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An Introduction to Dynamic Soft Sets: A Framework for Modeling Temporal Uncertainty

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Abstract. In engineering, healthcare, and decision-making systems, effectively handling uncertainty remains a fundamental research challenge. Existing frameworks, including fuzzy sets, rough sets, and soft sets, typically operate on static parameter spaces with fixed approximation mappings, which limits their ability to model real-world environments where relevant parameters and their interpretations evolve over time. To address this limitation, this paper introduces a dynamic extension of soft set theory and makes the following principal contributions. **(1)** A formal definition of **Dynamic Soft Sets** is proposed, in which both the active parameter subsets $A_t \subseteq E$ and the approximation mappings $F_t : A_t \to \mathcal{P}(U)$ are explicitly indexed by time or context $t \in T$, enabling the representation of parameters that activate, deactivate, or change relevance over time as a natural extension of Molodtsov's soft sets. **(2)** This framework is further generalized to **Dynamic Hypersoft Sets**, allowing time-varying multi-attribute Cartesian product parameter spaces $A_t = A_{t1} \times \cdots \times A_{tn}$, thereby extending classical hypersoft sets to dynamic environments. **(3)** A complete set-theoretic operator framework for Dynamic Soft Sets is developed, including union, intersection, complement, restricted and extended operators, as well as logical AND/OR operations, all defined in a time-dependent setting with fundamental algebraic properties established. **(4)** A systematic comparative analysis is presented to distinguish Dynamic Soft Sets from classical soft sets and hypersoft sets, highlighting differences in parameter dynamics and structural flexibility through structured tables. **(5)** The practical applicability of the proposed framework is demonstrated through a healthcare monitoring case study, showing how time-varying sensor availability and evolving clinical conditions can be naturally modeled using Dynamic Soft Sets. Unlike classical soft sets with static parameters and hypersoft sets with fixed Cartesian product domains, Dynamic Soft Sets introduce explicit time- or context-indexing of both parameter subsets and approximation mappings, achieving dual structural and temporal dynamism. The proposed framework provides a unified and

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mathematically rigorous foundation for time- and context-adaptive uncertainty modeling and decision-making in dynamic real-world applications.

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1. INTRODUCTION

The uncertainty is a natural feature of contemporary data-driven systems experienced in intelligent decision support [24, 3], engineering design [2], medical diagnostics, economic modelling [20], and artificial intelligence. Vagueness and incomplete information have been long modeled using classical mathematical approaches like probability theory [1, 23], fuzzy set theory [26] and rough set theory [17]. These methods can, however, usually assume accurate probability distributions, fixed membership functions or pre-specified equivalence relationships, which are not easy to characterize or explain in a complex and dynamically changing real world.

Soft set theory [11] is a parameterized and flexible form of uncertainty modeling that does not need any other assumptions, like membership grades or probabilistic measures. Soft sets, which model complex systems where parameter relevance is of primary importance by using families of approximating sets indexed by parameters, allow the intuitive and flexible modelling of uncertainty. Due to this structural elasticity, soft set theory has received particular interest and has been developed extensively in theory and practice in the past 20 years.

There are many extensions to soft set theory suggested to make it more expressive. Fuzzy soft sets include graded membership information, intuitionistic fuzzy soft sets model hesitation simultaneous membership and non-membership degrees [8], and neutrosophic soft sets more generally include indeterminacy [9]. Hypersoft sets are an extension of the classical framework of soft sets by adding multi-parameter spaces of Cartesian product parameters that allow the fine-grained modelling of interdependent criteria in multi-criteria decision-making (MCDM) problems. Saeed et al. [18] defined soft sets, soft members, and soft elements and generalized classical soft set theory so that elements and memberships are both parameterized soft objects. This framework increases the capacity of soft sets to model structured and layered uncertainty in complicated situations of decision-making. Saeed et al. [19] suggested soft algebraic structures with soft members and soft elements, a generalization of classical algebraic systems where uncertainty is parameterized, both at the structural and element levels. Fatimah et al. [5] Probabilistic and dual probabilistic soft sets. Soft set theory is applicable to uncertainty decision-making, and probabilistic and dual probabilistic soft sets are extensions that include probability measures to make appropriate decisions. Musa and Asaad [13] Bipolar m-parametrized n-soft sets are suggested as an enhanced soft set model that represents dual (positive-negative) information and allows the use of a more informed and flexible decision-making process. Saeed et al. cubic Pythagorean fuzzy soft set frame has been conceptually elaborated to handle the higher-order uncertainty and has been successfully applied to the decision-making issues of a multi-attribute problem. All these have greatly increased the value space as well as the structural depth of soft-set-based uncertainty models.

In addition to using soft-set-based models, there is a significant amount of recent research in MCDM that concentrates on more expressive fuzzy and hesitant models. There are vector similarity measures of picture type-2 hesitant fuzzy sets, bipolar-valued complex hesitant fuzzy Dombi aggregating operators, generalized Dice-type similarity measures of single-valued neutrosophic type-2 hesitant fuzzy sets, and interval-valued q-rung orthopair hesitant fuzzy Choquet aggregating operators, all of which are aimed at reflecting high-order uncertainty, interaction among criteria, and subtle expert hesitation in MCDM settings [16, 15, 14]. They are complementary to the current work: as much as such frameworks enhance the value space with the aid of complex membership and aggregation schemes, Dynamic Soft Sets are mainly designed to work in the structural and time dimensions, offering the ability of time-indexed parameter subsets and approximations, and can theoretically be integrated with advanced fuzzy/hesitant operators to create temporally adaptive MCDM systems.

Classical soft sets define uncertainty with single-level parameters and thus they are not able to define the problems that have parameters with multiple related attributes. To address this, structural generalizations of soft sets were proposed in [10] with Hypersoft Sets where parameters are represented as tuples of the Cartesian product of disjoint attribute domains. $C = A_1 \times A_2 \times \cdots \times A_n$ and a mapping $F : C \to \mathcal{P}(U)$ is used to define the parameter space of a hypersoft set framework and a mapping is used to assign subsets of the universe to each multi-attribute tuple. Analysis has been conducted on the N-bipolar hypersoft sets introduced by Musa [12] to improve the decision-making algorithms through the modeling of multi-attribute information using layered bipolar uncertainty. Yolcu et al. [25] It has been suggested that fuzzy hypersoft sets are a functional extension of fuzzy set theory in which the multi-attribute uncertainty of decision-making processes has been demonstrated to be effectively dealt with. Debnath et al. [4] Fuzzy hypersoft sets Fuzzy hypersoft sets along with a weightage operator have been designed to enhance the process of aggregation and ranking of the alternative in decision-making problems. Rahman et al. presented the theory of bijective hypersoft sets and used in decision-making problems, which is a systematic method of dealing with the relationships between parameters of one to one. The theory of neutrosophic hypersoft sets on matrices has been created and implemented on multi-attribute, multi-criteria decision-making to increase computation speed and analytical clarity.

Although such progress has been made, the majority of classical and extended soft set models are necessarily static models in that the sets of parameters are fixed and the mapping of approximations is non-dynamical. To deal with time issues of uncertainty, various time-sensitive variants are suggested. Time Fuzzy Soft Sets (T-FSS) are a fuzzy membership with discrete time values, which allow the summation of the historical information to be used in decision-making processes. Generalized temporal intuitionistic fuzzy soft sets (GTIFSS) [6] are an extension of this landscape adding time-dependent generalized intuitionistic membership and non-membership degrees to each parameter of a fixed parameter set to provide an intuitionistic environment with time consciousness to MCDM. Dynamic Soft Sets (DSS), on the contrary, are intended to complement and extend these

time-dependent structures by enabling the active parameter set to vary over time, structurally representing situations where attributes can emerge, disappear, or become relevant and so temporally depending on parameters not treated by GTIFSS and T-FSS whose sets of parameters are constant.
Nevertheless, such models fill the value representation of uncertainty significantly, but they usually assume that sets of parameters and attributes are fixed. This makes them fail to directly discuss the structural dynamics of parameter relevance and availability that are often found in real-world systems, where attributes can emerge, vanish, and more or less become important over time.

1.1. **Motivation for Dynamic Soft Sets.** The uncertainty in most real world application is not only time varying in terms of data values but is also dynamic in structure. In intelligent healthcare monitoring, such as sensor availability can be inconsistent because of technical malfunctions, patient movement or medical practices as the symptoms and biomarkers change with disease progression and treatment phases. Equally, in financial systems, stable market conditions are often characterized by an interest in price trends and interest rates, and a volatile, liquid and systemic risk indicator is of interest during a period of instability or crisis. Similar structural dynamics exist in environmental surveillance and cybersecurity, where the attributes of interest can often arise, fall out of favor, or gain importance based on the time and context.
Most of the classical soft sets and their extensions use a fixed parameter set $A \subseteq E$, meaning that they cannot model such changing parameter landscapes. Even hypersoft sets, although allowing multi-attribute modeling by use of Cartesian product parameter spaces, are based on fixed attribute definitions and are thus unable to allow change in parameter relevance or availability over time. Thereupon, available frameworks have difficulties in modeling systems where the structure of parameters as well as the corresponding approximations change with time.
These observations indicate that dynamic uncertainty modeling has three basic problems:

- **Temporal Parameter Dynamics**: Attributes can be activated or deactivated with variations in the state of the system,
- **Context-Dependent Approximations**: the meaning of parameters and their interpretation is changing as new information is being received,
- **Real-Time Decision Requirements**: static models are not flexible enough to keep track of the constant monitoring and adjustive decisions.

These difficulties drive the construction of an active subsystem of soft sets where the subsets of active parameters as well as the mapping of the approximations are explicitly indexed by time or context so as to represent structurally changing uncertainty faithfully.

1.2. **Identified Research Gaps.** The literature review reveals a number of inherent constraints in the existing soft set-based uncertainty models. Many extensions of the representational power of the soft set theory have been added including fuzzy, intuitionistic, neutrosophic, and hypersoft sets, but in most cases, they maintain a fixed structural premise. Specifically, the underlying parameter universe, as well as the mappings used to

do the approximation, take the form of a fixed set of parameters usually maintained constant throughout the analysis.
The current temporal variants mainly focus on the issue of time dependency, making membership values or evaluations change over discrete time points but still based on an invariant parameter set. Consequently, these methods cannot be applied to model those situations where parameters themselves dynamically appear, disappear, or become irrelevant with time.
Therefore, in the present body of research there is a lack of:

- a formally defined soft set structure that explicitly includes subsets of parameters indexed by time or context $A_t \subseteq E$,
- a complete framework of set-theoretic operators that retain algebraic consistency with the dynamically varying structure of parameters,
- a single theoretical stance that explains the association of such dynamic models with classical soft sets, hypersoft sets and current temporal extensions of fuzzy sets.

1.3. **Novelty and Contributions.** To fill the gaps observed, this paper proposes the structures of DSS and DHSS. The main results of this work comprise the following:

- **Formal Definition of Dynamic Soft Sets**: A strict formulation, where groupings of parameters A_t and approximation mappings F_t depend on time or context.
- **Dynamic Hypersoft Extension**: Time-varying generalization of hypersoft sets of multi-attribute Cartesian products into dynamically evolving sets.
- **Comprehensive Operator Framework**: Creation of union, intersection, complement, logical AND/OR and restricted and extended operators on dynamic settings along with their algebraic properties.
- **Comparative Theoretical Analysis**: Syntactic placement of DSS and DHSS with respect to current soft set, hypersoft set and temporal uncertainty models.
- **Algorithmic and Practical Validation**: A versatile decision-making algorithm and practical case studies of applicability in the constantly changing real life.

As shown in Figure 1, proposed DSS framework will create a systematic connection between the uncertainty situation in the real world and the theoretical model behind it, its dynamic parameter structure, fundamental properties, and decision-support mechanism. The flow diagram also shows the way the conceptual framework is implemented in an algorithmic process with time consideration, which allows making adaptive assessment and uniform decision-making with varying parameter relevance.

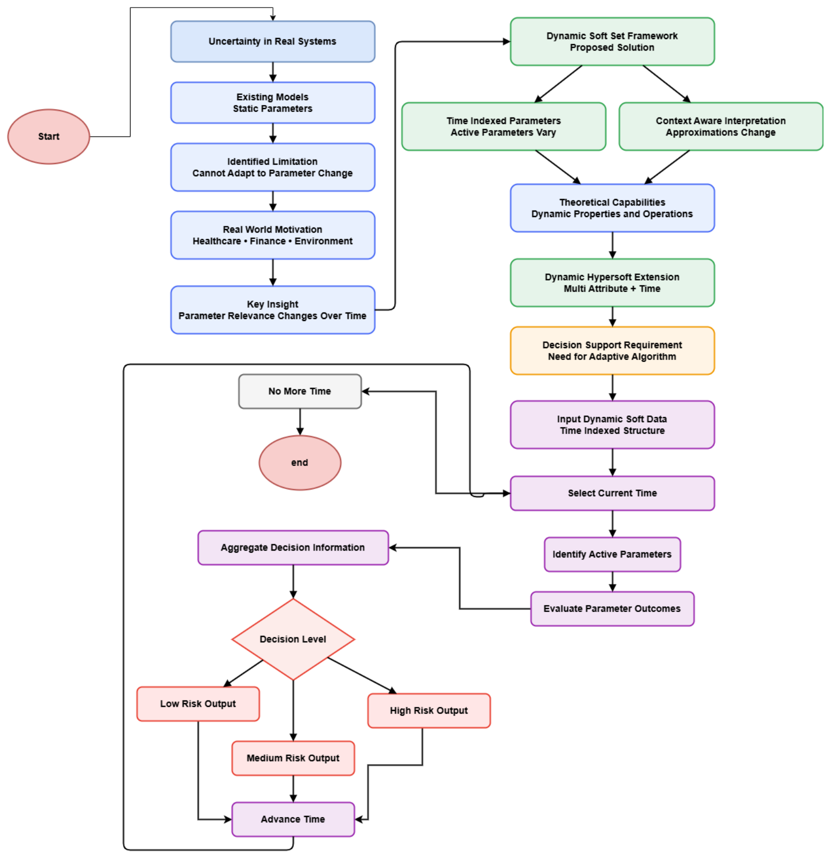

FIGURE 1. Hybrid flowchart representation of the proposed dynamic soft set–based framework

The proposed framework, through these contributions, provides a foundation step in the direction of fully dynamic and structurally adaptive soft set theory, and with wide impact on decision support as well as uncertainty modelling in time-dependent systems.

1.4. **Merits and Limitations Analysis.** To place the proposed DSS framework within the wider context of soft-set-based uncertainty models, this subsection will provide the comparative analysis of the structural features, benefits, and inherent limitations of each of them. It compares the various extensions on their approach to uncertainty regarding parameter structure, temporal adaptation and computational scalability and thus explains the contribution and trade-offs brought about by DSS.

TABLE 1. Comparative Analysis of Soft Set Extensions

Framework	**Parameter Structure**	**Key Strengths**	**Principal Limitations**
Classical Soft Set (SS)	Fixed parameter subset $A \subseteq E$	Simple representation; low computational cost; clear semantics	Inability to model temporal variation or evolving parameter relevance
Hypersoft Set (HSS)	Fixed Cartesian product $C = A_1 \times \cdots \times A_n$	Supports multi-attribute and interdependent parameter modeling	Parameter space remains static; unsuitable for time-varying contexts
Temporal Fuzzy Models	Fixed parameters with time-indexed membership values	Captures gradual temporal changes in uncertainty levels	Structural parameter dynamics are not supported
Time Fuzzy Soft Sets (T-FSS)	Fixed A, fuzzy memberships $\mu_t(x)$ over time	Enables fuzzy temporal aggregation for decision-making	Assumes constant attribute availability and relevance
Dynamic Soft Sets (DSS)	Time-varying parameter subsets $A_t \subseteq E$	**Explicit modeling of evolving parameters and interpretations; backward compatibility with SS**	**Increased computational and storage complexity; requires careful design of A_t**

Merits of Dynamic Soft Sets:

- **Dual dynamism**: DSS records both the time evolution in both the parameter relevance and approximation mappings.
- **Structural flexibility**: Parameters can work on, off or change over time, which represents realistic information environments.

- **Backward compatibility**: Classical soft sets occur as a special case when $A_t = A$ and $F_t = F$ for all $t \in T$.
- **Natural temporal modeling**: DSS offers a straightforward and understandable framework of a time/context-sensitive decision support mechanism.

Limitations of Dynamic Soft Sets:

- **Computational overhead**: Complexity of storage and processing increases as $O(|T| \cdot |A_{\max}| \cdot |U|)$ in the worst case scenarios.
- **Domain dependence**: Time-varying parameter subsets A_t are commonly constructed well with the help of expert knowledge or application specific rules.
- **Reduction and optimization**: The use of standardized reduction, compression, and parameter-selection strategies of DSS has been an unresolved research problem.

The rest of the paper is structured in the following way. The section A2 presents the preliminaries needed. Section 3introduces the definition of DSSs and gives some examples. Section 3.2 constructs the corresponding set-theoretic operator framework, defines basic properties, and gives a comparative study. Section 4 is the introduction of DHSSs, along with a demonstrative example and a comparative discussion. Section 5 provides dynamic decision-making algorithms that are done depending on the proposed framework. Section 6 presents the findings focusing on the theoretical aspects and practical implications. Lastly, 7 is a conclusion of the paper that provides the direction of future research.

Abbreviations and Notations. For clarity and consistency, the main abbreviations and mathematical notations used throughout this paper are summarized in Table 2.

TABLE 2. Abbreviations and Mathematical Notations

Abbreviation	**Description**
	Soft Set Frameworks
SS	Soft Set
HSS	Hypersoft Set
DSS	Dynamic Soft Set
DHSS	Dynamic Hypersoft Set
	Operators and Methods
AND	Logical AND operator
OR	Logical OR operator
RU	Restricted Union
RI	Restricted Intersection
EU	Extended Union
EI	Extended Intersection
MCDM	Multi-Criteria Decision-Making
	Mathematical Symbols
U	Universe of discourse
E	Global set of parameters
A_t	Active parameter subset at time/context t
F_t	Approximation mapping at time t
T	Set of time or context indices
$\mathcal{P}(U)$	Power set of U
$\emptyset$	Empty set
$\subseteq$	Subset relation
$\cup$	Union
$\cap$	Intersection
$\times$	Cartesian product
$\setminus$	Set difference
$\lvert \cdot \rvert$	Cardinality of a set

2. PRELIMINARIES

This section reviews the essential definitions and mathematical structures underlying soft set and hypersoft set theory. These preliminaries provide the formal groundwork necessary for introducing time-indexed parameter spaces and dynamic approximation mappings in later sections.

Definition 2.0.1. [11] *Let U be a universal set and E be a set of parameters. A pair (F, A) is called a **soft set** over U, where $A \subseteq E$ and $F : A \to \mathcal{P}(U)$ is a mapping that assigns to each parameter $e \in A$ a subset $F(e) \subseteq U$.*

Definition 2.0.2. [10] *Let U be a universe of discourse and let $E = \{A_1, A_2, \dots, A_n\}$ be a family of* pairwise disjoint *attribute (parameter) domains, that is, $A_i \cap A_j = \varnothing \quad$ for all $i \neq$*

*j. Let $C = A_1 \times A_2 \times \cdots \times A_n$ denote the corresponding multi-attribute Cartesian product domain. A **hypersoft set** over U is defined as a mapping $F^{HS} : C \to \mathcal{P}(U)$, where each parameter is an n-tuple $(a_1, a_2, \ldots, a_n)$ with $a_i \in A_i$ for all $i = 1, 2, \ldots, n$.*

To obtain a detailed theoretical background of soft set theory and its further developments like fuzzy soft sets, further structural formulations and hypersoft set models, the reader is suggested to consult the foundational and survey articles in [11, 7, 18, 22, 21]. These sources include some elaborate coverage of the underlying definitions, algebraic operations, multi-attribute parameter space extensions, and the development of soft-set-based systems of uncertainty modelling and decision-making systems.

3. Dynamic Soft Sets

This section introduces the concept of **Dynamic Soft Sets** as a natural extension of classical soft set theory to time- or context-dependent environments. Formal definitions, illustrative examples, and fundamental set-theoretic operations are presented to demonstrate how dynamic soft sets capture the evolution of active parameters and their approximations, thereby distinguishing them from traditional static soft set models.

Definition 3.0.1. *Let U be a (possibly finite) universe of discourse, E be a set of parameters, and T be a set of time points, system states, or scenario indices (discrete or continuous). A **dynamic soft set** over U is defined as a family*

$$\mathcal{S} = \left\{ (t, e, F_t(e)) \ : \ t \in T, \ e \in A_t \right\},$$

where, for each $t \in T$, $A_t \subseteq E$ denotes the set of active parameters at time (or context) t, and

$$F_t : A_t \to \mathcal{P}(U)$$

is a mapping that assigns to each parameter $e \in A_t$ a subset $F_t(e) \subseteq U$.

Remark 3.0.1. *A dynamic soft set can be viewed as a time- or context-indexed family of classical soft sets, in which both the parameter subsets and their corresponding approximations are allowed to evolve. In particular, the proposed structure reduces to a classical soft set when $A_t = A$ and $F_t = F$ for all $t \in T$.*

Example 3.0.2. ***(Case 1: Smart Healthcare Monitoring)*** *This example illustrates the application of a dynamic soft set in a clinical monitoring scenario. Let the universe of patient states be*

$$U = \{\textit{Normal}, \textit{At Risk}, \textit{Critical}\},$$

the parameter set be

$$E = \{\textit{BP (Blood Pressure)}, \textit{HR (Heart Rate)}, \textit{SpO}_2 \textit{ (Oxygen Saturation)}\},$$

and the time index set be

$$T = \{\textit{Morning}, \textit{Evening}\}.$$

For each $t \in T$, the active parameter subset $A_t \subseteq E$ and the corresponding approximation mapping $F_t : A_t \to \mathcal{P}(U)$ are defined as follows.

*In the **Morning**,*

$$A_{Morning} = \{BP, HR, SpO_2\},$$

with

$$F_{Morning}(BP) = \{Normal, At\ Risk\},$$
$$F_{Morning}(HR) = \{Normal\},$$
$$F_{Morning}(SpO_2) = \{Normal, At\ Risk\}.$$

*In the **Evening**, the oxygen saturation sensor becomes inactive, and hence*

$$A_{Evening} = \{BP, HR\},$$

with

$$F_{Evening}(BP) = \{At\ Risk\},$$
$$F_{Evening}(HR) = \{At\ Risk, Critical\}.$$

Thus, the resulting dynamic soft set $\mathcal{S}$ over (U, E, T) is given by

$$\begin{aligned}\mathcal{S} = \big\{ \ &(Morning,\ BP,\ \{Normal, At\ Risk\}),\\ &(Morning,\ HR,\ \{Normal\}),\\ &(Morning,\ SpO_2,\ \{Normal, At\ Risk\}),\\ &(Evening,\ BP,\ \{At\ Risk\}),\\ &(Evening,\ HR,\ \{At\ Risk, Critical\}) \ \big\}.\end{aligned}$$

This example demonstrates how both sensor availability and medical interpretations evolve over time within the dynamic soft set framework.

Example 3.0.3. ***(Case 2: Financial Market Modeling)*** *This example presents a dynamic soft set in a financial decision-making context, where the relevance of market indicators varies across different regimes. Let the universe of alternatives be*

$$U = \{High\ Return\ Stocks, Moderate\ Return\ Stocks, Low\ Return\ Stocks\},$$

the parameter set be

$$E = \{Stock\ Price, Interest\ Rate, Volatility\},$$

and the regime (time) index be

$$T = \{Stable\ Market, Volatile\ Market\}.$$

For each $t \in T$, let $A_t \subseteq E$ denote the set of active indicators and $F_t : A_t \to \mathcal{P}(U)$ be the associated approximation mapping.

*In a **Stable Market**,*

$$A_{Stable} = \{Stock\ Price, Interest\ Rate\},$$

with

$$F_{\mathit{Stable}}(\mathit{Stock\ Price}) = \{\mathit{High\ Return\ Stocks}, \mathit{Moderate\ Return\ Stocks}\},$$
$$F_{\mathit{Stable}}(\mathit{Interest\ Rate}) = \{\mathit{Moderate\ Return\ Stocks}\}.$$

In a ***Volatile Market****,*

$$A_{\mathit{Volatile}} = \{\mathit{Stock\ Price}, \mathit{Volatility}\},$$

with

$$F_{\mathit{Volatile}}(\mathit{Stock\ Price}) = \{\mathit{Moderate\ Return\ Stocks}\},$$
$$F_{\mathit{Volatile}}(\mathit{Volatility}) = \{\mathit{Low\ Return\ Stocks}\}.$$

Hence, the corresponding dynamic soft set $\mathcal{S}$ is expressed as

$$\begin{aligned}\mathcal{S} = \Big\{ & (\mathit{Stable\ Market}, \mathit{Stock\ Price}, \{\mathit{High\ Return\ Stocks}, \mathit{Moderate\ Return\ Stocks}\}),\\ & (\mathit{Stable\ Market}, \mathit{Interest\ Rate}, \{\mathit{Moderate\ Return\ Stocks}\}),\\ & (\mathit{Volatile\ Market}, \mathit{Stock\ Price}, \{\mathit{Moderate\ Return\ Stocks}\}),\\ & (\mathit{Volatile\ Market}, \mathit{Volatility}, \{\mathit{Low\ Return\ Stocks}\}) \Big\}.\end{aligned}$$

This formulation highlights how both the set of active parameters and their induced classifications adapt dynamically to changing market conditions.

3.1. **Properties of Dynamic Soft Sets.** In this subsection, a few of the basic relational properties of dynamic soft sets are addressed, such as dynamic soft subsets, equality, null dynamic soft sets, and whole dynamic soft sets. These ideas constitute the ordering structure needed to make coherent comparisons and additional set-theoretic operations.

Definition 3.1.1. *Let $\mathcal{S} = \big\{ (t, e, F_t(e)) \ : \ t \in T,\ e \in A_t \big\}$ and $\mathcal{G} = \big\{ (t, e, G_t(e)) \ : \ t \in T,\ e \in B_t \big\}$ be two dynamic soft sets over the same universe U and time index set T, where for each $t \in T$,*

$$F_t : A_t \to \mathcal{P}(U), \qquad G_t : B_t \to \mathcal{P}(U).$$

The dynamic soft set $\mathcal{S}$ is said to be a ***dynamic soft subset*** *of $\mathcal{G}$, denoted by $\mathcal{S} \subseteq \mathcal{G}$, if and only if for every $t \in T$ the following conditions hold:*

(1) $A_t \subseteq B_t$,
(2) $F_t(e) \subseteq G_t(e)$ *for all* $e \in A_t$.

This definition ensures that, at each time or contextual index t, both the active parameter set and the corresponding approximations of $\mathcal{S}$ are contained within those of $\mathcal{G}$.

Example 3.1.2. *Let the universe of patient states be*

$$U = \{\mathit{Normal}, \mathit{At\ Risk}, \mathit{Critical}\},$$

and the time index set be

$$T = \{\mathit{Morning}, \mathit{Evening}\}.$$

Define two dynamic soft sets

$$\mathcal{S} = \{(F_t, A_t)\}_{t\in T} \quad \textit{and} \quad \mathcal{G} = \{(G_t, B_t)\}_{t\in T}.$$

For $t =$ Morning,

$$A_{Morning} = \{BP, HR\}, \qquad B_{Morning} = \{BP, HR, SpO_2\},$$

with approximation mappings

$$F_{Morning}(BP) = \{Normal\}, \qquad G_{Morning}(BP) = \{Normal, At\ Risk\},$$
$$F_{Morning}(HR) = \{Normal, At\ Risk\}, \qquad G_{Morning}(HR) = \{Normal, At\ Risk, Critical\},$$
$$G_{Morning}(SpO_2) = \{Normal\}.$$

For $t =$ Evening,

$$A_{Evening} = \{BP\}, \qquad B_{Evening} = \{BP, HR\},$$

with

$$F_{Evening}(BP) = \{At\ Risk\}, \qquad G_{Evening}(BP) = \{Normal, At\ Risk\},$$
$$G_{Evening}(HR) = \{Normal\}.$$

Since $A_t \subseteq B_t$ for each $t \in T$ and

$$F_t(e) \subseteq G_t(e) \quad \textit{for all } e \in A_t,$$

it follows that $\mathcal{S} \subseteq \mathcal{G}$.

Remark 3.1.1. *There is a natural interpretation of this concept of a* ***dynamic soft subset*** *in a clinical monitoring context in which observations are taken at the discrete time points labeled by the time T and the health status of a patient is described by the universe U. At a time slice $t \in T$, the parameter sets A_t and B_t would be the active clinical indicators of interest, and the relation $\mathcal{S} \subseteq \mathcal{G}$ would indicate a structured inclusion between two dynamical monitoring systems.*
In particular, condition $A_t \subseteq B_t$ and the differentiating condition $F_t(e) \subseteq G_t(e)$ for all $e \in A_t$ allow one to be sure that the health evaluations generated by $\mathcal{S}$ are always more exact or narrow than those of $\mathcal{G}$. This association offers a temporal meaning of informational granularity, where $\mathcal{S}$ signifies an attentive observation framework that is embedded in a larger diagnostic situation.
In addition, with parameter sets and their approximations allowed to alter over time, the dynamic soft subset relation will enable the alteration in monitoring scope and patient condition. This relation, therefore, is an essential cornerstone of the definition of algebraic operations, the ability to perform logical inference, and the hierarchical decision-making in dynamic and uncertain settings.

Definition 3.1.3. *Let $\mathcal{N} = \{ (t, e, F_t(e)) \ : \ t \in T, \ e \in A_t \}$ be a dynamic soft set over a universe U and time index set T. The dynamic soft set $\mathcal{N}$ is called a* ***dynamic null soft set*** *if, for every $t \in T$ and every $e \in A_t$, $F_t(e) = \varnothing$.*
A dynamic null soft set represents the complete absence of information across all time instances and parameters. It serves as the identity element for union-type operations and

provides a baseline structure for defining complements and difference operations in the dynamic soft set framework.

Definition 3.1.4. *Let* $\mathcal{W} = \big\{ (t, e, F_t(e)) \ : \ t \in T, \ e \in A_t \big\}$ *be a dynamic soft set over a universe* U *and time index set* T*. The dynamic soft set* $\mathcal{W}$ *is called a* ***whole (or absolute) dynamic soft set*** *if, for every* $t \in T$ *and every* $e \in A_t$*,* $F_t(e) = U$*.*
The whole dynamic soft set represents maximal information at every time instance, where each active parameter fully supports the entire universe. It acts as the identity element for intersection-type operations and forms the upper bound of the lattice structure induced by dynamic soft subset relations.

3.2. **Set-theoretic Operators on Dynamic Soft Sets.** In the subsection, we present some basic set-theoretic operators of dynamic soft sets, such as union, intersection, complement and logical conjunction/disjunction (AND/OR). These operators are designed with respect to temporal indexing, which allows uniform manipulation and reasoning with time-varying parameter set in dynamic decision-making settings.

Definition 3.2.1. *Let* $\mathcal{S} = \{(t, e, F_t(e)) : t \in T, \ e \in A_t\}$ *and* $\mathcal{G} = \{(t, e, G_t(e)) : t \in T, \ e \in B_t\}$ *be two dynamic soft sets over a common universe* U *and time index set* T*. The* ***union*** *of* $\mathcal{S}$ *and* $\mathcal{G}$*, denoted by* $\mathcal{S} \cup \mathcal{G}$*, is defined as the dynamic soft set*

$$\mathcal{H} = \{(t, e, H_t(e)) : t \in T, \ e \in C_t\},$$

where, for each $t \in T$*,*

$$C_t = A_t \cup B_t,$$

and the mapping $H_t : C_t \to \mathcal{P}(U)$ *is given by*

$$H_t(e) = \begin{cases} F_t(e), & \text{if } e \in A_t \setminus B_t, \\ G_t(e), & \text{if } e \in B_t \setminus A_t, \\ F_t(e) \cup G_t(e), & \text{if } e \in A_t \cap B_t. \end{cases}$$

Definition 3.2.2. *Let* $\mathcal{S} = \{(t, e, F_t(e)) : t \in T, \ e \in A_t\}$ *and* $\mathcal{G} = \{(t, e, G_t(e)) : t \in T, \ e \in B_t\}$ *be two dynamic soft sets defined over a common universe* U *and indexed by the same time set* T*. The* ***intersection*** *of* $\mathcal{S}$ *and* $\mathcal{G}$*, denoted by* $\mathcal{S} \cap \mathcal{G}$*, is the dynamic soft set*

$$\mathcal{I} = \{(t, e, I_t(e)) : t \in T, \ e \in D_t\},$$

where, for each $t \in T$*,*

$$D_t = A_t \cap B_t,$$

and the corresponding mapping $I_t : D_t \to \mathcal{P}(U)$ *is defined by*

$$I_t(e) = F_t(e) \cap G_t(e), \quad \text{for all } e \in D_t.$$

Definition 3.2.3. *Let* $\mathcal{S} = \{(t, e, F_t(e)) : t \in T, \ e \in A_t\}$ *and* $\mathcal{G} = \{(t, e, G_t(e)) : t \in T, \ e \in B_t\}$ *be two dynamic soft sets over a common universe* U *and time index set* T*. The* ***restricted union*** *of* $\mathcal{S}$ *and* $\mathcal{G}$ *is defined as the dynamic soft set*

$$\mathcal{R}_U = \{(t, e, R_{U,t}(e)) : t \in T, \ e \in R_t\},$$

where, for each $t \in T$*,*

$$R_t = A_t \cap B_t,$$

and the associated mapping $R_{U,t} : R_t \to \mathcal{P}(U)$ is given by

$$R_{U,t}(e) = F_t(e) \cup G_t(e), \quad \textit{for all } e \in R_t.$$

Definition 3.2.4. *Let $\mathcal{S} = \{(t, e, F_t(e)) : t \in T,\ e \in A_t\}$ and $\mathcal{G} = \{(t, e, G_t(e)) : t \in T,\ e \in B_t\}$ be two dynamic soft sets defined over the same universe U and time index set T. The **restricted intersection** of $\mathcal{S}$ and $\mathcal{G}$ is the dynamic soft set*

$$\mathcal{R}_I = \{(t, e, R_{I,t}(e)) : t \in T,\ e \in Q_t\},$$

where, for each $t \in T$,

$$Q_t = A_t \cap B_t,$$

and the mapping $R_{I,t} : Q_t \to \mathcal{P}(U)$ is defined by

$$R_{I,t}(e) = F_t(e) \cap G_t(e), \quad \textit{for all } e \in Q_t.$$

Definition 3.2.5. *Let $\mathcal{S} = \{(t, e, F_t(e)) : t \in T,\ e \in A_t\}$ and $\mathcal{G} = \{(t, e, G_t(e)) : t \in T,\ e \in B_t\}$ be two dynamic soft sets over a common universe U and time index set T. The **extended union** of $\mathcal{S}$ and $\mathcal{G}$ is defined as the dynamic soft set*

$$\mathcal{E}_U = \{(t, e, E_{U,t}(e)) : t \in T,\ e \in S_t\},$$

where, for each $t \in T$,

$$S_t = A_t \cup B_t,$$

and the mapping $E_{U,t} : S_t \to \mathcal{P}(U)$ is given by

$$E_{U,t}(e) = \begin{cases} F_t(e), & e \in A_t \setminus B_t, \\ G_t(e), & e \in B_t \setminus A_t, \\ F_t(e) \cup G_t(e), & e \in A_t \cap B_t. \end{cases}$$

Definition 3.2.6. *Let $\mathcal{S} = \{(t, e, F_t(e)) : t \in T,\ e \in A_t\}$ and $\mathcal{G} = \{(t, e, G_t(e)) : t \in T,\ e \in B_t\}$ be two dynamic soft sets defined over a common universe U and time index set T. The **extended intersection** of $\mathcal{S}$ and $\mathcal{G}$ is the dynamic soft set*

$$\mathcal{E}_I = \{(t, e, E_{I,t}(e)) : t \in T,\ e \in M_t\},$$

where, for each $t \in T$,

$$M_t = A_t \cup B_t,$$

and the mapping $E_{I,t} : M_t \to \mathcal{P}(U)$ is defined by

$$E_{I,t}(e) = \begin{cases} F_t(e) \cap G_t(e), & e \in A_t \cap B_t, \\ \emptyset, & \textit{otherwise}. \end{cases}$$

Remark 3.2.1. *The set-theoretic operations that have been established regarding dynamic soft sets have the following significant properties.*

- *The suggested dynamic operations, when fixed to time index $t \in T$, are the same , but with the classical soft set union and intersection and the restricted or extended forms all being applied pointwise to the parameter sets.*
- *The restricted operations are only operated on the parameters that are shared in both dynamic soft sets at a given time, hence maintaining common informational content.*

- *The domain of parameters is covered by the extended operations that include all active parameters and empty approximate sets to inactive ones to be structurally complete.*
- *The dynamic soft set framework, which permits the parameter subsets A_t to timedependent or contextdependent, gives a natural and rigorous generalization to classical operations of a soft set, and so the operators are most appropriate to the representation of time-dependent and contextsensitive decision problems.*

Definition 3.2.7. *Let $\mathcal{S} = \{ (t, e, F_t(e)) : t \in T,\ e \in A_t \}$ be aDSS over the universe U with parameter set E and time index set T. The **complement** of $\mathcal{S}$, denoted by $\mathcal{S}^C$, is defined as the DSS*

$$\mathcal{S}^C = \{ (t, e, F_t^c(e)) : t \in T,\ e \in A_t \},$$

where for each $t \in T$ and each $e \in A_t$,

$$F_t^c(e) = U \setminus F_t(e).$$

Remark 3.2.2. *In the complement of a DSS, the active parameter set A_t remains unchanged at each time $t \in T$. The complementation is performed with respect to the fixed universe U, meaning that for every active parameter $e \in A_t$, the approximate value $F_t(e)$ is replaced by its set-theoretic complement $U \setminus F_t(e)$.*

Definition 3.2.8. *Let U be a universe, E a parameter set, and T a set of time (or context) indices. Consider two dynamic soft sets*

$$\mathcal{S} = \{(t, e, F_t(e)) : t \in T,\ e \in A_t\}, \qquad \mathcal{G} = \{(t, e, G_t(e)) : t \in T,\ e \in B_t\}.$$

*The **AND-operation** of $\mathcal{S}$ and $\mathcal{G}$ at a fixed time $t \in T$ is defined as*

$$(F_t, A_t) \wedge (G_t, B_t) = (H_t,\ A_t \times B_t),$$

where the mapping $H_t : A_t \times B_t \to \mathcal{P}(U)$ is given by

$$H_t(a, b) = F_t(a) \cap G_t(b), \qquad \forall (a, b) \in A_t \times B_t.$$

The resulting DSS is the family

$$\{(t, (a, b), H_t(a, b)) : t \in T,\ (a, b) \in A_t \times B_t\}.$$

Definition 3.2.9. *Let U be a universe, E a parameter set, and T a set of time (or context) indices. Consider two dynamic soft sets*

$$\mathcal{S} = \{(t, e, F_t(e)) : t \in T,\ e \in A_t\}, \qquad \mathcal{G} = \{(t, e, G_t(e)) : t \in T,\ e \in B_t\}.$$

*The **OR-operation** of $\mathcal{S}$ and $\mathcal{G}$ at a fixed time $t \in T$ is defined as*

$$(F_t, A_t) \vee (G_t, B_t) = (K_t,\ A_t \times B_t),$$

where the mapping $K_t : A_t \times B_t \to \mathcal{P}(U)$ is given by

$$K_t(a, b) = F_t(a) \cup G_t(b), \qquad \forall (a, b) \in A_t \times B_t.$$

The resulting DSS is the family

$$\{(t, (a, b), K_t(a, b)) : t \in T,\ (a, b) \in A_t \times B_t\}.$$

Remark 3.2.3. *The following observations clarify the structural properties of the dynamic AND/OR operations:*

- *For each fixed time $t \in T$, the resulting parameter set is given by the Cartesian product $A_t \times B_t$.*
- *For every parameter pair $(a, b) \in A_t \times B_t$, the associated approximation is obtained by applying the set-theoretic intersection (AND) or union (OR) to the corresponding value sets $F_t(a)$ and $G_t(b)$.*
- *The formulation is consistent with classical soft set operations when the time index T reduces to a singleton, while naturally extending to the DSS framework.*

3.3. **Algebraic Properties of Dynamic Soft Set Operators.** This subsection establishes the fundamental algebraic properties of the operators defined in Section 4, addressing monotonicity, boundedness, idempotency, absorption laws, and reduction formulas. All proofs are constructive and preserve the dynamic structure.

3.3.1. *Monotonicity Properties.*

Theorem 3.3.1. ***(Monotonicity of Union)*** *Let S, G be Dynamic Soft Sets (DSS) over (U, E, T). If $S \subseteq G$ (i.e., $A_t \subseteq B_t$ and $F_t(e) \subseteq G_t(e)\ \forall t \in T, e \in A_t$), then $S \subseteq (S \cup G)$.*

Proof. Let $H = S \cup G$. Then $C_t = A_t \cup B_t \supseteq A_t\ \forall t \in T$. For any $e \in A_t$:

- Case 1: $e \notin B_t \implies H_t(e) = F_t(e) \supseteq F_t(e)$
- Case 2: $e \in A_t \cap B_t \implies H_t(e) = F_t(e) \cup G_t(e) \supseteq F_t(e)$

Thus $F_t(e) \subseteq H_t(e)\ \forall e \in A_t, t \in T$, so $S \subseteq H$. □ □

Theorem 3.3.2. ***(Monotonicity of Intersection)*** *If $S \subseteq G$, then $(S \cap H) \subseteq (G \cap H)$ for any DSS H.*

Proof. Let $I_1 = S \cap H$, $I_2 = G \cap H$. Then $D_t = A_t \cap C_t \subseteq B_t \cap C_t = E_t\ \forall t \in T$. For $e \in D_t$:

$$I_{1,t}(e) = F_t(e) \cap H_t(e) \subseteq G_t(e) \cap H_t(e) = I_{2,t}(e)$$

Thus $I_1 \subseteq I_2$. □ □

3.4. **Boundedness Properties.** This subsection explores properties of the Dynamic Soft Sets being bounded within the dynamic soft subset relation. Specifically, it defines the presence of global lower and upper bounds as the null and absolute dynamic soft sets respectively.

Theorem 3.4.1. ***(Null Set as Absorber)*** *For any DSS S and null DSS N (where $N_t(e) = \emptyset$ $\forall t, e$), $N \subseteq S$.*

Proof. $\emptyset \subseteq F_t(e)\ \forall t \in T, e \in A_t$. □ □

Theorem 3.4.2. ***(Absolute Set as Universum)*** *For any DSS S over U and absolute DSS $\mathbb{U}$ (where $\mathbb{U}_t(e) = U\ \forall t, e$), $S \subseteq \mathbb{U}$.*

Proof. $F_t(e) \subseteq U\ \forall t, e$. □ □

Theorem 3.4.3. ***(Union with Null Set)*** *$S \cup N = S$ for null DSS N.*

Proof. $H_t(e) = F_t(e) \cup \emptyset = F_t(e)\ \forall t, e \in A_t$. □ □

Theorem 3.4.4. ***(Intersection with Absolute Set)*** *$S \cap \mathbb{U} = S$ for absolute DSS $\mathbb{U}$.*

Proof. $H_t(e) = F_t(e) \cap U = F_t(e)\ \forall t, e \in A_t$. □ □

3.4.1. *Idempotency and Absorption Laws.* Dynamic soft sets satisfy the classical idempotency, absorption, and involution properties of set-theoretic operations, adapted to the time-indexed framework. These properties are stated pointwise as follows:

- **Idempotency of Union:** For any DSS S over (U, E, T),
$$S \cup S = S.$$
- **Idempotency of Intersection:** For any DSS S over (U, E, T),
$$S \cap S = S.$$
- **Absorption Law 1:** For any DSSs S and G over (U, E, T),
$$S \cup (S \cap G) = S.$$
- **Absorption Law 2:** For any DSSs S and G over (U, E, T),
$$S \cap (S \cup G) = S.$$
- **Complement Involution:** For any DSS S over (U, E, T), taking the complement twice returns the original set,
$$(S^c)^c = S.$$

These identities ensure that the algebra of DSSs inherits the familiar structural behavior of classical set operations in a time-indexed setting.

3.4.2. *Reduction Properties.*

Definition 3.4.1. ***(Time-Slice Reduction)*** *For DSS $S = \{(t, F_t, A_t)\}_{t \in T}$ and $T' \subseteq T$, the time-slice reduction is $S|_{T'} = \{(t, F_t, A_t)\}_{t \in T'}$.*

Theorem 3.4.5. ***(Time-Slice Preservation)*** *Operations preserve time-slice reduction: $(S \cup G)|_{T'} = (S|_{T'}) \cup (G|_{T'})$.*

Proof. Both sides defined identically on T'. □ □

Definition 3.4.2. ***(Parametric Reduction)*** *Given relevance function $R_t \subseteq A_t$ preserving decision information, the parametric reduction is $S^R = \{(t, F_t|_{R_t}, R_t)\}_{t \in T}$.*

Theorem 3.4.6. ***(Parametric Reduction Compatibility)*** *If $R_t^S \subseteq A_t^S$, $R_t^G \subseteq A_t^G$ are valid reductions, then $(S \cup G)^{R_t^S \cup R_t^G} = S^{R_t^S} \cup G^{R_t^G}$.*

Proof. For $e \in R_t^S \cup R_t^G$, union defined consistently on reduced parameters. ☐ ☐

3.5. **Comparative Analysis: Soft Set, Hypersoft Set, and Dynamic Soft Set.** An effective application of soft sets in real-world decision-making issues would also require a clear understanding of the differences between classical, hypersoft, and dynamic soft sets. The comparative tables in this subsection are used to show the basic differences between these models in the parameter structure, expressive capability, and temporal adaptability in the key application domains.

The development of soft-set-based frameworks in the case of smart healthcare monitoring 3 can be seen to be the shift from the fixed character of patient data to dynamical structures that may reflect the availability and uncertainty of parameters in time. This development illustrates the importance of dynamic soft sets to simulate changing clinical situations and sensor-based environments.

TABLE 3. Comparison of Soft Set Types in Smart Healthcare Monitoring

Type	**Parameter Structure & Mapping**	**Example Scenario and Interpretation**
Soft Set (SS)	Fixed parameter set $E = \{\text{BP, HR, SpO}_2\}$. Single-level flat parameters; mapping $F: E \to \mathcal{P}(U)$ fixed at a single time with no index.	Patient health represented at a fixed time snapshot by Blood Pressure (BP), Heart Rate (HR), and Oxygen Saturation (SpO_2). Parameters remain constant over time or context.
Hypersoft Set (HS)	Parameters represented as multi-attribute tuples $C = A_1 \times A_2 \times \cdots \times A_n$, where each A_i is a disjoint attribute set. Mapping $F: C \to \mathcal{P}(U)$.	Attributes decomposed, e.g., $A_1 = \{\text{BP Levels: Low, Normal, High}\}$, $A_2 = \{\text{Age Group: Adult, Senior}\}$, $A_3 = \{\text{Symptom: Cough, Fever}\}$. HS models combined complex criteria simultaneously, such as patients having (High BP, Senior, Fever).
Dynamic Soft Set (DSS)	Time-indexed parameter subsets $A_t \subseteq E$ varying over $t \in T$; family of soft sets $\{(t, F_t, A_t) : t \in T\}$ with $F_t : A_t \to \mathcal{P}(U)$. Parameters and mappings change dynamically.	For example, at morning time t_1, $A_{t_1} = \{\text{BP, HR, SpO}_2\}$; at evening t_2, SpO_2 sensor fails and $A_{t_2} = \{\text{BP, HR}\}$. DSS models evolving sensor availability and clinical measures over time.

Similarly, Table 4 illustrates how dynamic parameterization enhances the representation of financial indicators under changing market regimes, where the relevance of parameters is inherently time-dependent.

TABLE 4. Comparison of Soft Set Types in Financial Modeling

Type	**Parameter Structure & Mapping**	**Example Scenario and Interpretation**
Soft Set (SS)	Fixed parameter set, e.g., $E = \{\text{Stock Price}, \text{Interest Rate}, \text{Volatility}\}$ Static mapping $F : E \to \mathcal{P}(U)$ with no temporal indexing.	Financial asset classification at a fixed snapshot where parameters like stock price, interest rate, and volatility are considered constant throughout the analysis period.
Hypersoft Set (HS)	Multi-attribute parameter tuples from disjoint attribute sets, e.g., $A_1 =$ {Market Sector: Tech, Finance, Health}, $A_2 =${Risk Level: Low, Medium, High}, $A_3 =${Time Horizon: Short, Long}.	Models interaction of sector, risk, and investment horizon jointly, allowing simultaneous multi-criteria-asset evaluation based on combined tuple attributes.
Dynamic Soft Set (DSS)	Time-varying parameter subsets $A_t \subseteq E$ and mappings $F_t : A_t \to \mathcal{P}(U)$ indexed by $t \in T$. Models changing relevance or availability of financial indicators.	During different market regimes, parameter relevance changes; e.g., $A_{t_1} =$ {Stock Price, Interest Rate} in stable markets; $A_{t_2} = \{\text{Stock Price}, \text{Volatility}\}$ in crisis periods. DSS captures this evolution succinctly.

The analysis of the similarities and differences between DSS and HS with an accent on the conceptual basis and modeling possibilities. The structural distinctions of paramount importance are summarized in Table 5, which makes it clear that DSS mainly deals with time-varying parameter relativeness, while HS is structured to represent fixed combinations of multi-attribute parameters. The comparison outlines clear roles of the two frameworks in modelling uncertainty in dynamic and multi-criteria settings.

TABLE 5. Clear Differentiation Points Between DSS and HS

Aspect	**Dynamic Soft Set (DSS)**	**Hypersoft Set (HS)**
Parameter type	Single attributes from E (flat parameter set), possibly varying active parameter subsets $A_t \subseteq E$ over time $t \in T$.	Multi-attribute tuples from Cartesian product of disjoint attribute sets $C = A_1 \times A_2 \times \cdots \times A_n$.
Parameters decomposition	No decomposition; parameters are atomic elements of E.	Parameters inherently decomposed into independent attribute sets; structured, multi-dimensional tuples.
Time variation / dynamics	Explicitly dynamic; family of soft sets indexed by time or context $t \in T$, parameter sets A_t and mappings vary arbitrarily with t.	Typically static parameter domain C; dynamic hypersoft sets extend C with time but domain is fixed upfront as full Cartesian product.
Structural form	Collection of soft sets (F_t, A_t) with time-dependent parameter subsets and mappings.	Single mapping $F : C \to \mathcal{P}(U)$ defined on multi-attribute tuples, fixed parameter space.
Complexity / Expressiveness	Suitable for modeling evolving availability and relevance of single-level parameters over time.	Suitable for modeling complex multi-criteria, hierarchical attribute structures as fixed tuples.
Mathematical relation	Not a special case or substructure of HS without multi-attribute tuple formalism.	Hypersoft set generalizes soft sets only when parameter sets degenerate to single-level sets (rarely holds).
Use case focus	Time-adaptive parameter tracking, such as sensor availability changing dynamically.	Multi-criteria decision making and attribute combination analysis involving fixed multiple attributes.

4. DYNAMIC HYPERSOFT SETS

In this section, the author presents the idea of **Dynamic Hypersoft Sets**, that is, the extension of the classical hypersoft sets with the introduction of time- or context-dependent parameterization. The given framework enables the dynamic development of the underlying multi-attribute parameter domains as well as the approximations in place to model the complex systems whose relative attribute relevance and structure change.

Definition 4.0.1. *Let U be a universe of discourse, T be a set of time points or contextual indices, and let $E_t = \{A_{t1}, A_{t2}, \ldots, A_{tn}\}$ be a family of pairwise disjoint attribute* *domains associated with time $t \in T$. Let $A_t = A_{t1} \times A_{t2} \times \cdots \times A_{tn}$ denote the corresponding Cartesian product of attribute sets. A* ***dynamic hypersoft set*** *over U is defined as a family*

$$\mathcal{H} = \left\{ (t, e, F_t(e)) \ : \ t \in T, \ e \in A_t \right\},$$

where, for each $t \in T$, the mapping $F_t : A_t \to \mathcal{P}(U)$ assigns to every multi-attribute parameter tuple $e \in A_t$ a subset of the universe U.
A dynamic hypersoft set generalizes both classical hypersoft sets and dynamic soft sets by permitting simultaneous temporal variation in attribute domains, parameter combinations, and approximation mappings.

Example 4.0.2. ***(Dynamic Hypersoft Set in Healthcare Monitoring)*** *Consider a smart healthcare monitoring system in which patient conditions are assessed using multiple attribute dimensions that may change over time.*

Let the universe of patient states be

$$U = \{\textit{Normal}, \textit{At Risk}, \textit{Critical}\},$$

and let the time index set be

$$T = \{\textit{Morning}, \textit{Evening}\}.$$

At time $t = $ Morning, the parameter domain is defined as

$$A_{\textit{Morning}} = A_{\textit{Morning},1} \times A_{\textit{Morning},2} \times A_{\textit{Morning},3},$$

where

$$\begin{aligned} A_{\textit{Morning},1} &= \{\textit{BP Low}, \textit{BP Normal}, \textit{BP High}\}, \\ A_{\textit{Morning},2} &= \{\textit{Age Adult}, \textit{Age Senior}\}, \\ A_{\textit{Morning},3} &= \{\textit{Symptom None}, \textit{Symptom Fever}\}. \end{aligned}$$

The approximation mapping

$$F_{\textit{Morning}} : A_{\textit{Morning}} \to \mathcal{P}(U)$$

assigns patient states to each multi-attribute tuple. For example,

$$F_{\textit{Morning}}(\textit{BP High}, \textit{Age Senior}, \textit{Symptom Fever}) = \{\textit{Critical}\},$$

indicating a critical health condition for this attribute combination.

At time $t = $ Evening, symptom monitoring is no longer available, and the parameter domain reduces to

$$A_{\textit{Evening}} = A_{\textit{Evening},1} \times A_{\textit{Evening},2},$$

where

$$\begin{aligned} A_{\textit{Evening},1} &= \{\textit{BP Low}, \textit{BP Normal}\}, \\ A_{\textit{Evening},2} &= \{\textit{Age Adult}, \textit{Age Senior}\}. \end{aligned}$$

The corresponding mapping

$$F_{\textit{Evening}} : A_{\textit{Evening}} \to \mathcal{P}(U)$$

defines the approximations for the reduced attribute space. For instance,

$$F_{Evening}(BP\ Low, Age\ Adult) = \{Normal\}.$$

This example illustrates how dynamic hypersoft sets capture both multi-attribute dependency and temporal variation in healthcare monitoring systems.

Remark 4.0.1. *The previous example demonstrates the relevance of a Dynamic Hypersoft Set to a real-life smart healthcare monitoring scenario, where the status of patients depends on the combinations of various attributes, the relevance and the availability of which change through time. The universe U corresponds to discrete clinical states, where it is possible to classify them according to multi-attribute profiles changing over time, rather than single parameters.*
The attributes monitored at each time index are based on the current clinical priorities and the availability of data. Specifically, the more efficient parameter space $A_{Morning}$, which is a Cartesian product of blood pressure, age category, and symptom features, enables the specific context-sensitive evaluation by the mapping $F_{Morning}$. On the other hand, the smaller domain $A_{Evening}$ represents those cases where some attributes are not available temporarily, like the monitoring of symptoms and thus requires a revision in the approximation mapping $F_{Evening}$.
This is a temporal reorganization of attribute domains and mappings where a major difference can be found between Dynamic Hypersoft Sets and classical hypersoft sets whereby the parameter space is taken as fixed. DHSS supports a flexible and mathematically sound framework of modeling time-sensitive multi-attribute uncertainty by explicitly indexing both domains of attributes and approximation functions. This type of flexibility is not only necessary in healthcare monitoring but also in any decision-making setting where there are changing criteria and a sense of environmental vagueness.

4.1. **Comparison of Dynamic Hypersoft Sets and Classical Hypersoft Sets.** In this subsection, I will make a systematic comparison between classical HSS and DHSS. The comparison, as seen in Table 6, shows how the explicit representation of evolving attribute structures and uncertainty in DHSS can be done by the explicit use of temporal or contextual indexing in time sensitive contexts that cannot be effectively represented using hypersoft sets of traditional.

TABLE 6. Key Differences Between Hypersoft Set and Dynamic Hypersoft Set

Feature	**Hypersoft Set (HSS)**	**Dynamic Hypersoft Set (DHSS)**
Parameter Set	Fixed Cartesian product of disjoint attribute sets: $A = A_1 \times A_2 \times \cdots \times A_n$ where each A_i is static across analysis.	Time-indexed Cartesian product of disjoint attribute subsets: $A_t = A_{t1} \times A_{t2} \times \cdots \times A_{tn(t)}$ allowing attribute sets to evolve or vary with $t \in T$.
Mapping Function	Single mapping: $F : A \to \mathcal{P}(U)$ assigning subsets of universe U to fixed parameter tuples.	Family of time-dependent mappings: $F_t : A_t \to \mathcal{P}(U), \quad t \in T$ which can change values as parameters evolve over time/context.
Time Dimension	No explicit temporal or contextual index; parameters and mappings are static.	Explicit inclusion of a time or context index $t \in T$, capturing evolving uncertainty and parameter relevance.
Attribute Dynamics	Attributes are fixed; the set of parameters is static throughout the model.	Attribute sets can be added, removed, or modified at different times t, reflecting dynamic system changes (e.g. sensors failing or new symptoms appearing).
Applicability	Suited for static or one-time multi-attribute analysis, where the parameter configuration does not change.	Ideal for modeling real-world systems that require adaptive, temporal modeling of uncertain, multi-attribute data (e.g., smart healthcare monitoring).
Example Scenario	Patient health assessment at a single time with fixed parameters like blood pressure, age group, and symptom status.	Patient monitoring with varying parameters through the day, such as reduced symptom monitoring in the evening, dynamically altering parameter space and classification.
Mathematical Complexity	Generally simpler due to fixed parameter structure and one mapping function.	More complex due to indexing over time, requiring families of sets and mappings, and handling variable parameter dimensions.
Modeling Flexibility	Limited in handling evolving attribute presence or granularity.	Highly flexible, adapts to changes in attribute presence, granularity, and relevance, allowing refined temporal uncertainty modeling.

5. Algorithmic Framework

This part of the paper gives a concrete algorithm scheme to make a decision under Dynamic Soft Sets, showing how the previous sections' developed concepts can be converted into actual procedures. We are going to demonstrate the methodology in the example of healthcare monitoring and present pseudocode, complexity analysis, and a visual flowchart representation.

5.1. Algorithm 1: Dynamic Soft Set Based Patient Risk Assessment.

Patient monitoring systems in clinical decision-making should evaluate risk throughout the time and sensor availability and other important clinical parameters should vary dynamically. This process is formalized by algorithm: with DSS models of patient states, active parameters and their approximations at each time-point, the Algorithm 1 combines risk information and yields a risk classification at each time index.

Algorithm 1 Dynamic Soft Set Based Patient Risk Assessment

Require: Universe U (patient states), parameter set E (clinical attributes), time index set T, Dynamic Soft Set $S = \{(t, F_t, A_t) : t \in T\}$, risk thresholds $\theta_1, \theta_2 \in \mathbb{R}$ with $0 < \theta_1 < \theta_2$, weight function $w : E \to [0,1]$

Ensure: Risk classification sequence `RiskScore` = $[(t_i, \text{class}_i)]_{i=1}^{|T|}$

```
RiskScore ← []                                      ▷ Initialize empty list
for each t ∈ T do
    A_t ← GetActiveParameters(t)                    ▷ Retrieve active parameter set at time t
    S_t ← 0                                         ▷ Initialize aggregation score
    for each e ∈ A_t do
        RiskStates_e ← {Critical, At Risk}          ▷ Define risk states
        overlap ← |F_t(e) ∩ RiskStates_e|           ▷ Count overlapping states
        S_t ← S_t + w(e) · overlap                  ▷ Aggregate weighted score
    end for
    if S_t > θ_2 then
        class ← "High Risk"
    else if S_t > θ_1 then
        class ← "Medium Risk"
    else
        class ← "Low Risk"
    end if
    RiskScore.append((t, class))                    ▷ Record classification
end for
return RiskScore
```

Figure 2 presents a visual flowchart for Algorithm 1, illustrating the control flow and decision points in a professional format suitable for publication.

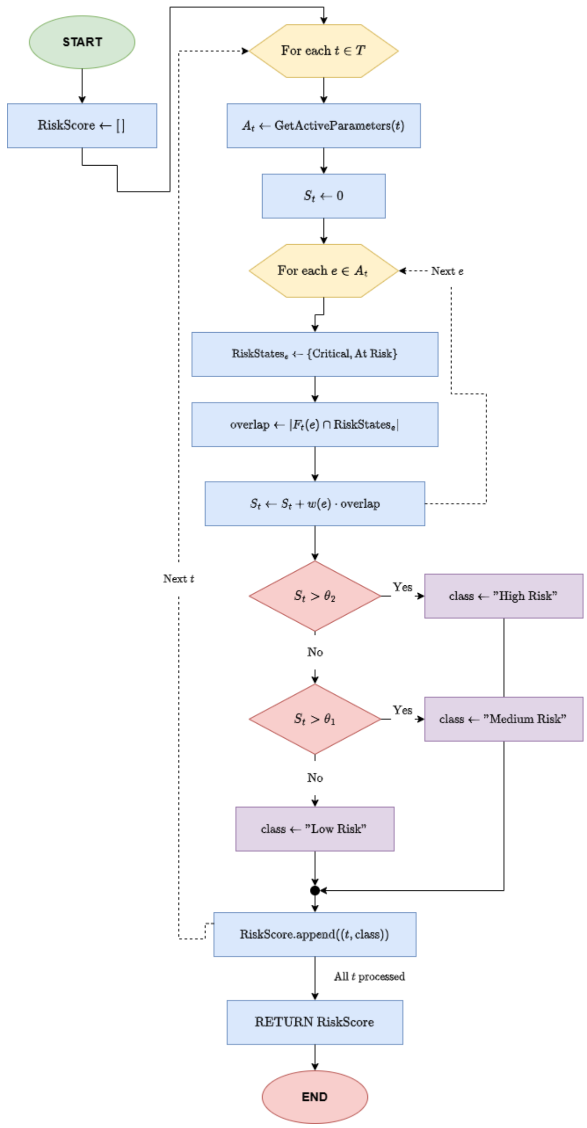

FIGURE 2. flowchart of Algorithm 1 Dynamic soft set based risk assessment over time

Example 5.1.1. *Consider the healthcare monitoring scenario from Section 3. Let*

$$U = \{Normal, At\ Risk, Critical\}, \qquad E = \{BP, HR, SpO_2\},$$
$$T = \{Morning, Evening\}, \qquad \theta_1 = 1.0, \quad \theta_2 = 2.0,$$

with uniform weights $w(e) = 1$ for all $e \in E$. At the Morning time point,

$$A_{Morning} = \{BP, HR, SpO_2\}, \qquad F_{Morning}(BP) = \{Normal, At\ Risk\},$$
$$F_{Morning}(HR) = \{Normal\}, \qquad F_{Morning}(SpO_2) = \{Normal, At\ Risk\}.$$

Applying Algorithm 1, we obtain

$$\begin{aligned} overlap_{BP} &= \left|F_{Morning}(BP) \cap \{Critical, At\ Risk\}\right| = 1, \\ overlap_{HR} &= \left|F_{Morning}(HR) \cap \{Critical, At\ Risk\}\right| = 0, \\ overlap_{SpO_2} &= \left|F_{Morning}(SpO_2) \cap \{Critical, At\ Risk\}\right| = 1, \\ S_{Morning} &= 1 \cdot 1 + 1 \cdot 0 + 1 \cdot 1 = 2. \end{aligned}$$

Since $S_{Morning} = 2 = \theta_2$, we classify the patient at Morning as borderline High Risk (or Medium Risk if using $S > \theta_2$ strictly). Similar calculations at Evening yield the complete RiskScore sequence, capturing how patient risk evolves dynamically.

Remark 5.1.1. ***(Complexity Analysis):*** *The time complexity of Algorithm 1 is*

$$\mathcal{O}\left(|T| \cdot |A_{\max}| \cdot \log |U|\right),$$

where $|A_{\max}| = \max_{t \in T} |A_t|$ is the maximum number of active parameters at any time point, and $\log |U|$ accounts for set intersection operations. The space complexity is $\mathcal{O}(|T| + |U|)$ for storing the input Dynamic Soft Set and output classifications. For moderate problem instances (e.g., $|T| \approx 100$, $|A_{\max}| \approx 20$, $|U| \approx 50$), execution is instantaneous on modern hardware.

5.2. **Implementation Considerations.** The Algorithm 1 nature, as different computations at different time indices A1 are independent of each other, allows immediate parallelization across multi-core platforms or distributed computing systems, allowing it to be scaled out.

(1) **Data Representation**: The DSS $\mathcal{S}$ is to be implemented with a sparse data format, e.g., nested hash maps, whereby every time index t maps to an active parameter set A_t, and parameter $e \in A_t$ maps to the subset of U it relates to. This model reduces the consumption of memory and eliminates extraneous dense models.
(2) **Weight Calibration**: The weights of the parameters of $w(e)$ must be domain-specific and can be obtained either by a process of expert elicitation or by data-driven learning based on historical data. A time-dependent or adaptive weighting mechanism can also contribute greatly to responsiveness in changing systems.
(3) **Threshold Determination**: Principled methods, including cross-validation, ROC analysis or consensus-based domain benchmarks, should be used to choose decision thresholds θ_1 and θ_2. Sensitivity analysis is to be used as robustness checks to determine stability when the threshold is varied.
(4) **Streaming and Incremental Updates**: In continuous or live time, Algorithm 1 can be extended to work for incremental updates of A1 and A1, with the arrival of new observations, preventing one from calculating A_t and F_t all the way back through the time horizon.

(5) **Parallel and Distributed Processing**: The algorithm's nature, as different computations at different time indices $t \in T$ are independent of each other, allows immediate parallelization across multi-core platforms or distributed computing systems, allowing it to be scaled out.

5.3. **Generalizations and Extensions.** It is possible to consider an Algorithm 1 as a specific example of a wider set of decision-making models that are dynamic and, at the same time, reflect the uncertainty. In the same formal framework, a number of meaningful extensions can be constructed:

- **Multi-Objective Decision Modeling**: The scalar risk measure S_t can be extended to a performance indicators vector so that MCDM methods of weighted aggregation, TOPSIS, or PROMETHEE could be used to rank or select tasks.
- **Fuzzy and Neutrosophic Generalizations**: The crisp approximations of $F_t(e)$ can be substituted with fuzzy membership functions of $\mu_t(e) : U \to [0, 1]$ or neutrosophic representations where the corresponding changes are done in the calculation of the aggregation and overlap.
- **Data-Driven Parameter Learning**: The supervised or semi-supervised learning methods of labeled historical data can be used to learn weight functions $w(e)$ and decision thresholds θ_1, θ_2, in order to enable the model to adapt to the empirical trends and to changing environments.
- **Temporal Dependency Modeling**: Aggregating scores of risk over successive time points by slipping-window averages, exponential smoothing, or by temporal operators that rely on recurrence can introduce temporal coherence.

These extensions indicate that the suggested framework can not be confined to the context of any fixed or purely theoretical context but offers a versatile base on which adaptive decision-support systems managing time-dependent uncertainty in the complex real-life scopes can be constructed.

6. Results and Discussion

This part summarizes the key results of the suggested DSS and DHSS models and explains their theoretical and practical importance. The results are presented in terms of structural properties, modelling capabilities, and computational characteristics as opposed to numerical experiments, owing to the fact that the study is more theoretical than experimental.

6.1. **Theoretical Implications.** The emergence of DSS and DHSS is what makes a single framework that incorporates both temporal variation and parameterized uncertainty modelling. The proposed structures have evolving uncertainty compared to the static soft, fuzzy or rough set models by letting the active parameter subsets A_t and the corresponding approximation mappings F_t be time or contextual index dependent. This definition allows the model to capture both information value changes and the changes in the relevance and

availability of the parameters.
One important theoretical finding is that the family of DSSs is in fact algebraically closed with respect to the basic set-theoretic and logical operators provided above, such as union, intersection, complement, AND, and OR operators. This closure property implies that combinations and compositions of dynamic operations are in the same mathematical system, thus making the reasoning contribution to time-dependent contexts uniform and rigorous.The structure of the framework provokes the following hierarchy of generalizations:

$$\text{Classical sets} \subset \text{Soft sets} \subset \text{Dynamic soft sets},$$

together with the parallel extension

$$\text{Soft sets} \subset \text{Hypersoft sets} \subset \text{Dynamic hypersoft sets}.$$

This hierarchy clarifies how temporal indexing and multi-attribute parameterization are progressively incorporated. Within this structure, DSS and DHSS serve as encompassing models that extend classical soft and hypersoft theories while preserving their foundational algebraic principles.

6.2. **Practical Implications.** The dynamic framework can be applied to a large variety of systems where the attributes and information sources that are relevant are dynamic. Time-indexed parameter sets can be a natural way to represent alterations in sensor presence, machine conditions, and quality-control parameters between production cycles in smart manufacturing settings to support adaptive monitoring and fault diagnosis.
When it comes to the application in the context of the environment and climate, the applicability of parameters like the type of pollutant, area of monitoring or even the variable of meteorology may be conditional on the seasonal conditions, extreme events, or priorities of the regulations. Such variability can be directly added to the uncertainty model by a dynamic parameter space indexed by time or scenario.
The same benefits could be said in other areas. In the context of cybersecurity, such indicators as traffic patterns, intrusion signatures, and system logs are dependent on the context of operation and the level of threat, and so context-dependent modelling is required. In supply chain management, the properties of relevance vary between the normal, stressed and disrupted situations and a dynamic soft structure helps in resilient decision-making in the face of uncertainty. Similarly, customized and accurate healthcare demands modeling of patient-specific factors that change throughout disease progression and treatment protocol, an environment which is inherently suited by both DSS and DHSS.

6.3. **Relation to Alternative Approaches.** A number of the approaches that exist deal with temporal uncertainty that focuses on the modelling assumption. The temporal fuzzy set models normally permit membership values to change over time with a constant set of parameters. Instead, DSS allows sets of parameters A_t, as well as the approximation mappings F_t to also evolve, capturing not only the magnitude of the uncertainty but also the relevance of the attributes.
Temporal rough set models accommodate changing granulations over time by using relations that depend on time but fail to explicitly parameterize uncertainty using families of attributes. Such approaches are complemented by the dynamic soft set framework that

offers a parameter-focused description. Classical time series approaches are based on numerical data and fixed characteristics, whilst DSS are based on qualitative and set-valued uncertainty, hence being appropriate when information is incomplete, heterogeneous, or linguistically characterized. Probabilistic models of time need accurate specifications of probabilities and structure, whereas DSS and DHSS provide a non-probabilistic alternative, which can be used in conjunction with probabilistic approaches where quantitative data is present.

6.4. **Computational Feasibility and Scalability.** Computationally, a DSS representation of a universe U, parameter set E and time index set T requires the storage of a family of soft sets (F_t, A_t) indexed by $t \in T$. The worst case is that the memory requirement increases on the order of

$$O\big(|T| \cdot |A_{\max}| \cdot |U|\big),$$

where $|A_{\max}| = \max_{t\in T} |A_t|$. In the case of moderate problem sizes, such a requirement is still manageable (assuming that sparse representations are used).

The elementary algorithms specified above can be executed with time scales that are linear in the size of the time index set and the size of the active parameter sets. As the operations at different time points are independent, the structure inherently provides incremental processing, window-based processing and parallel processing. Despite the fact that DSS and DHSS add complexity to the existing framework of the static models, this complexity is dictated by the dynamics of the information systems and can be computationally addressed based on proper data structures and algorithms.

7. Conclusion and Future Work

The frameworks of the DSS and DHSS have been introduced in this work as principled extensions of classical soft and hypersoft set theories to model the uncertainty in the time-varying and context-dependent environments. Through the possibility to evolve the active parameter subsets $A_t \subseteq E$ and the corresponding approximation mappings F_t through time, the proposed models address the limitation of the traditional approach of parameterization being inherent in the static nature of the structure.

It has been formally defined and shown to possess a complete set of set-theoretic and logical operators such as union, intersection, complement, AND/OR and restricted and extended versions, satisfying well-known algebraic axioms such as idempotency, absorption, monotonicity, boundedness and compatibility with reduction. These findings make DSS and DHSS algebraically closed and structurally consistent systems, appropriate to perform rigorous arguments in dynamic contexts. Moreover, the suggested algorithm of decision-making proves that the framework is computationally tractable, where linear complexity with the number of time indices, parameters, and elements of the universe can be used in real-time and streaming applications. The conceptual case illustrations also reveal that DSS can model changing parameter dynamics in areas like healthcare tracking and finance systems.

Limitations. Although the current framework is theoretically very general and adaptable, it has some limitations that can drive further research:

- The formulation used is mostly discrete in time, which might not adequately represent smoothly varying phenomena with time.
- Set membership is handled crisply and does not give expressiveness in situations with graded, ambiguous, or contradictory information.
- The algorithm that decides the weights and thresholds is externally defined, and it can therefore be subjective without data-driven calibration.
- Empirical validation is restricted to illustrative cases as opposed to massive real-life implementations.

Future Research Directions. Some of the avenues of prospective research arise naturally out of this work:

- **Fuzzy and Neutrosophic Extensions:** The graded and multi-valued uncertainty Gradual expansion of the DSS and DHSS into fuzzy, intuitionistic, spherical fuzzy, rough and neutrosophic with graded and multi-valued uncertainty into the dynamism of the parameter structure.
- **Continuous-Time Models:** Generalizing the discrete time index T to continuous spaces so as to facilitate smooth time evolution and the analysis of differentiation and integration of dynamic soft sets.
- **Data-Driven and Learning-Based Integration:** Integrating DSS and machine learning to learn automatically the relevance of parameters, weights and thresholds of streams of time-dependent data.
- **Empirical Evaluation:** Using DSS-based decision models on real-world data, e.g., intensive care monitoring or high-frequency financial data, to compare the performance of these models to that of other temporal models that are statistical or otherwise.
- **Topological and Measure-Theoretic Analysis:** Expanding the mathematical foundations of the dynamic soft set spaces by investigating topological structures, closure operators, compactness and measure-theoretic properties of the soft set space.

Generally, DSSs and DHSSs are a major advance to a fully temporalized soft set theory. The proposed framework integrates parameter dynamics with the modelling of uncertainties, which makes it a more generalized base of future research and practical decision-support systems that would apply in changing and uncertain environments.

CRediT Authorship Contribution's Statement

The author solely contributed to the conception, methodology, analysis, and writing of the manuscript. The author has read and approved the final version of the manuscript.

Declarations

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