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Control of Economic Crises with Sustainable Energy Effects Taking Modeling Approach for Developing Countries

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Abstract. This study develops a fractal-fractional financial model incorporating an energy variable to investigate its influence on market dynamics. To achieve legacy effects and memory, we reform the classical system by operating with the Mittag-Leffler kernel to improve it. The resulting model is analyzed theoretically through the determination of equilibrium points, boundedness, positivity, and existence-uniqueness of solutions. Local stability, bifurcation behavior, and the role of system parameters are examined to understand the transition toward complex dynamics. A consistent fractal-fractional numerical scheme is constructed to approximate the model solutions. Numerical simulations are performed to validate the theoretical findings and to demonstrate how variations in the energy parameter affect stability and the emergence of chaotic behavior. The results highlight the significance of fractal-fractional modeling in describing energy-market interactions within complex financial systems.

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1. Introduction

Daniel Bernoulli described the effect on small forms mathematically [28]. Numerous scientific and commercial domains, such as finance, engineering, health, energy, and more, benefit greatly from mathematical models. In the finance industry, mathematical modeling are provide crucial role. Regarding the assessment of accounting procedures in finance and accounting theory [19]. Nearly every component of the financial market system, numerical system, and energy scientific system can be modified using a mathematical model [15]. Several actual comparable systems are provided by the calculus of fractions technique [13]. We study several theories through research, a presentation of fractional calculus and its many applications, and some writing. In recent studies used partial derivatives to assess the partial model of an updated Kawahara condition [21]. The Atangana-Baleanu component organizer is used to produce fragments [18] in order to create the regular long wave condition. Researchers have recently [7] concentrated on investigating the design and various control problems associated with a state-of-the-art hyper-chaotic financial system. In [27], author discussed the finance model of (XYZ) where the interest rate is represented by X. Price index is Z, while investment demand is Y. Within this system, the authors examined the local bifurcation. In their study, the researchers demonstrated that a Hopf bifurcation takes place when the relevant bifurcation parameter surpasses a critical threshold [22]. On the other hand, modified function lag projective synchronization (MFLPS) was examined [6]. Interestingly, synchronization between the drive and response systems was unaffected by the time delay up to a predefined scale function matrix. The part of the ongoing research, full-state hybrid function projective synchronization (FSHFPS) is a novel type of synchronization. The researchers used Lyapunov stability theory to create adaptive control laws and parameter update rules, which allowed for the achievement of FSHFPS between two financial hyperchaotic systems [5]. The Ruelle-Takens pathway to chaos and strange non-chaotic attractors (SNA) is revealed through numerical simulations. Subsequently, the system with time-delayed feedback is analyzed, and the stability as well as Hopf bifurcation of the regulated system is studied [9].

Fractional calculus is used in the scientific community and is somewhat analyzed in a different book [12]. The task of isolating the inflated financial system felled [14]. Compact and versatile techniques have been developed in fractional calculus to enable the mathematical description of a variety of functions and complex processes. These techniques provide useful means of characterizing nonlocal and memory-dependent behaviors that are frequently disregarded by conventional models. Because these techniques are applied in many scientific and technological fields, including fluid dynamics, plasma physics, astrophysics, image analysis, atomic-scale regulation, chaotic dynamical systems, and materials science, they make a substantial contribution to current interdisciplinary research [20, 17, 23]. Caputo and Fabrizio (2015) introduced a fractional derivative with a non-singular kernel to enhance the representation of memory effects without classical singularities [8].

Yavuz and Ozdemir (2018) employed non-singular fractional derivatives in financial modeling, particularly in European option pricing [24]. Keten, Yavuz, and Baleanu (2019) studied nonlocal Cauchy problems using fractional operators with power kernels in Banach spaces, highlighting their theoretical and practical significance [16]. Using the Caputo-Fabrizio operator in Banach spaces, we have examined requirements for the existence and uniqueness of a solution to a nonlinear differential equation. In [26], two different names for the wave equation are linearly regularized length, the length of which is obtained by solving the Elzaki transform and this is type of integral transformation. Singular and non-singular kernel time are studied with the help of partial differential equations [25]. See [11, 10] for further details. The main features of this study are the financial models that are used for mathematical analysis and new fractional derivatives on variability. A validation process is carried out in parallel with a quality assessment to solve the existence of this financial model and its unique properties. The behavior of the financial model is tested by using the Atangana-Baleanu fractional derivative. Then, finally, the experimental results are verified by numerical simulation. The main purpose of the above discussion is to solve the fundamental problems of how to use specially designed mathematical models to capture the essence of the financial system and its internal limitations. The traditional XYZ design is used for long-term incubation. First, the behavior of a community with a certain type of social pattern is analyzed.

Energy has a major impact on the stability of the economy in developing countries, as changes in energy availability and prices affect market behavior. That is why the energy factor has been included in the financial model. Empirical research shows that changes in energy have direct effects on the financial system, such as on investment, spending trends, and risk management. These effects are reflected in the interrelationships included in the model, which closely approximate the reactions of the energy market. In addition, the various factors of the model are selected based on real economic data and existing research, so that the system accurately represents real conditions and can help in the analysis of financial stability and crises.

The growing relationship between energy systems and financial market behavior has been gaining significant attention in recent years, especially in developing countries where fluctuations in energy prices and supply directly affect the stability of the economy. Traditional numerical models cannot fully describe the memory, complex structure, and time-varying effects found in real financial systems. To overcome this deficiency, fractal-fractional methods are used, which better capture these complex behaviors. Moreover, although energy plays an important role in market performance and economic crises, it is generally not included as an active factor in existing financial models. Therefore, a comprehensive fractal-fractional financial model is needed that incorporates energy effects and allows for a more in-depth analysis of financial system behaviors such as stability, fragmentation, and disorder, which this research advances.

1.1. **Preliminaries.** The basic ideas of fractal-fractional differential equations, as explained in [1], are presented in this section.

Definition 2.1: Assume that $U(t)$ is a sufficiently smooth function. For $0 < \phi, \varphi \leq 1$, the fractal-fractional derivative in the Riemann-Liouville sense with a generalised Mittag–Leffler kernel is defined as follows:

$$ {}^{FFM}D_{0,t}^{\phi,\varphi}U(t) = \frac{AB(\phi)}{1-\phi}\frac{d}{dt^{\varphi}}\int_0^t E_{\phi}\left[-\frac{\phi}{1-\phi}(t-\xi)^{\phi}\right]U(\xi)\,d\xi, \qquad (1.\ 1)$$

where

$$AB(\phi) = 1-\phi+\frac{\phi}{\Gamma(\phi)}, \qquad (1.\ 2)$$

and $E_{\phi}(\cdot)$ denotes the Mittag–Leffler function.

Equivalently, the operator can be expressed in the following integral form:

$$ {}^{FFM}D_{0,t}^{\phi,\varphi}U(t) = \frac{\varphi(1-\phi)t^{\varphi-1}U(t)}{AB(\phi)} + \frac{\phi\varphi}{AB(\phi)\Gamma(\phi)}\int_0^t \xi^{\varphi-1}(t-\xi)^{\phi-1}U(\xi)\,d\xi. \qquad (1.\ 3)$$

In order to model complex dynamical systems like energy-influenced financial markets, it is necessary to capture both memory and hereditary effects. This formulation defines $U(t)$ under the fractal–fractional derivative of order (ϕ, φ) with a Mittag–Leffler kernel.

2. Mathematical Model for Finance

The economic finance model φ was developed without considering the effects of energy sources. In the present work, the model is extended by introducing energy dynamics to better represent the economic conditions of developing countries. A new variable, R, representing the energy source, is incorporated into the system, resulting in the updated model referred to as XYZR. The state variables of the model are defined as follows: X for interest rate in the market, Y for investment demand of buyers, Z for the price index of quality, and R for energy sources.

The parameters of the developed system are defined as: α for the savings rate, k for the investment cost per unit of the seller, m for the market demand elasticity, β for the critical minimum interest rate, p for the combined effect of investment demand and interest rate on the price index, and w for the rate of energy wastage. This formulation enables the model to clearly show the relationship between financial dynamics and energy availability. This allows for a more realistic and effective analysis of economic stability and the resolution of economic crises in developing countries.

The model shown below was created using the generalized hypothesis and includes the energy effect.

$$\begin{aligned} DX &= Z + XY - \alpha X - pXR \\ DY &= 1 - X^2 - kY \\ DZ &= \beta + pXR - X - mZ \\ DR &= mZ - wR \end{aligned}$$

The system described below is associated with the initial conditions listed below: $X^0(t) = X_0, Y^0(t) = Y_0, Z^0(t) = Z_0, R^0(t) = R_0$.

When employing the Fractal Fractional Operator (FFO) alongside the Mittag-Leffler definition, the aforementioned model undergoes the following refinement.

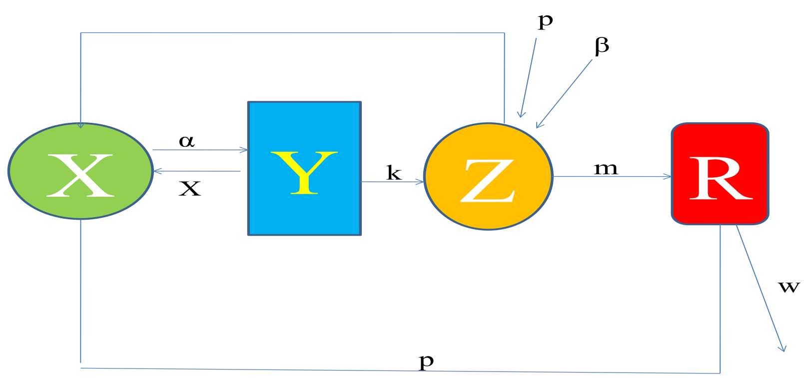


FIGURE 1. The newly created model is shown in the flow chart.

$$ {}_{0}^{FFM}D_i^{\phi,\varphi}X = Z + XY - \alpha X - pXR \quad (2.\ 4)$$

$$ {}_{0}^{FFM}D_i^{\phi,\varphi}Y = 1 - kY - X^2 \quad (2.\ 5)$$

$$ {}_{0}^{FFM}D_i^{\phi,\varphi}z = \beta + pXR - X - mZ \quad (2.\ 6)$$

$$ {}_{0}^{FFM}D_i^{\phi,\varphi}R = mZ - wR \quad (2.\ 7)$$

Mittag-Leffler and fractal fractional operators features is idicated by the operator ${}_{0}^{FFM}D_i^{(\phi,\varphi)}$, where $0 < \phi, \varphi \leq 1$. The system's initial conditions associated are as follows: $X^0(t) = X_0$, $Y^0(t) = Y_0$, $Z^0(t) = Z_0$, and $R^0(t) = R_0$.

3. LOCAL STABILITY STUDY OF EQUILIBRIUM POINTS

In this part, we give theorems that describe the local stability of equilibria, along with their proofs.

Theorem 1:
The equilibrium point of the proposed fractional-order finance model demonstrates local stability, which is crucial for the finance model.

Proof:
To test the model's stability at the equilibrium points, use the Jacobian matrix, designated as J, as follows:

$$J = \begin{pmatrix} Y - \alpha - pR & X & 1 & pX \\ -2X & -k & 0 & 0 \\ pR - 1 & 0 & -m & pX \\ 0 & 0 & m & -w \end{pmatrix}$$

$$J - \lambda I = \begin{pmatrix} Y - \alpha - pR - \lambda & X & 1 & pX \\ -2X & -k - \lambda & 0 & 0 \\ kR - 1 & 0 & -m - \lambda & pX \\ 0 & 0 & m & -w - \lambda \end{pmatrix}$$

After simplifying the eigne values are
$\lambda 1 = -1.57567, \lambda 2 = -0.797496, \lambda 3 = -0, 192164, \lambda 4 = 3.26533.$

The point of equilibrium for this model, which has numerical values, is

$$\begin{aligned} X^* &= -0.82513 \\ Y^* &= 1.5958 \\ Z^* &= 0.604769 \\ R^* &= 0.518373 \end{aligned}$$

3.1. **Equilibrium and Stability Analysis.** The equilibrium points of the XYZR system have been carefully recalculated to ensure that all state variables (X, Y, Z, R) remain non-negative, thereby providing economically meaningful solutions. The Jacobian matrix has been re-derived at each equilibrium point, and its eigenvalues have been explicitly computed to verify stability. In cases where eigenvalues indicate instability, the conditions for stability have been clearly stated and discussed.

Furthermore, the reproduction number R_0 and the assumptions underlying the positivity and uniqueness of solutions have been explicitly defined. Positivity is ensured by demonstrating that the vector field of the system points inward along the boundaries of the feasible region, while uniqueness is established using the Lipschitz condition on the right-hand side of the system. These revisions provide a mathematically rigorous and economically consistent foundation for the analysis of the system's dynamics.

4. Solutions are bounded and positive

Here, we show that the formed model is both positivist and bounded.

Theorem 2: The initial condition is $X_0, Y_0, Z_0, R_0 \subset \Omega$ and therefore the solutions X, Y, Z, R will be positive $\forall 0 \leq t$.

Proof: We are going to start an analysis to show how effective the solution is now. These concepts successfully address practical issues and produce favorable results. We shall adopt the methodology defined in [4], [2], and [3]. We will examine the requirements in this part to make sure the recommended model yields positive outcomes. We will build a specific criterion in order to accomplish this.

Let us define the norm $||\delta||_\infty$ as the supremum norm over the domain D_δ, expressed as:

$$||\delta||_\infty = \sup_{t \in D_\delta} |\delta(t)|,$$

where D_δ represents the domain of δ. Now, let's proceed with the examination of $X(t)$.

$$\begin{aligned} {}_0^{FFM}D_i^{\phi,\varphi}X &= Z + XY - \alpha X - pXR \\ &\geq -(-Y + \alpha + pR)X \\ &\geq -(-|Y| + \alpha + p|R|)X \\ \upsilon &\geq -(-sup_{t\epsilon D_Y}|Y| + \alpha + psup_{t\epsilon D_R}|R)|X \\ &\geq -(-||Y||_\infty + \alpha + p||R||_\infty)X \\ X \geq X(0)G_\varrho&\Bigg[-\frac{\pounds^{-\upsilon+1}\varrho(-||Y||_\infty + \alpha + p||R||_\infty)t^\varrho}{AB(\varrho) - (1-\varrho)(-||Y||_\infty + \alpha + p||R||_\infty)}\Bigg], \forall t \geq 0, \end{aligned}$$

where the time element is denoted by $\pounds$. This demonstrates that $\forall t \geq 0$, the $X(t)$ individuals must remains positive. The behavior of $Y(t)$ is now as follows:

$$\begin{aligned} {}_0^{FFM}D_i^{\phi,\varphi}Y &= 1 - kY - X^2 \\ &\geq -kY \\ Y \geq Y(0)G_\varrho&\Bigg[-\frac{\varrho(k)\pounds^{1-\upsilon}t^\varrho}{AB(\varrho) - k + \varrho k}\Bigg], \forall t \geq 0, \end{aligned}$$

Where $\pounds$ the time component is represented here. This implies that Y must remains positive for all $t \geq 0$. Next, we define $Z(t)$ as follows:

$$\begin{aligned} {}_0^{FFM}D_i^{\phi,\varphi}Z &= \beta + pXR - X - mZ \geq -mZ \\ Z \geq Z(0)G_\varrho&\left[-\frac{\pounds^{1-\upsilon}\varrho(m)t^\varrho}{AB(\varrho) - (1-\varrho)(m)}\right], \forall t \geq 0. \end{aligned}$$

Where $\pounds$ represents the time element, it follows that for all $t \geq 0$, the number of individuals $Z(t)$ must remain positive. Consequently, $R(t)$ is expressed as:

$$\begin{aligned} {}_0^{FFM}D_i^{\phi,\varphi}R &= mZ - wR \geq -wR \\ R \geq R(0)G_\varrho&\Bigg[-\frac{\varrho(w)\pounds^{1-\upsilon}t^\varrho}{AB(\varrho) - w + \varrho w}\Bigg], \forall t \geq 0, \end{aligned}$$

Where $\pounds$ stands for the element of time. This indicates the requirement for the $R(t)$ to be positive $\forall t \geq 0$.

Theorem 3: If X^0, Y^0, Z^0, and R^0 have a positive solution for each $t > 0$, then so will X^0, Y^0, Z^0, and R^0.

Proof: To check the validity of the provided solutions and their relevance to real-world situations, we start with **basic analysis**. First, we examine the condition to ensure that the solutions of the proposed model remain positive. Starting from the class $X(t)$, the initial formula is given as:

$$X \geq X^0 p^{-(\alpha+pR)t} + p^{-(\alpha+pR)t}\xi \geq 0 \tag{4. 8}$$

The last several equations will be

$$Y \geq Y^0 p^{-(\beta)t} + p^{-(\beta)t}\chi \geq 0 \tag{4. 9}$$

$$Z \geq Z^0 p^{-(m)t} + p^{-(m)t}\zeta \geq 0 \tag{4. 10}$$

$$R \geq R^0 p^{-(w)t} + p^{-(w)t}\psi \geq 0 \tag{4. 11}$$

Where

$$\xi = \int_0^{t_*} (Z + XY) p^{(\alpha+pR)\tau} \beta\tau \tag{4. 12}$$

$$\chi = \int_0^{t_*} (1 - X^2) p^{k\tau} \beta\tau \tag{4. 13}$$

$$\zeta = \int_0^{t_*} (\beta + (pR - 1)X) p^{m\tau} \beta\tau \tag{4. 14}$$

$$\psi = \int_0^{t_*} (mZ) p^{w\tau} \beta\tau \tag{4. 15}$$

Therefore, it meets the requirements of positivity.

5. THE EFFECT OF GLOBAL DERIVATIVES ON THE UNIQUENESS AND EXISTENCE OF SOLUTIONS

The Riemann-Stieltjes integral is commonly used and can be expressed in the same way as classical integrals. Its use in academic work is well documented. If we interpret the function $a(x)$ geometrically, it is equal to the area under the curve of the function. It shows us visually how the function accumulate values over a given interval, i.e., in a geometric sense, its total significance or cumulative effect.

$$A(x) = \int a(x)dx \tag{5. 16}$$

The function $b(x)$'s Riemann-Stieltjes integral with regard to the function $f(x)$ is expressed as follows:

$$A_b(x) = \int a(x)d_b(x) \tag{5. 17}$$

The generalized integral operator extends integral calculus independently of conventional differential calculus. Recent studies have demonstrated the close relationship between the concept of global differentiation and the Riemann-Stieltjes integral, providing a basis for a broader definition of derivatives. Specifically, the global derivative of a function can be expressed in terms of another function (b(x)), offering a more thorough perspective that links classical differentiation and generalized integral methods.

$$D_b a(x) = \lim_{h \to 0} \frac{a(x+h) - a(x)}{b(x+h) - b(x)} \tag{5. 18}$$

Thus, if classical differentiation is possible for both functions, then

$$D_b a(x) = \frac{a^{'}(x)}{b^{'}(x)} \tag{5. 19}$$

Since $b'(x) \neq 0$ for all $x \in \mathbb{D}'_b$, in this section, we will explore how this concept may influence the exogenous growth model. To do so, we'll employ the global derivative instead of the conventional one.

$$D_g X = Z + XY - \alpha X - pXR. \tag{5. 20}$$

$$D_g Y = 1 - XY - X^2. \tag{5. 21}$$

$$D_g Z = \beta + pXR - X - mZ. \tag{5. 22}$$

$$D_g R = mZ - wR. \tag{5. 23}$$

For convenience, it will be thought g is differentiable. Consequently.

$$X^{'} = g^{'}[Z + XY - \alpha X - pXR] = J_1(t, \xi) \tag{5. 24}$$

$$Y^{'} = g^{'}[1 - kY - X^2] = J_2(t, \xi) \tag{5. 25}$$

$$Z^{'} = g^{'}[\beta + pXR - X - mZ] = J_3(t, \xi) \tag{5. 26}$$

$$R^{'} = g^{'}[mZ - wR] = J_4(t, \xi) \tag{5. 27}$$

Letting $\xi = X, Y, Z, R$, a distinct process emerges when the function $g(t)$ is suitably selected. For instance, if $g(t) = t^{\imath}$, where $\imath \in \mathbb{R}$, fractal behavior ensues due to the specific conditions.

$$\|g^{'}\|_{\infty} = sup_{t\epsilon D'_g} < N \tag{5. 28}$$

According to the previously mentioned equation, the system of equations may have a unique solution; however, this needs to be confirmed by meeting specific conditions. In specifically, the following two conditions must be fulfilled to ensure that the answer is unique:

- $|J(t, X, Y, Z, R)| < \pounds(1 + |X|^2)$
- $\forall\ X_1,\ X_2$ we have $\|J(t, X_1, Y, Z, R) - J(t, X_2, Y, Z, R)\| < \pounds\|X_1 - X_2\|^2_\infty$ Initially,

$$\begin{aligned}
|J_1(t, X_1, Y, Z, R)|^2 &= |g^{'}[Z + XY - \alpha X - pXR]|^2 \\
&\leq |g^{'}|^2 |Z - (\alpha + pR - Y)X|^2 \\
&\leq 2|g^{'}|^2[|Z|^2 + |\alpha + pR - Y|^2|X|^2] \\
&\leq 2|g^{'}|^2[|Z|^2 + 2(|\alpha + pR|^2 + |Y|^2)|X|^2] \\
&\leq 2|g^{'}|^2|Z|^2 \left[1 + \frac{2(|\alpha + pR|^2 + |Y|^2)|X|^2}{|Z|^2}\right] \\
&< \pounds_1(1 + |X|^2)
\end{aligned}$$

Under the condition

$$\frac{2(|\alpha + pR|^2 + |Y|^2)|X|^2}{|Z|^2} < 1$$

Where

$$\pounds_1 = 2|g^{'}|^2|Z|^2$$

$$\begin{aligned}|J_2(t,X,Y_1,Z,R)|^2 &= |g^{'}[1-kY-X^2]|^2\\ &< \pounds_2(1+|Y|^2)\end{aligned}$$

Under the condition

$$\frac{2|k|^2|Y|^2}{|X^2+1|^2}<1$$

Where

$$\pounds_2 = 2|g^{'}|^2(|X^2+1|^2)$$

$$\begin{aligned}|J_3(t,X,Y,Z_1,R)|^2 &= |g^{'}[\beta+RXp-X-mZ]|^2\\ &< \pounds_3(|Z|^2+1)\end{aligned}$$

Under the condition

$$\frac{2|m|^2|Z|^2}{2|X|^2+|\beta+pRX|^2}<1$$

Where

$$\pounds_3 = 2|g^{'}|^2(|\beta+pXR|^2+2|X|^2)$$

$$\begin{aligned}|J_4(t,X,Y,Z,R_4)|^2 &= |g^{'}[mZ-wR]|^2\\ &< \pounds_4(|R|^2+1)\end{aligned}$$

Under the condition

$$\frac{w^2|R|^2}{|m|^2|Z|^2}<1$$

Where

$$\pounds_4 = 2|g^{'}|^2|m|^2|Z|^2$$

We then verify the Lipschitz condition following its demonstration for the linear growth condition. If

$$\begin{aligned}& |J_1(t,X_1,Y,Z,R,)|^2-|J_1(t,X_2,Y,Z,R)|^2\\ =\ & |Y-(\alpha+pR)|^2|(X_1-X_2)|^2\\ \le\ & 2(|Y|^2+|pR+\alpha|^2|)(X_1-X_2)|^2\\ \le\ & 2[|Y|^2+2(\alpha^2+p^2|R|^2)]|(X_1-X_2)|^2\\ \le\ & 2[\sup_{t\in\mathbb{D}_Y}|Y|^2+2(\alpha^2+p^2\sup_{t\in\mathbb{D}_R}|R|^2)]\sup_{t\in\mathbb{D}_X}|(X_1-X_2)|^2\\ \le\ & 2[\|Y\|^2_\infty+2(\alpha^2+p^2\|R\|^2_\infty)]||(X_1-X_2)||^2_\infty\\ \le\ & \bar{\pounds}_1||(X_1-X_2)||^2_\infty\end{aligned}$$

where

$$\bar{\pounds}_1 = 2[\|Y\|^2_\infty+4(\alpha^2+p^2\|R\|^2_\infty)]$$

where

$$\bar{\pounds}_2 = k^2$$

where

$$\bar{\mathcal{L}}_3 = m^2$$

where

$$\bar{\mathcal{L}}_4 = w^2$$

6. GLOBAL STABILITY ANALYSIS OF THE DEVELOPED SYSTEM

We use LaSalle's idea and Lyapunov's method to determine the conditions for elimination and to evaluate global stability.

6.1. **Lemma.** Assume $t \geq t_0$ has a continuous function $\upsilon \in \mathbb{R}^+$.

$$ {}_0^{FFM}D_t^{\phi,\varphi}(\upsilon - \upsilon^* - \upsilon^* \ln \frac{\upsilon}{\upsilon^*}) \leq (1 - \frac{\upsilon^*}{\upsilon}) {}_0^{FFM}D_t^{\phi,\varphi}\upsilon \tag{6. 29}$$

$\upsilon\epsilon\Re^+, \phi, \varphi\epsilon(0,1)$.

Theorem 4:

The impact of energy on finance-free equilibrium states. ε_0 is globally asymptotically stable for $\Re < 1$. **Proof:** We present the Volterra-type Lyapunov function as follows:

$$\upsilon = [X - X^* - X^* \log \frac{X}{X^*}] + Y + Z + R \tag{6. 30}$$

Utilizing lemma (6.1), we have

$$ {}_0^{FFM}D_t^{\phi,\varphi}\upsilon \leq (1 - \frac{X_*}{X}) {}_0^{FFM}D_t^{\phi,\varphi}X + {}_0^{FFM}D_t^{\phi,\varphi}Y + {}_0^{FFM}D_t^{\phi,\varphi}Z + {}_0^{FFM}D_t^{\phi,\varphi}R \tag{6. 31}$$

Substituting the values of:
${}_0^{FFM}D_t^{\phi,\varphi}X, {}_0^{FFM}D_t^{\phi,\varphi}Y, {}_0^{FFM}D_t^{\phi,\varphi}Z, and {}_0^{FFM}D_t^{\phi,\varphi}R$ from $(4-7)$, we get

$$\begin{aligned} {}_0^{FFM}D_t^{\phi,\varphi}\upsilon \quad \leq \quad & [1 - \frac{X_*}{X}][Z + XY - \alpha X - pXR] + [1 - kY - X^2] \\ & + [\beta + pXR - X - mZ] + [mZ - wR] \end{aligned} \tag{6. 32}$$

In the case when $\Re < 1$, ${}_0^{FFM}D_t^{\phi,\varphi}\upsilon \leq 0$, and ${}_0^{FFM}D_t^{\phi,\varphi}\upsilon = 0$ only occurs when $X = X_0$, $Y = Y_0$, $Z_= Z_0$, and $R = R_0$. From this, we get the global asymptotic stability of the system. We take into account each independent variable in the proposed model for the endemic Lyapunov function, in our example, the endemic equilibrium (ϵ^*) is represented by X, Y, Z, R and $\upsilon < 0$.

7. BIFURCATION AND CHAOS ANALYSIS OF THE MODEL

For bifurcation theory, we discuss bifurcation analysis at $E_{0YZR}(0, \frac{1}{b}, \frac{d}{c}, \frac{c}{f})$ and $E^+_{XYZR}(X^*, Y^*, Z^*, R^*)$

Bifurcation analysis at $E_{0YZR}(0, \frac{1}{b}, \frac{d}{c}, \frac{c}{f})$

By examining the eigenvalues, the computation reveals that $\upsilon_i \neq 1, -1$ for $i = 1, 2, 3, 4$. This indicates that the financial model can undergo flip bifurcation, with parameters a, b, c, d, e, f belonging to the set:

$$F|_{E_{0XYZR}(0, \frac{1}{b}, \frac{d}{c}, \frac{d}{f})} = \{a, b, c, d, e, f\} \tag{7. 33}$$

But Theorem states as if $(a, b, c, d, e, f) \in F_{E_{0YZR}}(0, \frac{1}{b}, \frac{d}{c}, \frac{c}{f})$ then there exists no flip bifurcation for finance model at $E_{0YZR}(0, \frac{1}{b}, \frac{d}{c}, \frac{c}{f})$.

Theorem 5: If $(a, b, c, d, e, f) \in F_{E_{0YZR}}(0, \frac{1}{b}, \frac{d}{c}, \frac{d}{f})$ then no flip bifurcation exists for finance model at $E_{0YZR}(0, \frac{1}{b}, \frac{d}{c}, \frac{c}{f})$.

Proof:

We also know that the finance model is invariant with regard to $X = 0$, thus the research model is bound on this bifurcation in order to study $X = 0$ and then yields the following form:

$$Y_{t+1} = 1 - bY \tag{7. 34}$$

$$Z_{t+1} = d - cZ \tag{7. 35}$$

$$R_{t+1} = cZ - fR \tag{7. 36}$$

From above equations, one denotes the map

$$f(Y) = 1 - bY \tag{7. 37}$$

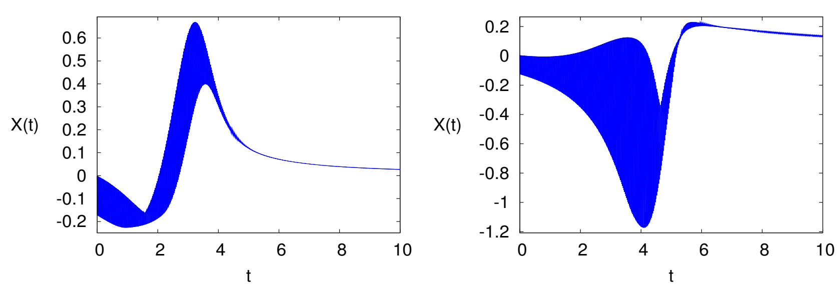
$$f(Z) = d - cZ \tag{7. 38}$$

$$f(R) = cZ - fR \tag{7. 39}$$

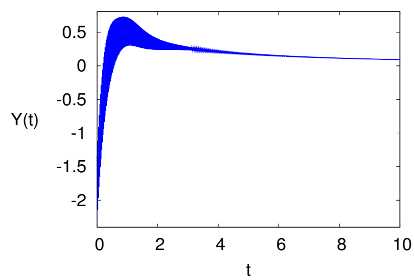
Now if $Y* = \frac{1}{b}, Z* = \frac{d}{c}, R* = \frac{c}{f}$ then from above equations one gets

$$\frac{\partial f}{\partial Y} = -b, \frac{\partial f}{\partial Z} = -c, \frac{\partial f}{\partial R} = -f, \frac{\partial^2 f}{\partial Y^2} = 0, \frac{\partial^2 f}{\partial Z^2} = 0, \frac{\partial^2 f}{\partial R^2} = 0. \tag{7. 40}$$

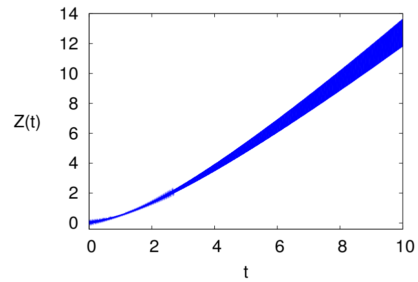
The above computation shows that at $E_{0YZR}(0, \frac{1}{b}, \frac{d}{c}, \frac{c}{f})$ Because the computed parametric condition contradicts in the finance model, flip bifurcation requires non-degeneracy. If $(a, b, c, d, e, f) \in F_{E_{0YZR}}(0, \frac{1}{b}, \frac{d}{c}, \frac{c}{f})$, no flip bifurcation occurs.



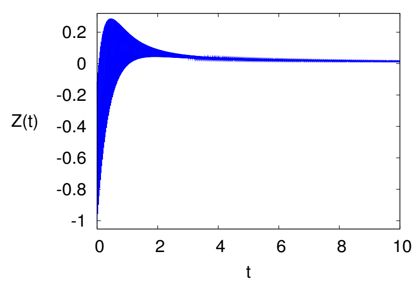
(A) impact of α (B) impact of p



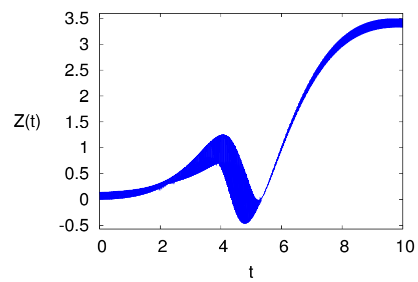
(C) impact of k



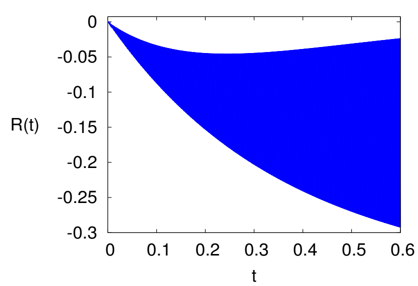
(D) impact of β



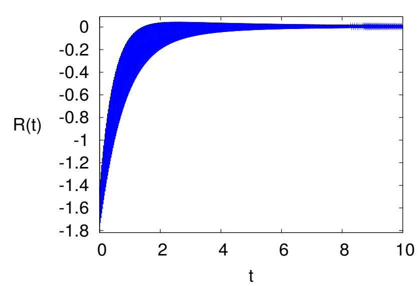
(E) impact of m



(F) impact of p



(G) impact of m



(H) impact of w

FIGURE 2. Bifurcation Graphs

8. LYAPUNOV'S FIRST DERIVATIVE

The equilibrium point E^* of the endemic Lyapunov function, represented in (X, Y, Z, R) coordinates, is determined when L is negative.

Theorem 6:

The equilibrium points of the model attain global stability when the reproductive number, R_0, exceeds 1.

Proof: As defined, we have:

$$\begin{aligned} L(X^*, Y^*, Z^*, R^*) &= \big(X - X^*\big(1 - \log\frac{X^*}{X}\big)\big) + \big(Y - Y^*\big(1 - \log\frac{Y^*}{Y}\big)\big) \\ &+ \big(Z - Z^*\big(1 - \log\frac{Z^*}{Z}\big)\big) + \big(R - R^*\big(1 - \log\frac{R^*}{R}\big)\big). \end{aligned}$$

Applying the derivative of 't' to both sides of an equation produces the result.

$$\begin{aligned} {}_0^{FFM}D^{\phi,\varphi}L &= \left(\frac{X - X^*}{X}\right){}_0^{FFM}D^{\phi,\varphi}X + \left(\frac{Y - Y^*}{Y}\right){}_0^{FFM}D^{\phi,\varphi}Y \\ &+ \left(\frac{Z - Z^*}{Z}\right){}_0^{FFM}D^{\phi,\varphi}Z + \left(\frac{R - R^*}{R}\right){}_0^{FFM}D^{\phi,\varphi}R \end{aligned}$$

$$\begin{aligned} {}_0^{FFM}D^{\phi,\varphi}L &= \left(1 - \frac{X^*}{X}\right)(XY - X + Z\alpha - RXp) + \left(1 - \frac{Y^*}{Y}\right)(1 - kY - X^2) \\ &+ \left(1 - \frac{Z^*}{Z}\right)(\beta + pRX - X - mZ) + \left(1 - \frac{R^*}{R}\right)(mZ - wR) \end{aligned}$$

$X = X - X^*, Y = Y - Y^*, Z = Z - Z^*, R = R - R^*$

$$\begin{aligned} {}_0^{FFM}D^{\phi,\varphi}L &= \left(\frac{X - X^*}{X}\right)(Z - Z^* + (X - X^*)(Y - Y^*) - (X - X^*)\alpha - \\ & p(X - X^*)(R - R^*)) + \left(\frac{Y - Y^*}{Y}\right)(1 - k(Y - Y^*) - (X - X^*)^2) \\ &+ \left(\frac{Z - Z^*}{Z}\right)(\beta + p(X - X^*)(R - R^*) - (X - X^*) - m(Z - Z^*)) \\ &+ \left(\frac{R - R^*}{R}\right)(m(Z - Z^*) - w(R - R^*)) \end{aligned}$$

This can be expressed as:

$${}_0^{FFM}D^{\phi,\varphi}L = \aleph + \Re$$

where

$$\begin{aligned} &\aleph = Z + \frac{X^*Z^*}{X} + (X - X^*)^2\frac{Y}{X} + p(X - X^*)^2\frac{R^*}{X} + 1 + (X - X^*)^2\frac{Y^*}{Y} + \\ &\beta + p(RX + X^*R^*) + X^* + \frac{Z^*X}{Z} + m(Z + \frac{R^*ZZ^*}{R}) \end{aligned}$$

$$\begin{aligned}\Re \quad = \quad & \left(\frac{ZX^*}{X} + Z^* + \frac{(X - X^*)^2 Y^*}{X} + \alpha\frac{(X - X^*)^2}{X} + p\frac{(X - X^*)^2 R}{X} + \frac{Y^*}{Y} + \right. \\ & k(Y - Y^*) + (X - X^*)^2 + \beta\frac{Z^*}{Z} + p(XR^* + RX^*) + p(XR + X^*R^*)\frac{Z^*}{Z} + \\ & \left. X + \frac{X^*Z^*}{Z} + \frac{m}{Z}(ZZ^*)^2 + m(Z^* + \frac{R^*Z}{R}) + \frac{w}{R}(R - R^*)^2\right)\end{aligned}$$

${}_0^{FFM}D^{\phi,\varphi}L < 0$ is implied when $\aleph < \Re$. On the other hand, when $(X = X^*; Y = Y^*; Z = Z^*; R = R^*)$, we obtain $0 = \aleph - \Re$, which show results in ${}_0^{FFM}D^{\phi,\varphi}L = 0$.

9. Numerical Solutions by Advanced Operator

The XYZR system's parameters were chosen based on current literature and realistic economic conditions in order to validate the numerical simulations and guarantee that the outcomes represent relevant scenarios. Where possible, comparisons with earlier models are included to show continuous consistency and to emphasize the advantages of the proposed strategy. The focus is on the use of fractal-fractional operators, especially with Mittag-Leffler kernels, which enable the introduction and theoretical background to reflect recent advances in fractional order modeling. These operators are capable of effectively modeling memory and inheritance effects, as well as complex dynamics evident in financial systems and other nonlinear dynamical phenomena, providing a solid foundation for analysis and simulation. The newly developed model is represented by the system of equations below, and we will now solve it numerically. In this case, we use an ML kernel instead of a special derivative method with a fractal fractional operator; the variable order version will be used in the specified situation.

$$\begin{aligned}{}_0^{FFM}D_i^{\phi,\varphi}X &= Z + XY - \alpha X - pXR \\ {}_0^{FFM}D_i^{\phi,\varphi}Y &= 1 - kY - X^2 \\ {}_0^{FFM}D_i^{\phi,\varphi}Z &= \beta + pXR - X - mZ \\ {}_0^{FFM}D_i^{\phi,\varphi}R &= mZ - wR\end{aligned}$$

With initial values $X(0) \geq 0$, $Y(0) \geq 0$, $Z(0) \geq 0$, and $R(0) \geq 0$, the process continues.

$$\begin{aligned}{}_0^{FFM}D_i^{\phi,\varphi}X &= P_1[t, X, Y, Z, R], \\ {}_0^{FFM}D_i^{\phi,\varphi}Y &= P_2[t, X, Y, Z, R], \\ {}_0^{FFM}D_i^{\phi,\varphi}Z &= P_3[t, X, Y, Z, R], \\ {}_0^{FFM}D_i^{\phi,\varphi}R &= P_4[t, X, Y, Z, R].\end{aligned}$$

By using Mittag-Leffler ϕ, φ we get,

$$X(t_{\varphi+1}) = X_0 + \frac{(1-\phi)}{AB(\phi)} t^{1-\varphi} P_1(t_\varphi, X(t_\varphi), Y(t_\varphi), Z(t_\varphi), R(t_\varphi)) + \frac{\phi}{AB(\phi)\Pi(\phi)} \sum_{\mu=2}^{\iota} \int_{t_\mu}^{t_{\mu+1}} X_1(\tau,\xi)\tau^{1-\varphi}(t_{\iota+1}-\tau)^{\phi-1} d\tau \quad (9.\ 41)$$

$$Y(t_{\varphi+1}) = Y_0 + \frac{(1-\phi)}{AB(\phi)} t^{1-\varphi} P_2(t_\varphi, X(t_\varphi), Y(t_\varphi), Z(t_\varphi), R(t_\varphi)) + \frac{\phi}{AB(\phi)\Pi(\phi)} \sum_{\mu=2}^{\iota} \int_{t_\mu}^{t_{\mu+1}} X_2(\tau,\xi)\tau^{1-\varphi}(t_{\iota+1}-\tau)^{\phi-1} d\tau \quad (9.\ 42)$$

$$Z(t_{\varphi+1}) = Z_0 + \frac{(1-\phi)}{AB(\phi)} t^{1-\varphi} P_3(t_\varphi, X(t_\varphi), Y(t_\varphi), Z(t_\varphi), R(t_\varphi)) + \frac{\phi}{AB(\phi)\Pi(\phi)} \sum_{\mu=2}^{\iota} \int_{t_\mu}^{t_{\mu+1}} X_3(\tau,\xi)\tau^{1-\varphi}(t_{\iota+1}-\tau)^{\phi-1} d\tau \quad (9.\ 43)$$

$$R(t_{\varphi+1}) = R_0 + \frac{(1-\phi)}{AB(\phi)} t^{1-\varphi} P_4(t_\varphi, X(t_\varphi), Y(t_\varphi), Z(t_\varphi), R(t_\varphi)) + \frac{\phi}{AB(\phi)\Pi(\phi)} \sum_{\mu=2}^{\iota} \int_{t_\mu}^{t_{\mu+1}} X_4(\tau,\xi)\tau^{1-\varphi}(t_{\iota+1}-\tau)^{\phi-1} d\tau \quad (9.\ 44)$$

Here we recall the Newton polynomial:

$$\begin{aligned}\Upsilon(t,X,Y,Z,R) \simeq\ & \Upsilon(t_{j-2},X_{j-2},Y_{j-2},Z_{j-2},R_{j-2}) + \frac{1}{\triangle\ t}\Big\{\Upsilon(t_{j-1},X_{j-1},Y_{j-1},\\ & Z_{j-1},R_{j-1}) - \Upsilon(t_{j-2},X_{j-2},Y_{j-2},Z_{j-2},R_{j-2})\Big\} \times (\tau - t_{\varphi-2})\\ & +\frac{1}{\triangle\ t^2}\Big\{\Upsilon(t_j,X_j,Y_j,Z_j,R_j) - 2\Upsilon(t_{j-1},X_{j-1},Y_{j-1},Z_{j-1},\\ & R_{j-1}) + \Upsilon(t_{j-2},X_{j-2},Y_{j-2},Z_{j-2},R_{j-2})\Big\}\\ & (\tau - t_{\varphi-1})(\tau - t_{\varphi-2}). \end{aligned} \quad (9.\ 45)$$

By replacing the Newton polynomial (45) into the equations (41)-(44), we get

$$X(t_{\varphi+1}) = X_0 + \frac{(1-\phi)}{AB(\phi)} t_\iota^{1-\varphi} P_1[t_\iota, X(t_\iota), Y(t_\iota), Z(t_\iota), R(t_\iota)] + \frac{\phi}{AB(\phi)\Pi(\phi)} \sum_{\mu=2}^{\iota} P_1[t_{\mu-2}, X^{\mu-2}, Y^{\mu-2}, Z^{\mu-2}, R^{\mu-2}] t_{\mu-2}^{1-\varphi}$$

$$
\begin{aligned}
&\times \int_{t_\mu}^{t_{\mu+1}} (t_{\iota+1}-\tau)^{\phi-1} d\tau \\
&+ \frac{\phi}{AB(\phi)\Pi(\phi)} \sum_{\mu=2}^{\iota} \frac{1}{\triangle t} \Big\{ t_{\mu-1}^{1-\varphi} P_1[t_{\mu-2}, X^{\mu-2}, Y^{\mu-2}, Z^{\mu-2}, R^{\mu-2}] \\
&- t_{\mu-2}^{1-\varphi} p_1[t_{\mu-2}, X^{\mu-2}, Y^{\mu-2}, Z^{\mu-2}, R^{\mu-2}] \Big\} \\
&\times \int_{t_\mu}^{t_{\mu+1}} (\tau - t_{\mu-2})(t_{\iota+1}-\tau)^{\phi-1} d\tau \\
&+ \frac{\phi}{AB(\phi)\Pi(\phi)} \sum_{\mu=2}^{\iota} \frac{1}{2 \triangle t^2} \Big\{ t_{\mu}^{1-\varphi} P_1[t_{\mu}, X^{\mu-2}, Y^{\mu}, Z^{\mu}, R^{\mu}] \\
&- 2t_{\mu-1}^{1-\varphi} p_1[t_{\mu-1}, X^{\mu-1}, Y^{\mu-1}, Z^{\mu-1}, R^{\mu-1}] \\
&+ t_{\mu-2}^{1-\varphi} p_1[t_{\mu-2}, X^{\mu-2}, Y^{\mu-2}, Z^{\mu-2}, R^{\mu-2}] \Big\} \\
&\times \int_{t_\mu}^{t_{\mu+1}} (\tau - t_{\mu-2})(\tau - t_{\mu-1})(t_{\iota+1}-\tau)^{\phi-1} d\tau \qquad (9.\ 46)
\end{aligned}
$$

$$
\begin{aligned}
Y(t_{\varphi+1}) &= Y_0 + \frac{(1-\phi)}{AB(\phi)} t_{\iota}^{1-\varphi} P_2[t_\iota, X(t_\iota), Y(t_\iota), Z(t_\iota), R(t_\iota)] \\
&+ \frac{\phi}{AB(\phi)\Pi(\phi)} \sum_{\mu=2}^{\iota} P_2[t_{\mu-2}, X^{\mu-2}, Y^{\mu-2}, Z^{\mu-2}, R^{\mu-2}] t_{\mu-2}^{1-\varphi} \\
&\times \int_{t_\mu}^{t_{\mu+1}} (t_{\iota+1}-\tau)^{\phi-1} d\tau \\
&+ \frac{\phi}{AB(\phi)\Pi(\phi)} \sum_{\mu=2}^{\iota} \frac{1}{\triangle t} \Big\{ t_{\mu-1}^{1-\varphi} P_2[t_{\mu-2}, X^{\mu-2}, Y^{\mu-2}, Z^{\mu-2}, R^{\mu-2}] \\
&- t_{\mu-2}^{1-\varphi} P_2[t_{\mu-2}, X^{\mu-2}, Y^{\mu-2}, Z^{\mu-2}, R^{\mu-2}] \Big\} \\
&\times \int_{t_\mu}^{t_{\mu+1}} (\tau - t_{\mu-2})(t_{\iota+1}-\tau)^{\phi-1} d\tau \\
&+ \frac{\phi}{AB(\phi)\Pi(\phi)} \sum_{\mu=2}^{\iota} \frac{1}{2 \triangle t^2} \Big\{ t_{\mu}^{1-\varphi} p_2[t_{\mu}, X^{\mu-2}, Y^{\mu}, Z^{\mu}, R^{\mu}] \\
&- 2t_{\mu-1}^{1-\varphi} p_2[t_{\mu-1}, X^{\mu-1}, Y^{\mu-1}, Z^{\mu-1}, R^{\mu-1}] \\
&+ t_{\mu-2}^{1-\varphi} p_2[t_{\mu-2}, X^{\mu-2}, Y^{\mu-2}, Z^{\mu-2}, R^{\mu-2}] \Big\} \\
&\times \int_{t_\mu}^{t_{\mu+1}} (\tau - t_{\mu-2})(t_{\iota+1}-\tau)^{\phi-1} d\tau \qquad (9.\ 47)
\end{aligned}
$$

we write down the numerical scheme of $X(t_{\varphi+1})$ and $Y(t_{\varphi+1})$ therefore we omitted the numerical scheme of $Z(t_{\varphi+1})$ and $R(t_{\varphi+1})$ because it similar do.

The following computations may be performed for the integral in the aforementioned equations (44)-(47).

$$\begin{aligned}\int_{t_\mu}^{t_{\mu+1}}(t_{\iota+1}-\tau)^{\phi-1}d\tau &= \frac{(\triangle\ t)^{\phi}}{\phi}[(\iota-\mu+1)^{\phi}-(\iota-\mu)^{\phi}]\\ \int_{t_\mu}^{t_{1+\mu}}(\tau-t_{\mu-2})(t_{1+\iota}-\tau)^{\phi-1}d\tau &= \frac{(\triangle\ t)^{1+\phi}}{\phi(1+\phi)}[(1+\iota-\mu)^{\phi}(3+\iota-\mu+2\phi)\\ &- (\iota-\mu)^{\phi}(3+\iota-\mu+3\phi)]\end{aligned}$$

$$\begin{aligned}&\int_{t_\mu}^{t_{1+\mu}}(\iota-t_{\mu-2})(\iota-t_{\mu-1})(t_{1+\iota}-\tau)^{\phi-1}d\tau=\frac{(\triangle\ t)^{2+\phi}}{(1+\phi)(2+\phi)\phi}\\ &\times\Big[(1+\iota-\mu)^{\phi}\Big\{2(\iota-\mu)^2+(3\phi+10)(12+\iota-\mu)+2\phi^2+9\phi\Big\}\\ -\quad &(\iota-\mu)^{\phi}\Big\{2(\iota-\mu)^2+(10+5\phi)(\iota-\mu)+6\phi^2+18\phi+12\Big\}\Big]\end{aligned}$$

Hence, we get finally

$$\begin{aligned}X(t_{\varphi+1}) &= X_0+\frac{(1-\phi)}{AB(\phi)}t_{\iota}^{1-\varphi}P_1[t_\iota,X(t_\iota),Y(t_\iota),Z(t_\iota),R(t_\iota)]\\ &+ \frac{\phi(\triangle\ t)^{\phi}}{AB(\phi)\Pi(1+\phi)}\sum_{\mu=2}^{\iota}P_1[t_{\mu-2},X^{\mu-2},Y^{\mu-2},Z^{\mu-2},R^{\mu-2}]t_{\mu-2}^{1-\varphi}\end{aligned}$$

$$\begin{aligned}&\times\Big\{(\iota-\mu+1)^{\phi}-(\iota-\mu)^{\phi}\Big\}\\ +\quad &\frac{\phi(\triangle\ t)^{\phi}}{AB(\phi)\Pi(\phi+2)}\\ &\sum_{\mu=2}^{\iota}\Big\{P_1[t_{\mu-1},X^{\mu-1},Y^{\mu-1},Z^{\mu-1},R^{\mu-1}]\\ -\quad &t_{\mu-2}^{1-\varphi}P_1[t_{\mu-2},X^{\mu-2},Y^{\mu-2},Z^{\mu-2},R^{\mu-2}]\Big\}\times\mathcal{A}_2+\frac{\phi(\triangle\ t)^{\phi}}{2AB(\phi)\Pi(3+\phi)}\end{aligned}$$

$$\begin{aligned}&\sum_{\mu=2}^{\iota}\Big\{t_{\mu}^{1-\varphi}P_1[t_\mu,X^{\mu},Y^{\mu},Z^{\mu},R^{\mu}]\\ &P_1[t_{\mu-1},X^{\mu-1},Y^{\mu-1},Z^{\mu-1},R^{\mu-1}]+\\ &t_{\mu-2}^{1-\varphi}P_1[t_{\mu-2},X^{\mu-2},Y^{\mu-2},Z^{\mu-2},R^{\mu-2}]\Big\}\times\mathcal{A}_3\end{aligned}\qquad (9.\ 48)$$

$$\begin{aligned}
Y(t_{\varphi+1}) &= Y_0 + \frac{(1-\phi)}{AB(\phi)} t_\iota^{1-\varphi} P_2[t_\iota, X(t_\iota), Y(t_\iota), Z(t_\iota), R(t_\iota)] \\
&+ \frac{\phi(\triangle\, t)^\phi}{AB(\phi)\Pi(1+\phi)} \\
&\sum_{\mu=2}^{\iota} P_2[t_{\mu-2}, X^{\mu-2}, Y^{\mu-2}, Z^{\mu-2}, R^{\mu-2}] t_{\mu-2}^{1-\varphi} \\
&\times \Big\{(1+\iota-\mu)^\phi - (\iota-\mu)^\phi\Big\} + \frac{\phi(\triangle\, t)^\phi}{AB(\phi)\Pi(\phi+2)} \\
&\sum_{\mu=2}^{\iota} \Big\{P_2[t_{\mu-1}, X^{\mu-1}, Y^{\mu-1}, Z^{\mu-1}, R^{\mu-1}] \\
&- t_{\mu-2}^{1-\varphi} P_2[t_{\mu-2}, X^{\mu-2}, Y^{\mu-2}, Z^{\mu-2}, R^{\mu-2}]\Big\} \times \mathcal{A}_2 + \frac{\phi(\triangle\, t)^\phi}{2AB(\phi)\Pi(3+\phi)} \\
&\sum_{\mu=2}^{\iota} \Big\{t_\mu^{1-\varphi} P_2[t_\mu, X^\mu, Y^\mu, Z^\mu, R^\mu] - 2t_{\mu-1}^{1-\varphi} \\
&P_2[t_{\mu-1}, X^{\mu-1}, Y^{\mu-1}, Z^{\mu-1}, R^{\mu-1}] \\
&+ t_{\mu-2}^{1-\varphi} P_2[t_{\mu-2}, X^{\mu-2}, Y^{\mu-2}, Z^{\mu-2}, R^{\mu-2}]\Big\} \times \mathcal{A}_3 \qquad (9.\ 49)
\end{aligned}$$

we write down the numerical scheme of $X(t_{\varphi+1})$ and $Y(t_{\varphi+1})$ therefore we omitted the numerical scheme of $Z(t_{\varphi+1})$ and $R(t_{\varphi+1})$ because it similar do.

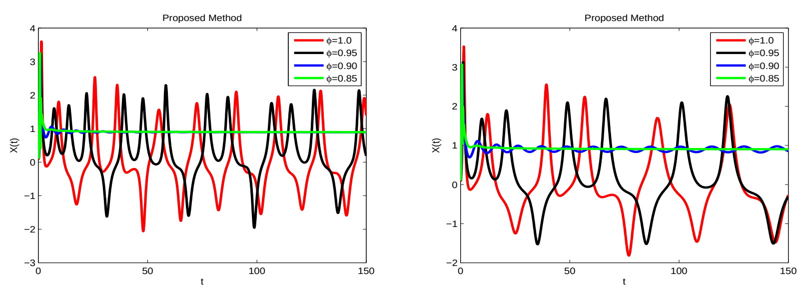
where

$$\begin{aligned}
\mathcal{A}_2 &= [(1+\iota-\mu)^\phi(3+\iota-\mu+2\phi) - (\iota-\mu)^\phi(3+\iota-\mu+3\phi)] \\
\mathcal{A}_3 &= \Big[(1+\iota-\mu)^\phi\Big\{2(\iota-\mu)^2 + (10+3\phi)(\iota-\mu) + 2\phi^2 + 9\phi + 12\Big\} \\
&- (\iota-\mu)^\phi\Big\{2(\iota-\mu)^2 + (5\phi+10)(\iota-\mu) + 6\phi^2 + 18\phi + 12\Big\}\Big].
\end{aligned}$$

10. Simulation Explanation

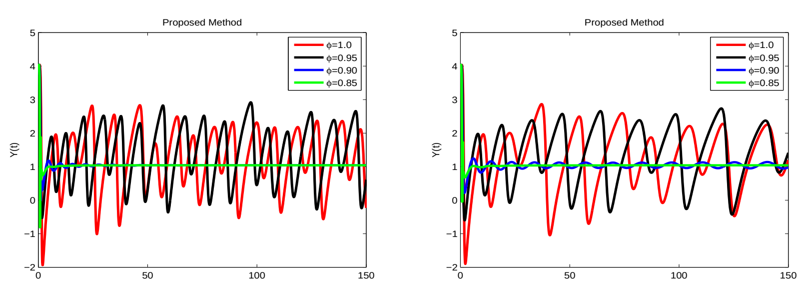
The given statement conveys that we prove the credibility of our derived theories by presenting practical examples that follows here. We present a mathematical analysis of financial problems, yielding intriguing findings by employing non-integer parametric choices. The examples below demonstrate the effectiveness of the theoretical results gained. A mathematical investigation of finance in connection to energy yields crucial insights, with particularly dramatic outcomes shown when non-integer parameters are applied, revealing the complex implications of fractional-order models on financial dynamics. The solution in Figure 3-6 The obtained result is achieved by converting the fractional values to the desired value. MATLAB is used to create a numerical simulation of fractional order finance with energy implications. The initial values for the system components are $X(0) = 0.1$,

$Y(0) = 4$, $Z(0) = 0.5$, and $R(0) = 0$. We examine the findings produced using the fractal-fractional operator to evaluate its performance and impacts of its integer orders in Figures 3-6. Investment demand and interest rate become stable and normalize after certain time by decreasing the fractional values as can be seen in figure 3 and 4 with different dimensions. But the price index and energy sources fluctuates with the passage of time and become stable with certain normalized losses which can be oberted in figure 5 and 6 using different dimensions. It has been noted that in order to mitigate the economic risks in developing nations, we must increase energy sources. These figures show the financial model's graphical representation using the suggested numerical method, taking energy influence into account. Additionally, it has been noted that the solutions for every compartment become more accurate and dependable as fractional values and dimensions are reduced. The numerical results demonstrate the distinct behavior of the dynamics in fractal-fractional parameters, and these operators are found to be more efficient than current non-integer order models.



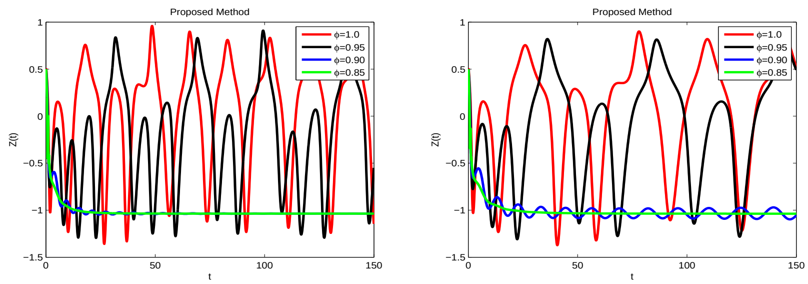
(A) 0.9 dimension (B) 0.8 dimension

FIGURE 3. Simulation of X using fractal fractional operator



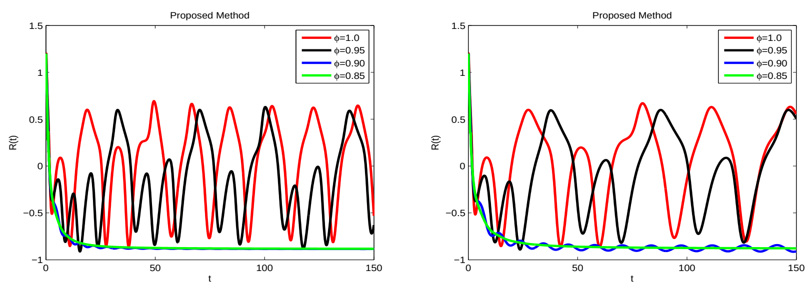
(A) 0.9 dimension (B) 0.8 dimension

FIGURE 4. Simulation of Y using fractal fractional operator



(A) 0.9 dimension (B) 0.8 dimension

FIGURE 5. Simulation of Z using fractal fractional operator



(A) 0.9 dimension (B) 0.8 dimension

FIGURE 6. Simulation of R using fractal fractional operator

CONCLUSION

In order to examine the dynamics of financial systems in developing nations, a fractal-fractional financial model incorporating energy effects was created. By using the fractal–fractional operator to incorporate memory and hereditary effects, the model offers a more realistic framework. The conditions under which the system stays stable or changes to complex and chaotic behaviour were determined by theoretical analyses, such as equilibrium, stability, and bifurcation investigations. Numerical simulations showed how energy parameters affect market dynamics and play an important role in controlling financial instability and protecting against crises. The results indicate that energy integration can have a profound impact on system behavior and provide useful information for monitoring and managing financial risks. Overall, this work offers a robust methodology for understanding the complex energy-market relationship and provides valuable guidance for further research and policymaking in the financial systems of developing countries.

CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

Safdar Abbas, Evren Hincal: Conceptualization.
Safdar Abbas, Abdul Ghaffar : Methodology
Safdar Abbas, Aqeel Ahmad: Model Development and Analysis
Aqeel Ahmad: Software/ Simulation
Safdar Abbas, Abdul Ghaffar: Validation
Safdar Abbas: Formal Analysis
Safdar Abbas, Ghulam Mustafa: Investigation
Abdul Ghaffar: Data Curation
Safdar Abbas: Writing-Original Draft Preparation
Safdar Abbas, Evren Hincal, Ghulam Mustafa. Writing-Review & Editing
Aqeel Ahmad: Visualization
Evren Hincal: Supervision
Evren Hincal: Project Administration

DECLARATIONS

Conflict of Interest: The authors declare that there is no conflict of interest regarding the publication of this manuscript.
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