x^5; \int_{\alpha}^{\beta} x^5dx = \frac{(\beta^6 - \alpha^6)}{5} = w_0\alpha^4 + w_1(\frac{\alpha + \beta}{2})^5 + 5\alpha^4 w_2 + 5w_3\beta^4$$

By solving above system using the method of undetermined coefficients, we were able to find the weight coefficients:$w_0 = 0, w_1 = (\beta - \alpha), w_2 = -\frac{(\beta-\alpha)}{24}^2 and w_3 = \frac{(\beta-\alpha)}{3}^2$. The following quadrature rule is thus obtained:

$$\int_{\alpha}^{\beta} g(x)dx \approx w_0 g(\alpha) + w_1 g\frac{(\alpha + \beta)}{2} + [w_2 g'(\alpha) + w_3 g'(\beta)] \tag{2. 23}$$

The proof of theorem 5 completes here. □

Theorem 6. The precision degree of two-point DSONC3 is three.

Proof. This theorem is demonstrated by confirming that the new method (2. 23) is accurate for $g(x) = x^n, n = 0, 1, 2, 3$. The exact and approximate integrals results using DSONC3 integral is given as:

$$\int_{\alpha}^{\beta} g(x)\, dx = \frac{(\beta^{n+1} - \alpha^{n+1})}{n + 1} \tag{2. 24}$$

However, the DSONC3 approximation for $g(x) = x^4$ is defined in (2. 25).

$$\int_{\alpha}^{\beta} x^4\, dx = \frac{1}{48}[7\alpha^4\beta - 14\alpha^3\beta^2 + 14\alpha^2\beta^3 - \alpha\beta^4 - 11\alpha^5 + 11\beta^5] \tag{2. 25}$$

which is not exact. That is:

$$\int_{\alpha}^{\beta} x^4\, dx \neq \frac{(\beta^5 - \alpha^5)}{5} \tag{2. 26}$$

This demonstrates that the suggested approach DSONC3 has a three degree of precision. The proof of theorem 6 completes here. □

2.4. **Two-Point Derivative-Based SONC Rule-4 (DSONC4.** By employing derivative of order 1 of the function at every point of the interval of integration within the quadrature formula, a method possing higher degree of precision than the two-point conventional semi-open NC rule is suggested, all the while preserving the improvement in accuracy order. The derivation and precision degree of modified DSONC4 are detailed in Theorems 7-8, respectively.

Theorem 7. The two-point DSONC4 approach in its basic form is:

$$\int_{\alpha}^{\beta} g(x)\,dx = \frac{(\beta-\alpha)}{15}[7g(\alpha)+8g(\frac{\alpha+\beta}{2})]+\frac{1}{90}(\beta-\alpha)^2[5g'(\alpha)+14g'(\frac{\alpha+\beta}{2})+2g'(\beta)] \quad (2.\ 27)$$

Proof. DSONC4 uses five evaluations (as evident from (2. 27), so its precision is at most 4. This results in DSONC4 requiring that the monomials of type x^k where $k = 0, 1, ..., 4$ be accurate. Consequently, a five-by-five system is created. Therefore, we have a 4th-order polynomial for n=1:

$$g(x) = w_0 + w_1 x + w_2 x^2 + w_3 x^3 + w_4 x^4 \quad (2.\ 28)$$

The overall structure of the scheme would be:

$$\int_{\alpha}^{\beta} g(x)dx \approx w_0 g(\alpha) + w_1 g\frac{(\alpha+\beta)}{2} + [w_2 g'(\alpha) + w_3 g'(\beta)] \quad (2.\ 29)$$

Following the pattern of terms in (2. 28), the following equations are produced for values of k:

$$g(x) = 1; \int_{\alpha}^{\beta} 1dx = (\beta-\alpha) = w_0 + w_1$$

$$g(x) = x; \int_{\alpha}^{\beta} xdx = \frac{(\beta^2-\alpha^2)}{2} = w_0\alpha + w_1\frac{(\alpha+\beta)}{2} + w_2 + w_3$$

$$g(x) = x^2; \int_{\alpha}^{\beta} x^2dx = \frac{(\beta^3-\alpha^3)}{3} = w_0\alpha^2 + w_1(\frac{\alpha+\beta}{2})^2 + 2\alpha w_2 + 2w_3\beta + 2w_4\beta$$

$$g(x) = x^3; \int_{\alpha}^{\beta} x^3dx = \frac{(\beta^4-\alpha^4)}{4} = w_0\alpha^3 + w_1(\frac{\alpha+\beta}{2})^3 + 3\alpha^2 w_2 + 3w_3\beta^2 + 3w_4\beta^2$$

$$g(x) = x^4; \int_{\alpha}^{\beta} x^4dx = \frac{(\beta^5-\alpha^5)}{5} = w_0\alpha^4 + w_1(\frac{\alpha+\beta}{2})^4 + 4\alpha^3 w_2 + 4w_3\beta^3 + 4w_4\beta^3$$

$$g(x) = x^5; \int_{\alpha}^{\beta} x^5dx = \frac{(\beta^6-\alpha^6)}{5} = w_0\alpha^4 + w_1(\frac{\alpha+\beta}{2})^5 + 5\alpha^4 w_2 + 5w_3\beta^4 + 5w_4\beta^4$$

By solving above system using the method of undetermined coefficients, we were able to find the weight coefficients:$w_0 = \frac{7}{15}(\beta-\alpha), w_1 = \frac{8}{15}(\beta-\alpha), w_2 = -\frac{1}{18}(\beta-\alpha)^2, w_3 = \frac{7}{45}(\beta-\alpha)^2andw_4 = \frac{1}{45}(\beta-\alpha)^2$. The following quadrature rule is thus obtained:

$$\int_{\alpha}^{\beta} g(x)\,dx = \frac{(\beta-\alpha)}{15}[7g(\alpha)+8g(\frac{\alpha+\beta}{2})]+\frac{1}{90}(\beta-\alpha)^2[5g'(\alpha)+14g'(\frac{\alpha+\beta}{2})+2g'(\beta)] \quad (2.\ 30)$$

The proof of theorem 7 completes here. □

Theorem 8. The precision degree of two-point DSONC4 is four.

Proof. This theorem will be demonstrated by confirming that the new method (2. 30) is accurate for $g(x) = x^n, n = 0, 1, 2, 3, 4$. The exact and approximate integrals results using DSONC4 integral for $n = 0, 1, 2, 3, 4$. is given as:

$$\int_{\alpha}^{\beta} g(x)\,dx = \frac{(\beta^{n+1} - \alpha^{n+1})}{n+1} \tag{2. 31}$$

However, the DSONC4 approximation for $g(x) = x^5$ is defined in (2. 32).

$$\int_{\alpha}^{\beta} x^5\,dx = \frac{1}{720}[-42\alpha^5\beta+105\alpha^4\beta^2-105\alpha^2\beta^4-42\alpha\beta^5+127\beta^6-113\alpha^6+140\alpha^3\beta^3+210\alpha^2\beta^5] \tag{2. 32}$$

which is not exact. That is:

$$\int_{\alpha}^{\beta} x^5\,dx \neq \frac{(\beta^6 - \alpha^6)}{6} \tag{2. 33}$$

This demonstrates that the suggested approach DSONC4 has a four degree of precision. The proof of theorem 8 completes here. □

3. Error Analysis

This section provides the error terms for each of the recommended two-point approaches, whether they be gloabal or local errors. Using the remainder of the improved quadrature approach for the monomial $x^{(p+1)}/(p+1)!$ and the exact solution of $\frac{1}{(p+1)!}\int_a^b x^{(p+1)}\,dx$ where p is the method's precision [13]. The error terms and the accuracy order of the suggested rules are defined in basic forms in the upcoming theorems. The accuracy and errors of two-point DSONC1-4 in basic form are covered in Theorems 9–12.

Theorem 9. The local order of accuracy is five, and the local error term of two-point DSONC1 is as follows:

$$E_{[DSONC1]} = \frac{1}{720}(\beta - \alpha)^5 g^{(4)}(\epsilon) \tag{3. 34}$$

where $\epsilon \in (\alpha, \beta)$

Proof. Since integration of all monomials of order $0, 1, 2$and3 can be accurately calculated using the proposed method DSONC1, the Taylor's series term of $g(x)$ of order four is:

$$g(x) = \frac{1}{4!}(x - x_0)^4 g^{(4)}(x_0) \tag{3. 35}$$

The error term of proposed method can be expressed as follows:

$$E_{[DSONC1]} = [Exact(\frac{x^4}{4!}; \alpha, \beta) - DSONC1(\frac{x^4}{4!}; \alpha, \beta)]g^{(4)}(\epsilon) \tag{3. 36}$$

The exact value of integral value is shown in (3. 37) as:

$$Exact(\frac{x^4}{4!}; \alpha, \beta) = \frac{\beta^5 - \alpha^5}{120} \tag{3. 37}$$

And the integral's approximate value as determined by DSONC1 is shown in (3. 38) as:

$$[DSONC1](\frac{x^4}{4!};\alpha,\beta)=\frac{1}{44}[(\alpha+\beta)^3+4\alpha^3][(\alpha-\beta)^2-\alpha^4(\alpha-\beta)] \tag{3. 38}$$

Using (3. 37) and (3. 38) in (3. 36), we get the following equation:

$$E_{[DSONC1]}=\frac{1}{720}(\beta-\alpha)^5 g^{(4)}(\epsilon) \tag{3. 39}$$

Where, for $n=1$, $h=\frac{(\beta-\alpha)}{2}$. Hence (3. 39) will be:

$$E_{[DSONC1]}=\frac{2}{45}h^5 g^{(4)}(\epsilon) \tag{3. 40}$$

where $\epsilon\in(\alpha,\beta)$. Hence the method's accuracy is five. □

Theorem 10. The local order of accuracy is four, and the local error term of two-point DSONC2 is as follows:

$$E_{[DSONC2]}=\frac{7}{144}(\beta-\alpha)^4 g^{(3)}(\epsilon) \tag{3. 41}$$

where $\epsilon\in(\alpha,\beta)$

Proof. Since the integration of all monomials of order $0, 1$and 2 can be accurately calculated using the proposed method DSONC2, the Taylor's series term of $g(x)$ of order three is:

$$g(x)=\frac{1}{3!}(x-x_0)^3 g^{(3)}(x_0) \tag{3. 42}$$

The error term of proposed method can be expressed as follows:

$$E_{[DSONC2]}=[Exact(\frac{x^3}{3!};\alpha,\beta)-DSONC2(\frac{x^3}{3!};\alpha,\beta)]g^{(3)}(\epsilon) \tag{3. 43}$$

The exact value of integral value is shown in (3. 44) as:

$$Exact(\frac{x^3}{3!};\alpha,\beta)=\frac{\beta^4-\alpha^4}{24} \tag{3. 44}$$

And the integral's approximate value as determined by DSONC2 is shown in (3. 45) as:

$$[DSONC2](\frac{x^3}{3!};\alpha,\beta)=[-\frac{\alpha^4}{24}+\frac{7}{6}(\alpha^3\beta)+\frac{7}{4}(\alpha^3\beta^2)+\frac{7}{6}(\alpha\beta^3)+\frac{\beta^4}{24}] \tag{3. 45}$$

Using (3. 44) and (3. 45) in (3. 43), we get the following equation:

$$E_{[DSONC2]}=\frac{7}{144}(\beta-\alpha)^4 g^{(3)}(\epsilon) \tag{3. 46}$$

Where, for $n=1$, $h=\frac{(\beta-\alpha)}{2}$. Hence (3. 46) will be:

$$E_{[DSONC2]}=\frac{7}{9}h^4 g^{(3)}(\epsilon) \tag{3. 47}$$

where $\epsilon\in(\alpha,\beta)$. Hence the method's accuracy is four. □

Theorem 11. The local order of accuracy is five, and the local error term of two-point DSONC3 is as follows:

$$E_{[DSONC3]} = -\frac{7}{28800}(\beta-\alpha)^5 g^{(4)}(\epsilon) \tag{3. 48}$$

where $\epsilon \in (\alpha, \beta)$

Proof. Since integration of all monomials of order $0, 1,$ 2and3 can be accurately calculated using the proposed method DSONC3, the Taylor's series term of $g(x)$ of order four is:

$$g(x) = \frac{1}{4!}(x-x_0)^4 g^{(4)}(x_0) \tag{3. 49}$$

The error term of proposed method can be expressed as follows:

$$E_{[DSONC3]} = [Exact(\frac{x^4}{4!};\alpha,\beta) - DSONC3(\frac{x^4}{4!};\alpha,\beta)]g^{(4)}(\epsilon) \tag{3. 50}$$

The exact value of integral value is shown in (3. 51) as:

$$Exact(\frac{x^4}{4!};\alpha,\beta) = \frac{\beta^5-\alpha^5}{120} \tag{3. 51}$$

And the integral's approximate value as determined by DSONC3 is shown in (3. 52) as:

$$[DSONC3](\frac{x^4}{4!};\alpha,\beta) = \frac{1}{48}[7\alpha^4\beta - 14\alpha^3\beta^2 + 14\alpha^2\beta^3 - \alpha\beta^4 - 11\alpha^5 + 11\beta^5 \tag{3. 52}$$

Using (3. 51) and (3. 52) in (3. 50), we get the following equation:

$$E_{[DSONC3]} = -\frac{7}{28800}(\beta-\alpha)^5 g^{(4)}(\epsilon) \tag{3. 53}$$

Where, for $n = 1$, $h = \frac{(\beta-\alpha)}{2}$. Hence (3. 53) will be:

$$E_{[DSONC3]} = -\frac{7}{180}h^5 g^{(4)}(\epsilon) \tag{3. 54}$$

where $\epsilon \in (\alpha, \beta)$. Hence the method's accuracy is five. □

Theorem 12. The local order of accuracy is six, and the local error term of two-point DSONC4 is as follows:

$$E_{[DSONC4]} = \frac{7}{6750}(\beta-\alpha)^5 g^{(4)}(\epsilon) \tag{3. 55}$$

where $\epsilon \in (\alpha, \beta)$

Proof. Since integration of all monomials of order $0, 1, 2, 3,$and4 can be accurately calculated using the proposed method DSONC4, the Taylor's series term of $g(x)$ of order five is:

$$g(x) = \frac{1}{5!}(x-x_0)^5 g^{(5)}(x_0) \tag{3. 56}$$

The error term of proposed method can be expressed as follows:

$$E_{[DSONC4]} = [Exact(\frac{x^5}{5!};\alpha,\beta) - DSONC4(\frac{x^5}{5!};\alpha,\beta)]g^{(5)}(\epsilon) \tag{3. 57}$$

The exact value of integral value is shown in (3. 58) as:

$$Exact(\frac{x^5}{5!};\alpha,\beta)=\frac{\beta^6-\alpha^6}{720} \tag{3. 58}$$

And the integral's approximate value as determined by DSONC4 is shown in (3. 59) as:

$$[DSONC4](\frac{x^4}{4!};\alpha,\beta)=\frac{1}{720}[-42\alpha^5\beta+105\alpha^4\beta^2-105\alpha^2\beta^4-42\alpha\beta^5+127\beta^6-113\alpha^6+140\alpha^3\beta^3+210\alpha^2\beta^5] \tag{3. 59}$$

Using (3. 58) and (3. 59) in (3. 57), we get the following equation:

$$E_{[DSONC4]}=\frac{7}{6750}(\beta-\alpha)^6 g^{(5)}(\epsilon) \tag{3. 60}$$

Where, for $n=1$, $h=\frac{(\beta-\alpha)}{2}$. Hence (3. 60) will be:

$$E_{[DONC4]}=\frac{7}{432000}h^6 g^{(5)}(\epsilon) \tag{3. 61}$$

where $\epsilon\in(\alpha,\beta)$. Hence the method's accuracy is five. □

4. Results and Discussions

The correctness of the theoretical conclusions has been confirmed by a number of numerical tests that have been performed on the new derivative-based quadrature DSONC 1-4 in comparison to the current customary rules SONC. For each scheme, five different integrals are solved taken from [35], [6]-[7]. The exact values of these methods were found using double precision arithmetic software called MATLAB $(R2014b)$. Every result was recorded using an Intel (R) Core $i5$ laptop with1.8 GHz processing speed and 8.00 GB of RAM. Additionally, for each integral, the absolute errors and the computational order of accuracy were calculated. To support our findings, the following integrals are examined. For every case, the exact integral values are displayed against each integral.

(1) Integral 1 $\int_0^1 xe^{-x}\,dx=0.264241117657115$.
(2) Integral 2 $\int_0^{pi/4} cos^2x\,dx=0.642699081698724$.
(3) Integral 3 $\int_0^1 \frac{1}{(1+x)}\,dx=0.693147180559945$.
(4) Integral 4 $\int_0^{pi/4} e^{cos(x)}\,dx=1.939734850623649$.
(5) Integral 5 $\int_0^1 \frac{xln(1+x)}{(1+x^2)}\,dx=0.162865005917789$.

To illustrate the unique role of the performance of the modified techniques, the comparative analysis of the proposed methods is presented in a number of ways. In this section we will first analyze the computational order of accuracy by means of the formula specified in [11], where we saw the increasing precision degrees and accuracy orders of techniques, together with their fast-declining error patterns and distributions.

The computational accuracy order of the classical rules is indicated by the columns beneath each SONC heading in Tables 1-5, whereas the computational accuracy order of the new two-point derivative-based SONC rules is indicated by the columns beneath the headings of DSONC1, DSONC2, DSONC3, and DSONC4. These findings also support each method's theoretical and computational order of correctness. In contrast to the classical approach SONC, which has an accuracy order of 1, the modified derivative-based schemes DSONC

1-4 have accuracy orders of $4, 3, 4,$and 5, respectively. This indicates that the suggested methods are more efficient than the traditional ones.

In Figures 1-5, we compared the absolute error distributions with relation to the number of strips (i.e., from 1 to 20) between the modified and classical approaches. The absolute error is obtained by taking the exact and approximate values of the proposed methods for each integral 1-5 in the absolute sense. Therefore, for all integrals, these figures display diminishing absolute error distributions, and the trends showing that the suggested methods converge more rapidly, and the generated error terms agree with the accuracy order. The results produced by the new methods verify that they have reduced errors than the SONC methods.

Lastly, the overall computational cost as well as the pre-specified error tolerance is attained by observing the average C.P.U. time (in seconds) every integration step. i.e.,$1E-07$. A quadrature rule may offer acceptable accuracy in fewer steps because to the greater number of functional estimates at every step of integration, but it may also be less efficient and computationally more costly than alternative methods. In order to calculate the computing costs of each test problem, we first calculate the computational costs for all SONC rules in Table 6 for general strips m, by summarizing the total number of functional calculations needed per m for the modified and original methods. We compared the total computational cost for the integrals 1-5 with conventional method one-point SONC, and proposed methods DSONC 1-4. Additionally, We compared the total computational cost for the integrals 1-5 with conventional method one-point SONC, and proposed methods DSONC1-4. Additionally, we compared our results with those reported in the recent work by Mahesar et al. (2022) [17]. The comparative results are presented in Tables 7 and 8. The numerical findings show that the suggested methods incur lower computational cost than the both conventional approach and the modified techniques discussed in [17] recently. Therefore, the proposed methods are more computationally efficient and economical.

We also examined the average CPU times (in seconds), which are utilized to calculate the processor's runtime in MATLAB software for every method code independently in order to achieve the predetermined accuracy level, following the computational ascent in terms of computing expenses to reach a maximum $1E-07$ preset error. The execution timings take into consideration both functional and derivative evaluations in order to ultimately ascertain the approaches' time-efficiency. We have already observed that the proposed formulas use fewer evaluations (both functional and derivative) overall than the derivative free classical SONC rules regarding computational expenses, the executing time duration facilitates show that the processing power needed for derivative and functional analyses in the suggested quadrature formulas is consistently lower than that needed by the traditional rules with merely a large number of functional evaluations. Because derivatives can often be more difficult in calculation than functions alone, it is crucial to look into whether the suggested rules with derivatives place greater burden on the processor while analyzing the execution timings. The CPU times (in seconds) for each approach are listed in Table 8. We note that the suggested formulas outperform the conventional $SONC$ and modified methods $MSONC1-4$ in this regard as well. The suggested techniques are therefore also time efficient.

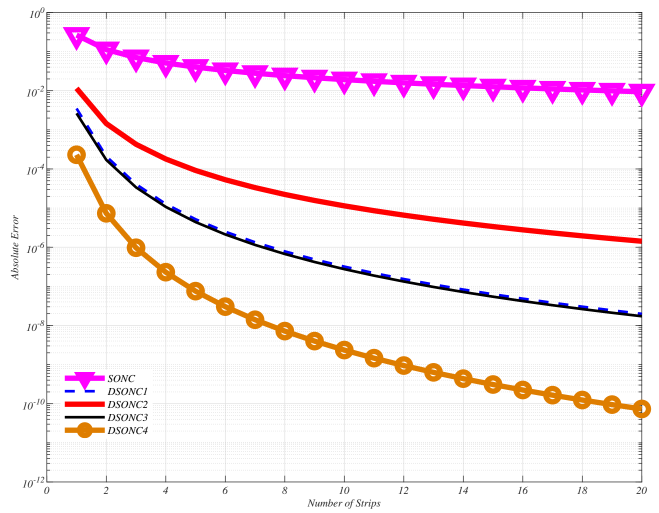

FIGURE 1. Absolute errors versus number of strips for Integral 1

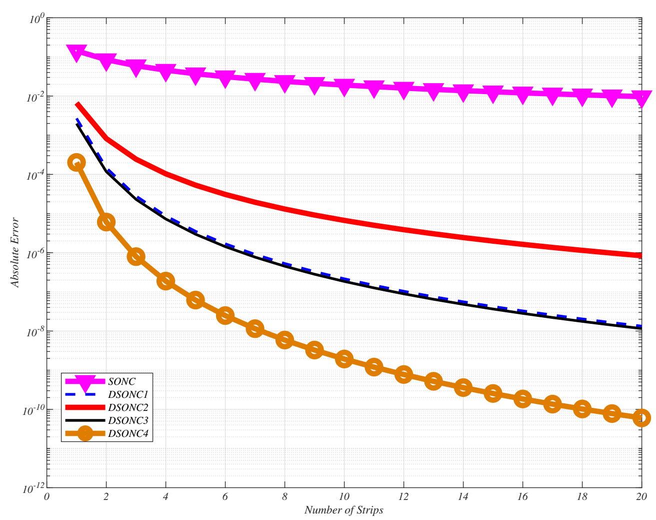

FIGURE 2. Absolute errors versus number of strips for Integral 2

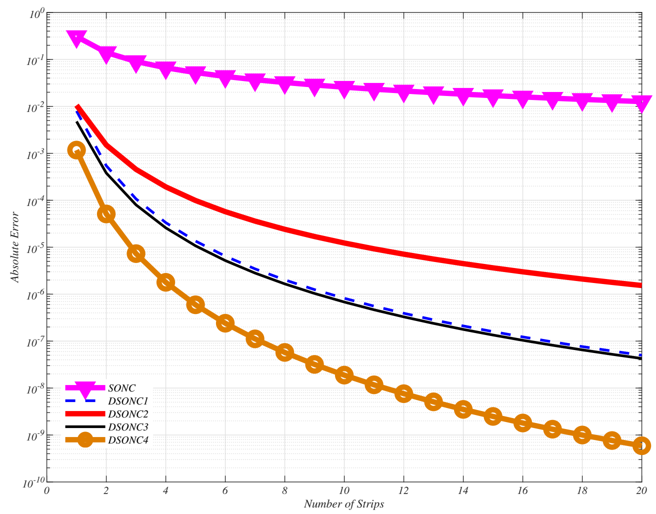

FIGURE 3. Absolute errors versus number of strips for Integral 3

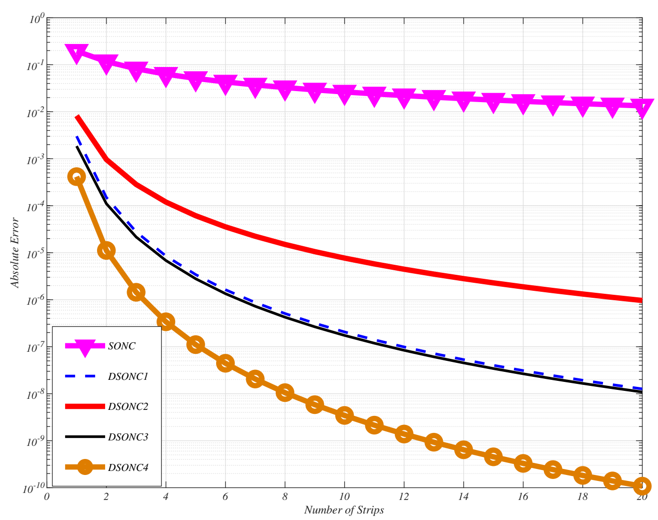

FIGURE 4. Absolute errors versus number of strips for Integral 4

TABLE 1. Global computational order of accuracy for integral 1

m	n	$SONC$	$DSONC1$	$DSONC2$	$DSONC3$	$DSONC4$
1	2	NA	NA	NA	NA	NA
2	4	1.2305	4.0626	2.9928	3.9563	4.9720
4	8	1.1376	4.0442	3.0104	3.9889	4.9973
8	16	1.0750	4.0254	3.0088	3.9972	5.0015
16	32	1.0750	4.0135	3.0053	3.9993	5.0015
32	64	1.0199	4.0069	3.0028	3.9998	5.0009
64	128	1.0101	4.0035	3.0015	3.9999	5.0006

TABLE 2. Global computational order of accuracy for integral 2

m	n	$SONC$	$DSONC1$	$DSONC2$	$DSONC3$	$DSONC4$
1	2	NA	NA	NA	NA	NA
2	4	0.7442	4.1928	3.0204	4.0731	5.0585
4	8	0. 8932	4.0824	2.9904	4.0177	5.0083
8	16	0. 9501	4.0381	2.9905	4.0044	4.9991
16	32	0. 9757	4.0183	2.9940	4.0011	4.9983
32	64	0. 9880	4.0090	2.9967	4.0002	4.9988
64	128	0. 9940	4.0044	2.9983	4.0000	5.0002

TABLE 3. Global computational order of accuracy for integral 3

m	n	$SONC$	$DSONC1$	$DSONC2$	$DSONC3$	$DSONC4$
1	2	NA	NA	NA	NA	NA
2	4	1.1302	3.87344	2.8138	3.6512	4.5204
4	8	1.0785	4.0256	2.9606	3.8770	4.8365
8	16	1.0425	4.0496	3.0024	3.9646	4.9600
16	32	1.0219	4.0356	3.0077	3.9907	4.9939
32	64	1.0111	4.0205	3.0055	3.9976	5.0007
64	128	1.0056	4.0109	3.0031	3.9994	5.0012

5. CONCLUSION

Four effective quadrature formulas that used derivatives to calculate integrals have been developed in this study. These formulas are variations of the traditional two-point semi-open Newton-Cotes quadrature rule that are efficient in terms of computation, cost, and time. For each of the suggested approaches, theorems were established concerning advances in the degree of precision, error terms and the order of accuracy. With the objective to compare all proposed DSONC1-4 methods to the traditional derivative-free SONC and derivative based MSONC1-4 rules, computational results were also calculated for five test

TABLE 4. Global computational order of accuracy for integral 4

m	n	$SONC$	$DSONC1$	$DSONC2$	$DSONC3$	$DSONC4$
1	2	NA	NA	NA	NA	NA
2	4	0. 7389	4.3269	3.0951	4.0721	5.2172
4	8	0.88932	4.1402	3.0078	4.0182	5.0427
8	16	0.94815	4.0676	2.9959	4.0045	5.0087
16	32	0. 9748	4.0335	2.9960	4.0011	5.0013
32	64	0. 9875	4.0167	2.9975	4.0002	4.9998
64	128	0. 9938	4.0083	2.9986	4.0000	5.0004

TABLE 5. Global computational order of accuracy for integral 5

m	n	$SONC$	$DSONC1$	$DSONC2$	$DSONC3$	$DSONC4$
1	2	NA	NA	NA	NA	NA
2	4	0.9939	5.4461	3.2192	5.0247	3.9288
4	8	0.9598	3.7301	3.1035	4.1743	5.3770
8	16	0.9783	3.8789	3.0274	4.0239	5.0805
16	32	0.9891	3.9424	3.0097	4.0056	5.0212
32	64	0.9945	3.9713	3.0039	4.0013	5.0063
64	128	0.9972	3.9856	3.0017	4.0003	5.0021

TABLE 6. Computational costs for m strips of all two-point SONC rules

$Methods$	FE	DE	Total	Total $(m = 1)$
$SONC$	m	0	1	1
$DSONC1$	m	$2m$	$3m$	3
$DSONC2$	$2m$	m	$3m$	3
$DSONC3$	m	2	$m + 2$	3
$DSONC4$	$2m$	$2m + 1$	$4m + 1$	5

TABLE 7. Computational costs comparison to achieve $1E - 07$ absolute error for Integrals $1 - 5$

n =1	$SONC$	$DSONC1$	$DSONC2$	$DSONC3$	$DSONC4$	$SONC$[17]	$MSONC1$[17]	$MSONC2$[17]	$MSONC3$[17]	$MSONC4$[17]
$Integral1$	1839390	13	48	12	5	919698	981747	1250000	1360000	3465800
$Integral2$	1963496	12	40	11	4	1290	1014	1118	1214	646
$Integral3$	2500000	17	50	17	8	912	718	792	860	456
$Integral4$	2715000	9	43	12	6	87	75	93	79	101
$Integral5$	6931600	14	55	14	6	18	21	27	18	21

problems from the existing literature in terms of observed orders of accuracy, computational expenses, average C.P.U times (in seconds), and absolute error drops. The results analysis showed the effectiveness of the updated methods when compared to the traditional

TABLE 8. Comparing the average CPU time (in seconds) required to obtain an absolute error of $1E-07$ for Integrals 1–5

$n=1$	$SONC$	$DSONC1$	$DSONC2$	$DSONC3$	$DSONC4$	$SONC$[17]	$MSONC1$[17]	$MSONC2$[17]	$MSONC3$[17]	$MSONC4$[17]
$Integral1$	0.5683	0.4502	0.4273	0.4545	0.4357	1.0190	1.0238	0.8856	1.046	20.5804
$Integral2$	1.0688	0.4200	0.4653	0.4449	0.4127	0.1623	0.16191	0.1074	0.1561	0.2571
$Integral3$	0.8988	0.2934	0.2897	0.2944	0.2889	0.4316	0.44322	0.2968	0.4175	0.5388
$Integral4$	1.4435	0.4120	0.4055	0.4053	0.4301	0.4229	0.41843	0.2976	0.4215	0.5191
$Integral5$	20.1369	0.3541	0.3353	0.3202	0.3457	0.4360	0.43715	0.3180	0.4162	0.5442

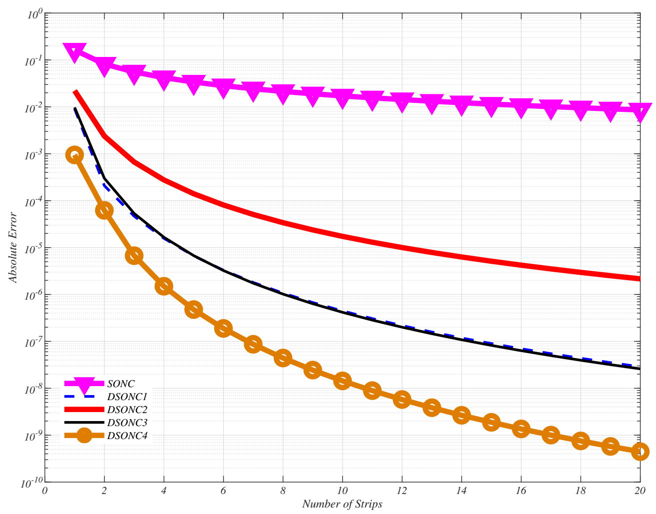

FIGURE 5. Absolute errors versus number of strips for Integral 5

approaches. This study highlighted how the proposed derivative-based approaches were more theoretically advanced and more cost- and time-efficient than conventional SONC methods.

FUNDING STATEMENT

This research did not receive funding.

CONFLICT OF INTEREST

All authors declare no conflict of interest with any person, institute, or funding agency.

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