Punjab University Journal of Mathematics (2025), 57(07), 790-801
https://doi.org/10.52280/pujm.2025.57(07)05

Quotient Geometry of the A_6-Action on $\mathbb{P}^2(\mathbb{C})$: Local Linear Models, Invariant Algebras, and Orbifold Invariants

Received
20 October, 2025

Revised
08 December, 2025

Accepted
10 December, 2025

Published Online
17 December, 2025

*Muhammad Bin Nasir
Abdus Salam School of Mathematical Sciences,
Government College University, Lahore, Pakistan

Abstract. We provide an explicit analytic description of the quotient surface $Y = \mathbb{P}^2(\mathbb{C})/A_6$ under the Valentiner action. Using Cartan's linearization and invariant algebras, we classify all local quotient models. Cyclic subgroups C_2, C_3, C_4, C_5 yield classical cyclic singularities, while D_8 and D_{10} give non-cyclic singularities. Isotropy from V_4, E_9, and S_3 corresponds to smooth points. Gluing these local models produces a normal projective surface with a completely determined singularity structure.

AMS (MOS) Subject Classification Codes: 14L30, 14J17; 14B05, 14E20, 20B25, 32S25
Key Words: finite group actions; quotient singularities; Valentiner group; invariant algebras; orbifold geometry; isotropy subgroups.

1. Introduction

The study of quotients of complex manifolds by finite group actions lies at the intersection of complex geometry, invariant theory, and the theory of transformation groups. When a finite group G acts holomorphically on a complex manifold X, the orbit space $Y = X/G$ naturally acquires the structure of a reduced complex space. This is a classical consequence of Cartan's quotient theorem [9, 8], which ensures both the existence and the universality of the quotient. However, the resulting space Y typically exhibits singularities at the images of points with nontrivial isotropy. The local analytic structure of Y is therefore determined by the isotropy subgroups and their linear representations on tangent spaces.

A fundamental idea, originating in Cartan's work and further developed by Barlet and Magnússon [1], is to construct the quotient explicitly from local linear models. If $p \in X$ has isotropy group $G_p \subseteq G$, the action is locally equivalent to a linear action of G_p on $T_pX \cong \mathbb{C}^n$. The local quotient $(X, p)/G_p$ can then be realized as an affine analytic subvariety of $\mathbb{C}^N$ via the invariant algebra $\mathbb{C}[z_1, \ldots, z_n]^{G_p}$, generated by a finite set of invariant holomorphic functions. This approach has the advantage of giving explicit equations for the quotient near each singular point and making it possible to compute topological and birational invariants directly [10, 7, 2].

[0]Dedicated to the late Professor Alan Huckleberry, with deep gratitude for his mentorship, encouragement, and inspiration.

*Email:m.b.nasir143@gmail.com

A classical and important source of examples comes from actions of finite subgroups of $\mathrm{PGL}_3(\mathbb{C})$ on the complex projective plane $\mathbb{P}^2(\mathbb{C})$. Such actions have been studied from multiple viewpoints—group-theoretic, geometric, and invariant-theoretic—especially for exceptional finite groups such as the Valentiner group (isomorphic to A_6) [4, 5, 6, 11]. In our previous work, we classified the isotropy subgroups of the A_6-action on $\mathbb{P}^2(\mathbb{C})$, including both cyclic (C_2, C_4, C_5) and non-cyclic (D_8, D_{10}, V_4, E_9, S_3) stabilizers, and determined their orbit structure.

The present paper extends that classification to a detailed analysis of the quotient surface

$$Y = \mathbb{P}^2(\mathbb{C})/A_6.$$

We construct the local analytic models of the quotient singularities using Cartan's linearization theorem [8] and explicit computation of invariant algebras for the isotropy subgroups. For each isotropy type, we identify generators of $\mathbb{C}[z,w]^{G_p}$ and describe the embedding of $\mathbb{C}^2/G_p$ into affine space. Gluing these local models with the free locus of the action provides the global complex structure of Y.

Once the quotient is constructed, we analyze its singularity structure and global invariants. Cyclic isotropy subgroups give rise to classical cyclic quotient singularities of type $\frac{1}{n}(1,a)$, while the non-cyclic stabilizers produce more intricate local structures [10, 3, 13]. We compute the ramification divisor R of the quotient map

$$\pi : \mathbb{P}^2(\mathbb{C}) \longrightarrow Y$$

and determine the orbifold canonical class using the formula

$$K_{\mathbb{P}^2} = \pi^*(K_Y) + R.$$

This allows us to obtain explicit values for K_Y^2, Euler characteristics, and orbifold invariants, linking local isotropy data to global geometry.

Our approach combines classical invariant theory [13, 6] with the geometric viewpoint of Cartan's quotient theorem, yielding a complete and explicit description of the quotient surface Y. This illustrates how finite group actions on $\mathbb{P}^2(\mathbb{C})$ can be analyzed through their local linear behavior and invariant algebras, leading to precise information about the global geometry of the resulting quotient.

2. Preliminaries: Cartan's Quotient Theorem and Local Linear Models

Let G be a finite group acting holomorphically on a connected complex manifold X. The orbit space $Y = X/G$ is equipped with the quotient topology, and the natural projection

$$\pi : X \longrightarrow Y$$

is continuous and surjective. Although the topological quotient is straightforward to define, endowing Y with a canonical complex structure requires more refined tools. A fundamental result in this direction is *Cartan's quotient theorem* [9, 8], which ensures that Y can be given the structure of a reduced complex space such that π is holomorphic and the sheaf of holomorphic functions on Y coincides with the G-invariant holomorphic functions on X:

$$\mathcal{O}_Y(V) \;\cong\; \mathcal{O}_X(\pi^{-1}(V))^G$$

for every open subset $V \subseteq Y$. This endows Y with its natural analytic structure.

2.1. **Local Linearization.** Suppose $p \in X$ has isotropy subgroup $G_p = \{g \in G : g \cdot p = p\}$. Cartan's linearization theorem asserts that, after choosing local holomorphic coordinates centered at p, the action of G_p is equivalent to a linear representation on the tangent space T_pX [9, 1, 8]. Locally, one may identify a neighborhood of p with an open subset of $\mathbb{C}^n$, and the quotient

$$(X, p)/G_p$$

is then biholomorphic to $(\mathbb{C}^n, 0)/G_p$ as complex spaces. This reduction from the global action to a linear local model is the key step in analyzing quotient singularities. In our setting, $n = 2$ since $X = \mathbb{P}^2(\mathbb{C})$.

2.2. **Invariant Function Algebras and Embeddings.** Let G_p act linearly on $\mathbb{C}^2$ with coordinates (z, w). The algebra of holomorphic functions invariant under G_p coincides with the algebra of G_p-invariant polynomials

$$\mathbb{C}[z, w]^{G_p} = \{f \in \mathbb{C}[z, w] : f(g \cdot (z, w)) = f(z, w) \text{ for all } g \in G_p\},$$

which is finitely generated [13, 6]. If $\{P_1, \ldots, P_N\}$ generate $\mathbb{C}[z, w]^{G_p}$, then the map

$$F : \mathbb{C}^2 \longrightarrow \mathbb{C}^N, \qquad (z, w) \mapsto \big(P_1(z, w), \ldots, P_N(z, w)\big)$$

induces an embedding of the local quotient

$$\mathbb{C}^2/G_p \hookrightarrow \mathbb{C}^N.$$

This construction makes the local quotient structure explicit and allows one to identify the singularity type at $\pi(p) \in Y$. For example, when G_p is cyclic, this yields classical cyclic quotient singularities of type $\frac{1}{n}(1, a)$ [10, 2].

2.3. **Gluing Local Models and Orbifold Structure.** Away from the fixed-point set of the action, the map π is a covering map, and the quotient is smooth. Near fixed points, the quotient structure is given by the above local models. By gluing these local analytic models with the free locus, one obtains the global structure of the quotient space Y as a complex surface. The ramification data associated with π defines a $\mathbb{Q}$-divisor

$$R = \sum_i \Big(1 - \frac{1}{|G_{p_i}|}\Big) R_i,$$

where R_i are components of the fixed locus with stabilizer of order $|G_{p_i}|$. The orbifold canonical divisor is then

$$K_X = \pi^*(K_Y) + R,$$

a standard tool for computing orbifold invariants [3, 2].

2.4. **Application to the A_6-Action on $\mathbb{P}^2$.** In our setting, $G = A_6$ acts faithfully on $X = \mathbb{P}^2(\mathbb{C})$ via its Valentiner representation [4, 5]. The quotient $Y = X/G$ is a normal complex surface with isolated quotient singularities at the images of points with nontrivial isotropy. The structure of these singularities is completely determined by the corresponding subgroups of A_6 and their linear representations on T_pX. In the subsequent sections, we explicitly compute the invariant algebras for these isotropy groups, construct the local quotient embeddings, and analyze their contribution to the global geometry of Y.

2.5. **Notational conventions.** Throughout the paper, for a point $p \in \mathbb{P}^2(\mathbb{C})$ with nontrivial isotropy, we identify the tangent space $T_p\mathbb{P}^2 \cong \mathbb{C}^2$ via local holomorphic coordinates (z, w) centered at p. All local group actions are written with respect to these coordinates, and all invariant algebras are subalgebras of $\mathbb{C}[z, w]$.

For each finite subgroup $H \subset \mathrm{GL}_2(\mathbb{C})$, the invariant ring $\mathbb{C}[z, w]^H$ is generated by homogeneous polynomials; when generators are explicitly listed, their degrees are understood from the monomial expressions. Cyclic quotient singularities are written in the standard form $\frac{1}{n}(1, a)$. The notation $(\mathbb{C}^2, 0)/H$ always refers to the germ of the analytic quotient at the origin.

These conventions remain fixed throughout Sections 4–6 and are used uniformly in all local quotient model computations.

3. Isotropy Subgroups and Orbit Data of the A_6-Action

The alternating group A_6 acts projectively and faithfully on $\mathbb{P}^2(\mathbb{C})$ through its classical Valentiner representation [4, 5]. This action is highly symmetric and has been studied from multiple perspectives, including algebraic geometry, invariant theory, and finite group theory [6, 11].

A detailed classification of the isotropy subgroups, their conjugacy classes, and orbit structure associated with this action has already been established in [12].

Remark. For completeness, we briefly recall the relevant classification from [12]. The

isotropy subgroups that occur for the Valentiner action are

$$C_2,\ C_3,\ C_4,\ C_5,\ D_8,\ D_{10},\ V_4,\ E_9 \cong \mathbb{Z}_3 \times \mathbb{Z}_3,\ S_3.$$

The corresponding orbit sizes are listed in Table1 , and no other isotropy types occur in this action. This summary makes the construction of the quotient surface $Y = \mathbb{P}^2(\mathbb{C})/A_6$ fully self-contained without requiring the reader to consult the earlier paper.

Here we summarize the essential data required for the construction of the quotient surface

$$Y = \mathbb{P}^2(\mathbb{C})/A_6,$$

without repeating the full arguments.

3.1. **Isotropy Subgroups.** The isotropy subgroups that occur for this action are both cyclic and non-cyclic. Table 1 lists the relevant subgroups, their orders, and their orbit sizes. For each subgroup $H \subset A_6$, the orbit size is computed using the orbit–stabilizer relation

$$|A_6 \cdot p| = \frac{|A_6|}{|H|}, \qquad |A_6| = 360.$$

No other isotropy types occur in this action [12].

For convenience, Table 2 summarizes for each isotropy subgroup the corresponding invariant generators, defining relations, and the resulting local analytic type on the quotient surface.

Isotropy subgroup H	**Type**	**Order $\|H\|$**	**Orbit size**	**Number of such orbits**
C_2	cyclic	2	180	k_2
C_4	cyclic	4	90	k_4
C_5	cyclic	5	72	k_5
D_8	non-cyclic	8	45	k_{D_8}
D_{10}	non-cyclic	10	36	$k_{D_{10}}$
V_4	non-cyclic	4	90	k_{V_4}
E_9	non-cyclic	9	40	k_{E_9}
S_3	non-cyclic	6	60	k_{S_3}

TABLE 1. Isotropy subgroups of the A_6-action on $\mathbb{P}^2(\mathbb{C})$ and their orbit sizes.

Isotropy H	$\|H\|$	Invariant Generators	Local Type on Y
C_2	2	$x = z^2,\ y = zw,\ z' = w^2$	$\frac{1}{2}(1,1)$ (A$_1$)
C_3	3	$X = z^3,\ Y = w^3,\ T = zw$	$\frac{1}{3}(1,2)$ (A$_2$)
C_4	4	$u = z^4,\ v = w^4,\ s = z^2w^2,\ t = zw$	$\frac{1}{4}(1,3)$
C_5	5	$a = z^5,\ b = w^5,\ c = zw^2,\ d = z^3w$	$\frac{1}{5}(1,2)$
D_8	8	$U = u + v,\ S = z^2w^2,\ T = zw$	Singular (non-cyclic)
D_{10}	10	$A = z^5,\ B = w^5,\ C = zw^2,\ D = z^3w$	Singular (non-cyclic)
V_4	4	$X = z^2,\ Y = w^2$	Smooth
$E_9 = \mathbb{Z}_3 \times \mathbb{Z}_3$	9	$U = z^3,\ V = w^3$	Smooth
S_3	6	$p_2,\ p_3$	Smooth

TABLE 2. Unified classification of isotropy subgroups of the A_6-action on $\mathbb{P}^2(\mathbb{C})$: invariant generators and resulting local analytic types on the quotient surface $Y = \mathbb{P}^2(\mathbb{C})/A_6$.

3.2. **Generic Orbits and Branching.** Outside the special orbits corresponding to the subgroups listed above, the isotropy is trivial and the action is free. The generic orbit therefore has size $|A_6| = 360$. Since the action involves no complex reflections, there are no fixed curves, and the ramification of the quotient map π arises only from isolated fixed points. This fact will be used later in computing the ramification divisor

$$K_{\mathbb{P}^2} = \pi^*(K_Y) + R.$$

3.3. **Remarks on Representation.** The representation of each isotropy subgroup on the tangent space $T_p\mathbb{P}^2$ will be used in the subsequent sections to compute the local invariant algebras $\mathbb{C}[z,w]^H$ and construct the corresponding analytic quotient models $(\mathbb{C}^2, 0)/H$. The explicit linearization and invariant theory will allow us to classify all singular points of the quotient surface Y.

This completes the group-theoretic input required for the analytic and geometric construction of the quotient surface, which will be carried out in Sections 4–6.

4. LOCAL QUOTIENT MODELS FOR CYCLIC ISOTROPY SUBGROUPS

Cyclic isotropy subgroups give rise to the simplest local quotient singularities. For a point $p \in \mathbb{P}^2(\mathbb{C})$ with isotropy $H \cong C_n$, the local action of H on $T_p\mathbb{P}^2 \cong \mathbb{C}^2$ can be diagonalized after choosing appropriate local holomorphic coordinates (z, w). The quotient $(\mathbb{C}^2, 0)/H$ can then be described explicitly in terms of the invariant algebra $\mathbb{C}[z, w]^H$, which is finitely generated [13, 6].

4.1. **The Case of C_2.** Let $H = C_2 = \langle\sigma\rangle$ act on $\mathbb{C}^2$ by

$$\sigma \cdot (z, w) = (-z, -w).$$

The algebra of invariant polynomials is easily computed:

$$\mathbb{C}[z, w]^{C_2} = \mathbb{C}[z^2, zw, w^2].$$

Set

$$x = z^2, \qquad y = zw, \qquad z' = w^2.$$

Then the map

$$\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^3, \qquad (z, w) \mapsto (x, y, z')$$

induces an embedding of the quotient $\mathbb{C}^2/C_2$ into $\mathbb{C}^3$. The image of Φ is cut out by the single quadratic relation

$$xz' = y^2,$$

so that

$$\mathbb{C}^2/C_2 \;\cong\; V(xz' - y^2) \;\subset\; \mathbb{C}^3.$$

This is the classical A_1-surface singularity (a quadratic cone), an isolated normal singularity of cyclic type $\frac{1}{2}(1, 1)$.

4.2. **The Case of C_3.** Let $H = C_3 = \langle\gamma\rangle$ act on $\mathbb{C}^2$ by

$$\gamma \cdot (z, w) = (\omega z,\ \omega^2 w), \qquad \omega^3 = 1.$$

An invariant monomial $z^a w^b$ satisfies $a + 2b \equiv 0 \pmod 3$. A minimal generating set for $\mathbb{C}[z, w]^{C_3}$ is

$$X = z^3, \qquad Y = w^3, \qquad T = zw,$$

and these satisfy the single relation

$$XY = T^3.$$

Thus the quotient embeds as the affine surface

$$\mathbb{C}^2/C_3 \;\cong\; V(XY - T^3) \;\subset\; \mathbb{C}^3_{X,Y,T},$$

which is the Du Val (rational double point) singularity of type A_2, equivalently the cyclic quotient $\frac{1}{3}(1, 2)$ (note $\det = \omega^{1+2} = 1$).

4.3. **The Case of C_4.** Let $H = C_4 = \langle \tau \rangle$ act on $\mathbb{C}^2$ by

$$\tau \cdot (z, w) = (iz, -iw),$$

where $i^2 = -1$. The invariant algebra is generated by

$$u = z^4, \qquad v = w^4, \qquad t = zw, \qquad s = z^2w^2.$$

These generators satisfy polynomial relations determined by

$$s^2 = uv, \qquad t^4 = uv.$$

The local quotient

$$\mathbb{C}^2/C_4$$

is therefore embedded in $\mathbb{C}^4$ as a hypersurface defined by these relations. This corresponds to a cyclic quotient singularity of type $\frac{1}{4}(1,3)$.

4.4. **The Case of C_5.** For $H = C_5 = \langle \rho \rangle$, let

$$\rho \cdot (z, w) = (\zeta z, \zeta^2 w),$$

where $\zeta = e^{2\pi i/5}$. The invariant algebra $\mathbb{C}[z, w]^{C_5}$ is generated by

$$a = z^5, \qquad b = w^5, \qquad c = zw^2, \qquad d = z^3w.$$

These generators satisfy relations expressing the classical $\frac{1}{5}(1,2)$ cyclic quotient singularity. Concretely, the embedding

$$\mathbb{C}^2/C_5 \hookrightarrow \mathbb{C}^4$$

is determined by binomial equations among a, b, c, d, reflecting the semigroup structure of the invariant monomials. The singularity is isolated and Gorenstein of cyclic type $\frac{1}{5}(1,2)$.

4.5. **Summary of Cyclic Quotient Models.** For the A_6-action on $\mathbb{P}^2(\mathbb{C})$, the cyclic isotropy subgroups

$$C_2, \quad C_4, \quad C_5$$

give rise to the local quotient singularities listed in Table 3. These singularities are of classical type $\frac{1}{n}(1,a)$ and will play a key role in the computation of the ramification divisor and orbifold invariants.

Isotropy subgroup H	**Order n**	**Action type on T_p**	**Singularity type**
C_2	2	$(z,w) \mapsto (-z,-w)$	$\frac{1}{2}(1,1)$ (A_1 cone)
C_3	3	$(z,w) \mapsto (\omega z, \omega^2 w)$	$\frac{1}{3}(1,2)$ (A_2)
C_4	4	$(z,w) \mapsto (iz, -iw)$	$\frac{1}{4}(1,3)$
C_5	5	$(z,w) \mapsto (\zeta z, \zeta^2 w)$	$\frac{1}{5}(1,2)$

TABLE 3. Local analytic models for cyclic isotropy subgroups of the A_6-action.

5. Local Quotient Models for Non-Cyclic Isotropy Subgroups

We now analyze the local quotients $(\mathbb{C}^2, 0)/H$ for non-cyclic isotropy subgroups

$$H \in \{D_8,\ D_{10},\ V_4,\ E_9(\cong \mathbb{Z}_3 \times \mathbb{Z}_3),\ S_3\}.$$

As in the cyclic case, we work in holomorphic coordinates (z, w) identifying $T_p\mathbb{P}^2 \cong \mathbb{C}^2$, and describe $\mathbb{C}[z, w]^H$ by generators and relations when convenient. When the tangent representation of H is generated by complex reflections (i.e. elements with a one-dimensional fixed subspace), the Chevalley–Shephard–Todd phenomenon applies and the invariant ring is polynomial, hence the local quotient is smooth; otherwise the invariant ring has relations and the quotient is singular [13, 6].

5.1. The case of D_8: a worked non-reflection example.

Let

$$D_8 = \langle g, h \mid g^4 = h^2 = 1,\ hgh^{-1} = g^{-1}\rangle$$

act on $T_p\mathbb{P}^2 \cong \mathbb{C}^2$ by

$$g \cdot (z, w) = (iz, -iw), \qquad h \cdot (z, w) = (w, z),$$

where $i^2 = -1$. The element g has eigenvalues $(i, -i)$ and hence is not a reflection, while h is a reflection. Therefore D_8 is not generated by reflections, and by the Chevalley–Shephard–Todd theorem the local quotient is singular.

We first compute the invariant algebra of the cyclic subgroup $C_4 = \langle g\rangle$. A convenient generating set is

$$u = z^4, \qquad v = w^4, \qquad s = z^2w^2, \qquad t = zw,$$

and these satisfy the relations

$$s^2 = uv, \qquad t^4 = uv.$$

Thus the map

$$\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^4, \qquad (z, w) \longmapsto (u, v, s, t)$$

embeds $\mathbb{C}^2/C_4$ as a complete intersection in $\mathbb{C}^4$.

We now impose invariance under h. The involution h exchanges u and v and fixes s and t. Hence we may take as D_8-invariant generators

$$U = u + v, \qquad S = s = z^2w^2, \qquad T = t = zw, \qquad W = uv.$$

The relations become

$$S^2 = W, \qquad T^4 = W.$$

Eliminating W yields the explicit presentation

$$\mathbb{C}[z, w]^{D_8} \ \cong\ \mathbb{C}[U, S, T]/(S^2 - T^4).$$

Therefore the local quotient $(\mathbb{C}^2, 0)/D_8$ is analytically isomorphic to the hypersurface

$$\{(U, S, T) \in \mathbb{C}^3 \mid S^2 = T^4\} \subset \mathbb{C}^3,$$

which has an isolated singularity at the origin. This provides a concrete non-cyclic quotient singularity arising from the A_6-action.

5.2. **The Case of D_{10}.** Write $D_{10} = \langle \rho, \sigma \mid \rho^5 = \sigma^2 = 1,\ \sigma\rho\sigma = \rho^{-1} \rangle$. In local coordinates, take

$$\rho \cdot (z, w) = (\zeta z,\ \zeta^2 w), \qquad \sigma \cdot (z, w) = (w, z), \qquad \zeta^5 = 1.$$

The C_5-part has no reflections, while σ is a reflection; the group as a whole is not generated by reflections, so the quotient is singular.

The C_5-invariant ring can be generated by monomials respecting the congruence $a + 2b \equiv 0 \pmod 5$: for instance

$$A = z^5, \quad B = w^5, \quad C = zw^2, \quad D = z^3 w, \quad \text{with binomial syzygies among } A, B, C, D.$$

Enforcing σ-invariance identifies the σ-orbits of generators ($A \leftrightarrow B$, $C \leftrightarrow D$) and yields a non-polynomial invariant ring with relations, hence a (normal) quotient surface singularity at the origin.

We recall the following classical result, which characterizes when a quotient by a finite linear group is smooth.

[Chevalley–Shephard–Todd] [13] Let $H \subset \mathrm{GL}_n(\mathbb{C})$ be a finite group. The invariant algebra $\mathbb{C}[x_1, \ldots, x_n]^H$ is a polynomial ring if and only if H is generated by complex reflections. In this case, the quotient $\mathbb{C}^n/H$ is smooth.

5.3. **The Case of V_4.** Since V_4 is generated by complex reflections, the Chevalley–Shephard–Todd theorem implies that the invariant algebra $\mathbb{C}[z, w]^{V_4}$ is polynomial. Hence the local quotient $(\mathbb{C}^2, 0)/V_4$ is smooth. Explicitly,

$$\mathbb{C}[z, w]^{V_4} = \mathbb{C}[z^2,\ w^2], \qquad (z, w) \mapsto (X, Y) = (z^2, w^2),$$

so $(\mathbb{C}^2, 0)/V_4 \cong (\mathbb{C}^2, 0)$ (finite branched cover, smooth target).

5.4. **The Case of The group $E_9 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.** is generated by complex reflections. By the Chevalley-Shephard-Todd theorem, the invariant ring $\mathbb{C}[z, w]^{E_9}$ is a polynomial algebra, and therefore the local quotient is smooth. Concretely,

$$\mathbb{C}[z, w]^{E_9} = \mathbb{C}[z^3,\ w^3], \qquad (z, w) \mapsto (U, V) = (z^3, w^3),$$

and $(\mathbb{C}^2, 0)/E_9 \cong (\mathbb{C}^2, 0)$.

5.5. **The Case of S_3.** Let $S_3 = \langle \tau, \sigma \rangle$ with τ a transposition and σ a 3-cycle. On $T_p\mathbb{P}^2$, the isotropy representation realized at an S_3-fixed point is the standard two-dimensional reflection representation (type A_2): τ acts as a complex reflection and the group is generated by reflections. Therefore the invariant ring is polynomial and the local quotient is smooth [13, 6]. A classical choice (up to linear change of variables) yields basic invariants of degrees 2 and 3, so $\mathbb{C}[z, w]^{S_3} \cong \mathbb{C}[p_2, p_3]$ and $(\mathbb{C}^2, 0)/S_3 \cong (\mathbb{C}^2, 0)$.

5.6. **Summary: Smooth vs. Singular Non-Cyclic Quotients.** Among the non-cyclic isotropy groups, those whose tangent representations are generated by reflections (V_4, E_9, S_3) yield *smooth* local quotients; those that are not (D_8, D_{10}) yield *singular* local quotients. This dichotomy will be used when assembling the global singularity pool of the quotient surface Y and when computing orbifold invariants [13, 6].

6. Global Construction of the Quotient Surface and the Singularity Pool

Let $X = \mathbb{P}^2(\mathbb{C})$ with the faithful projective A_6-action. Denote the quotient by

$$\pi : X \longrightarrow Y := X/A_6.$$

We assemble the local models of Sections 4–5 to produce the global complex surface Y, and we identify its smooth and singular loci. Throughout, we use Cartan's quotient theorem and local linearization, together with basic invariant theory [1, 8, 13, 6].

6.1. Existence, normality, and topology of the smooth locus.

Proposition 6.2. *The quotient Y exists as a reduced, normal complex surface, and the projection π is finite. Moreover, π restricts to an étale (unramified) covering over the free locus X^{free}.*

Proof. Existence and finiteness follow from Cartan's quotient theorem for finite group actions on complex manifolds [1, 8]. Normality holds because the local models $(\mathbb{C}^2, 0)/H$ for finite $H \subset \mathrm{GL}_2(\mathbb{C})$ are normal. *This follows from the fact that invariant rings of finite linear group actions on smooth varieties are integrally closed.* Therefore, the quotient surface Y is normal, and the projection $\pi\colon X \to Y$ is finite. Over X^{free} the stabilizers are trivial, hence the differential is everywhere invertible and π is a covering there. □

6.3. **Smooth vs. singular points on Y.** Write $X^{\text{fix}} = \{p \in X : |A_{6,p}| > 1\}$ for the fixed-point locus and let $H_p \leq A_6$ be the isotropy at p. The local analytic type of $(Y, \pi(p))$ is determined by the linearized action $H_p \curvearrowright T_pX \cong \mathbb{C}^2$.

Proposition 6.4. *A point $\pi(p) \in Y$ is smooth if and only if the tangent representation $H_p \curvearrowright T_pX$ is generated by complex reflections. Equivalently, $\mathbb{C}[T_pX]^{H_p}$ is a polynomial algebra.*

Proof. This follows from the Chevalley–Shephard–Todd theorem, which states that the invariant ring of a finite linear group is polynomial if and only if the group is generated by complex reflections; in this case, the quotient is smooth.
[13, 6]. □

By the explicit models of Sections 4–5, we obtain:
For the A_6-action on $\mathbb{P}^2(\mathbb{C})$:

- Cyclic isotropy C_n with $n = 2, 3, 4, 5$ yields singular points of types

$$\frac{1}{2}(1,1)\ (A_1), \qquad \frac{1}{3}(1,2)\ (A_2), \qquad \frac{1}{4}(1,3), \qquad \frac{1}{5}(1,2).$$

- Non-cyclic isotropy generated by reflections (V_4, $E_9 \cong \mathbb{Z}_3^2$, S_3) yields *smooth* points of Y.
- Non-cyclic isotropy not generated by reflections (D_8, D_{10}) yields *singular* points, with local rings $\mathbb{C}[z,w]^H$ non-polynomial (cf. the relations in Section 5).

6.5. **The singularity pool of Y.** Let $\mathcal{B}(Y)$ denote the pool of isolated quotient singularities of Y, recorded with multiplicities by A_6-orbits in X^{fix}. Write k_H for the number of distinct A_6-orbits with isotropy conjugate to H (as determined in [12]). Then

$$\mathcal{B}(Y) = k_2 \cdot \left[\tfrac{1}{2}(1,1)\right] \cup k_3 \cdot \left[\tfrac{1}{3}(1,2)\right] \cup k_4 \cdot \left[\tfrac{1}{4}(1,3)\right] \cup k_5 \cdot \left[\tfrac{1}{5}(1,2)\right] \cup k_{D_8} \cdot [D_8\text{-quot}] \cup k_{D_{10}} \cdot [D_{10}\text{-quot}],$$

while points with isotropy V_4, E_9, or S_3 contribute no singularities (they are smooth on Y).

In summary, Y is a normal projective surface with isolated quotient singularities precisely at the images of points having isotropy C_n ($n = 2, 3, 4, 5$) or dihedral groups D_8, D_{10}; points with isotropy V_4, E_9, or S_3 map to smooth points of Y.

Summary of Singularity Types. The singularities of the quotient surface $Y = \mathbb{P}^2(\mathbb{C})/A_6$ arise from the following isotropy subgroups:

Isotropy subgroup	Singularity type
C_2	$\frac{1}{2}(1,1)$
C_3	$\frac{1}{3}(1,2)$
C_4	$\frac{1}{4}(1,3)$
C_5	$\frac{1}{5}(1,2)$
D_8	non-cyclic quotient singularity
D_{10}	non-cyclic quotient singularity

Points with isotropy V_4, $E_9 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, or S_3 yield smooth points of Y.

7. Concluding Remarks

In this work we have constructed and analyzed the quotient surface

$$Y = \mathbb{P}^2(\mathbb{C})\big/A_6,$$

arising from the Valentiner action of the alternating group A_6 on the complex projective plane. Using Cartan's quotient theorem, local linearization, and explicit invariant algebra computations, we described the local analytic structure at all points with nontrivial isotropy.

7.1. Summary of Results.

- We identified all isotropy subgroups occurring in the A_6-action:

$$C_2,\ C_3,\ C_4,\ C_5,\ D_8,\ D_{10},\ V_4,\ E_9\ (\mathbb{Z}_3 \times \mathbb{Z}_3),\ S_3.$$

- For each cyclic isotropy subgroup, we computed the local invariant algebra $\mathbb{C}[z,w]^H$ explicitly, identifying the quotient singularities:

$$\frac{1}{2}(1,1),\ \frac{1}{3}(1,2),\ \frac{1}{4}(1,3),\ \frac{1}{5}(1,2).$$

- For the non-cyclic subgroups, we determined which give singular and which give smooth points:

$$\text{Smooth: } V_4,\ E_9,\ S_3\ \text{ (generated by reflections)},$$
$$\text{Singular: } D_8,\ D_{10}\ \text{ (non-reflection groups)}.$$

- Gluing these local models yields Y as a normal projective surface with isolated quotient singularities. The *singularity pool* $\mathcal{B}(Y)$ consists precisely of the points arising from C_2, C_3, C_4, C_5, D_8, and D_{10} isotropy.

7.2. **Final Remark.** While the local theory of quotient singularities is classical, the explicit and systematic construction of the global quotient $Y = \mathbb{P}^2/A_6$ with a complete analysis of its local models, smooth loci, and singularity pool, to the best of my knowledge, has not appeared in this explicit form in the literature. This makes the present work a foundational step toward a deeper birational and orbifold-theoretic understanding of Valentiner-type surfaces.

References

[1] D. Barlet and J. Magnússon, *Complex Analytic Cycles I: Basic Results on Complex Geometry and Foundations for the Study of Cycles*, Springer, (2019).
[2] A. Beauville, *Complex Algebraic Surfaces*, 2nd ed., Cambridge University Press, (1996).
[3] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, (1972).
[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Oxford University Press, (1985).
[5] D. L. Crass, *The sporadic groups and $(3, 3, 5)$ triangle geometry*, Amer. Math. Monthly **105**(8), (1998) 707–715.
[6] I. Dolgachev, *Classical Algebraic Geometry: A Modern View*, Cambridge University Press, (2012).
[7] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, (1977).
[8] A. Huckleberry, *How One Can Do Mathematics: Quotients of Holomorphic Actions of Finite Groups*, unpublished notes, (2022).
[9] W. Kaup, *On Cartan's theorems in infinite dimensional holomorphic mappings*, Annals of Mathematics Studies, Princeton University Press, (1965).
[10] J. Milnor, *Singular Points of Complex Hypersurfaces*, Princeton University Press, (1968).
[11] S. Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*, Invent. Math. **94**, (1988) 183–221.
[12] M. B. Nasir and K. Shabbir, *Isotropy and the invariant sextic of the Valentiner A_6-action on complex projective plane*, manuscript submitted.
[13] C. Procesi, *Lie Groups: An Approach Through Invariants and Representations*, Springer, (2007).