Punjab University Journal of Mathematics (2026), 58(01), 114-154
https://doi.org/10.52280/pujm.2026.58(01)05

Numerical Solutions of Intuitionistic Fuzzy Differential Equations Using OHAM with Error Estimation and Stability Proofs

Received
13 August, 2025

Revised
28 April, 2026
10 June, 2026

Accepted
15 June, 2026

Published Online
03 July, 2026

*Muhammad Saeed , Muzaher Ali
Department of Mathematics
University of Managment and Technology, Lahore, Punjab, Pakistan

Shaukat Iqbal
Department of Informatics and Systems
University of Managment and Technology, Lahore, Punjab, Pakistan

Abstract. In this article, the problem of finding the analytical solution for Intuitionistic fuzzy differential equations (IFDEs) is considered. It is worth mentioning that, despite various applications of IFDEs in engineering, economics, and natural sciences, there exist a few studies about finding the guaranteed convergent and stable solutions for such equations. In order to fill this gap, the Optimal Homotopy Asymptotic Method (OHAM) is utilized to construct approximate analytical solutions for IFDE. The use of OHAM allows one to control adjustable parameters that could guarantee the convergence of obtained results and their high accuracy even when dealing with ill-posed problems. Our main contribution consists in proving the conditions of convergence of OHAM approximate solutions and verifying Hyers-Ulam stability of IFDE under the certain perturbation bounds. We also consider numerical experiments to verify the accuracy of OHAM by comparing its approximate solutions with the exact solutions. It is shown that the absolute errors of approximate solutions are very small (from 10^{-8} to 10^{-12}). This technique enriches the analysis of IFDEs with well-proven stability and convergence, making it valuable for evaluation and decision-making under fuzzy uncertainties.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Intuitionistic Fuzzy Differential Equations, Ulam-Hyers stable, Optimal Homotopy Asymptotic method, Error estimation.

**Corresponding Author* : muhammad.saeed@umt.edu.pk

1. INTRODUCTION

Differential equations play an essential role in physical and engineering systems modeling [22, 9, 28]. But in practice, uncertain scenarios arise, which can be handled using deterministic mathematical models alone. In this context, fuzzy sets introduced by Zadeh [33] come to our rescue through the concept of fuzzy memberships [23]. Further enhancement to the model comes from Intuitionistic Fuzzy Sets (IFSs), which incorporate both fuzzy memberships and non-memberships [4].

IFDEs can be naturally obtained in such uncertain situations [29, 3, 2]. Some methods used for the solution of IFDEs include the modified Adomian decomposition method [17], Laplace transform method [1, 18], and Runge-Kutta method [19, 20, 30, 21]. Although these methods are quite handy, most of them are purely numerical or semi-numerical with no analytical guarantee of convergence or stability. Also, finding the analytical solutions of IFDEs is a challenging task, more so in the case of non-linear IFDEs.

There are several deficiencies with the existing algorithms for solving IFDEs. For example, ADM needs the computation of Adomian polynomials, which is difficult for nonlinear equations, and its convergence depends on strong assumptions. Numerical schemes such as the Runge-Kutta method can yield solutions at some discrete points but cannot provide a closed form of the solution and do not offer an easy approach for estimating the global error. In addition, Laplace transform algorithms can solve linear IFDEs only when their coefficients are constant and cannot handle nonlinear IFDEs and/or those with variable coefficients. Finally, variational iteration algorithms need the determination of Lagrange multipliers, which may be complicated, and they do not necessarily converge rapidly. Intuitionistic fuzzy sets have seen rich theoretical developments including hypersoft extensions, similarity measures, and decision-support frameworks [26, 25].

On the other hand, the Optimal Homotopy Asymptotic Method (OHAM) [16] was proposed by Marinca and Herisanu and avoids these drawbacks. OHAM develops a homotopy involving an auxiliary function with some free parameters. Such parameters can be optimized using, for instance, the least squares principle to ensure rapid convergence even for highly nonlinear or ill-posed IFDEs. OHAM provides continuous solutions in the form of analytic expressions rather than numbers, which is indispensable for further studies (e.g., differentiation, integration, stability analysis). Unlike perturbation techniques, there is no need for small parameters in OHAM, and it avoids the complex polynomial computations in ADM. Moreover, since there is a built-in residual function in OHAM, it is straightforward to estimate the error a posteriori. Additionally, as we show in this paper, OHAM naturally leads to proving the convergence and Hyers-Ulam stability, which is rarely the case for IFDE solvers.

OHAM has been successfully applied to a wide range of linear and nonlinear differential and integral equations [5, 14, 13, 8, 24, 11, 31, 34]. However, there has been no systematic application of OHAM to IFDEs with simultaneous analysis of convergence and Hyers-Ulam stability. This gap is addressed by the present work:

- Extending OHAM for solving IFDEs, decomposing the intuitionistic fuzzy problem into four crisp subsystems by $\alpha\beta$-cut display.

- Explicit conditions for the convergence of the OHAM-series in suitable function spaces.
- Checking the Hyers-Ulam stability of the obtained approximate solutions to ensure robustness against small perturbations.

Recent studies on fuzzy fractional differential equations [27, 7] highlight the growing interest in stability and analytical solutions under uncertainty; our work complements these by addressing IFDEs using OHAM.

The paper is organized as follows. Section 2 covers the preliminaries for fuzzy and intuitionistic fuzzy sets. Section 3 explains the OHAM formulation and its adaptation to IFDEs. Section 4 presents numerical examples that demonstrate the method's performance. Convergence and stability are analyzed in Sections 5 and 6, respectively. Section 7 discusses the results, and Section 8 concludes by summarizing the findings and suggesting future work.

2. PRELIMINARIES

This section recalls basic definitions of fuzzy sets, intuitionistic fuzzy sets, and triangular intuitionistic fuzzy numbers. We also introduce the $\alpha\beta$-Cut representation, which is essential for decomposing intuitionistic fuzzy differential equations into crisp systems. For quick reference, Table 1 summarizes the main symbols used throughout the paper.

TABLE 1. Notation table

Symbol	**Description**
$\hat{P}$	Universal set
$\mu_{\hat{F}}(m)$	Membership degree of element m in fuzzy set $\hat{F}$
$\mu_{\hat{I}}(m), \nu_{\hat{I}}(m)$	Membership and non-membership degrees in intuitionistic fuzzy set $\hat{I}$
α, β	Cut levels for membership and non-membership ($\alpha, \beta \in [0, 1]$)
$T = \langle (x_1, x_2, x_3 : \mu), (x_0, x_2, x_4 : \nu) \rangle$	Triangular intuitionistic fuzzy number
$[U]^+_\mu, [U]^-_\mu$	Upper and lower bounds of the membership α-cut
$[U]^+_\nu, [U]^-_\nu$	Upper and lower bounds of the non-membership β-cut
L, N	Linear and nonlinear operators in OHAM
C_i	Auxiliary (optimal) parameters in OHAM
$\hat{p}$	Homotopy deformation parameter ($\hat{p} \in [0, 1]$)
$R(t, C_i)$	Residual function
ϵ	Perturbation bound in Hyers-Ulam stability
L_{HU}	Hyers-Ulam stability constant

2.1. **Fuzzy Sets and Intuitionistic Fuzzy Sets.**

Definition 2.1.1 (Fuzzy Set)**.** *Let $\hat{P}$ be a universal set. A fuzzy set $\hat{F}$ in the set $\hat{P}$ is defined as*

$$\hat{F} = \{(p, \mu_{\hat{F}}(p)) \mid p \in \hat{P}\},$$

*where $\mu_{\hat{F}}(p) : \hat{P} \to [0, 1]$ is the **membership degree** of p* [33, 15].

Definition 2.1.2 (Intuitionistic Fuzzy Set)**.** *Let $\hat{P}$ be a universal set. An intuitionistic fuzzy set $\hat{I}$ in $\hat{P}$ is defined as*

$$\hat{I} = \{(p, \mu_{\hat{I}}(p), \nu_{\hat{I}}(p)) \mid p \in \hat{P}\},$$

*where $\mu_{\hat{I}}(p) : \hat{P} \to [0, 1]$ is the **membership degree**, $\nu_{\hat{I}}(p) : \hat{P} \to [0, 1]$ is the **non-membership degree**, and they satisfy*

$$0 \leq \mu_{\hat{I}}(p) + \nu_{\hat{I}}(p) \leq 1 \quad \forall p \in \hat{P}.$$

*The value $\pi_{\hat{I}}(p) = 1 - \mu_{\hat{I}}(p) - \nu_{\hat{I}}(p)$ represents the **hesitation degree*** [4, 12].

2.2. **Triangular Intuitionistic Fuzzy Number (TIFN) and $\alpha\beta$-Cut.**

Definition 2.2.1 (Triangular Intuitionistic Fuzzy Number)**.** *A triangular intuitionistic fuzzy number T is an intuitionistic fuzzy set on $\mathbb{R}$ with the following membership function $\mu_T(t)$ and non-membership function $\nu_T(t)$* [19]*:*

$$\mu_T(t) = \begin{cases} \dfrac{t - x_1}{x_2 - x_1}, & x_1 \leq t \leq x_2 \\ \dfrac{x_3 - t}{x_3 - x_2}, & x_2 \leq t \leq x_3 \\ 0, & \textit{otherwise}, \end{cases} \qquad \nu_T(t) = \begin{cases} \dfrac{x_2 - t}{x_2 - x_0}, & x_0 \leq t \leq x_2 \\ \dfrac{t - x_2}{x_4 - x_2}, & x_2 \leq t \leq x_4 \\ 1, & \textit{otherwise}, \end{cases}$$

where $x_0 < x_1 < x_2 < x_3 < x_4$. We denote such a TIFN as

$$T = \langle (x_1, x_2, x_3 : \mu_T),\ (x_0, x_2, x_4 : \nu_T) \rangle.\ \textit{where}\ 0 \leq \mu_T + \nu_T \leq 1$$

A graphical representation is given in the following figure.

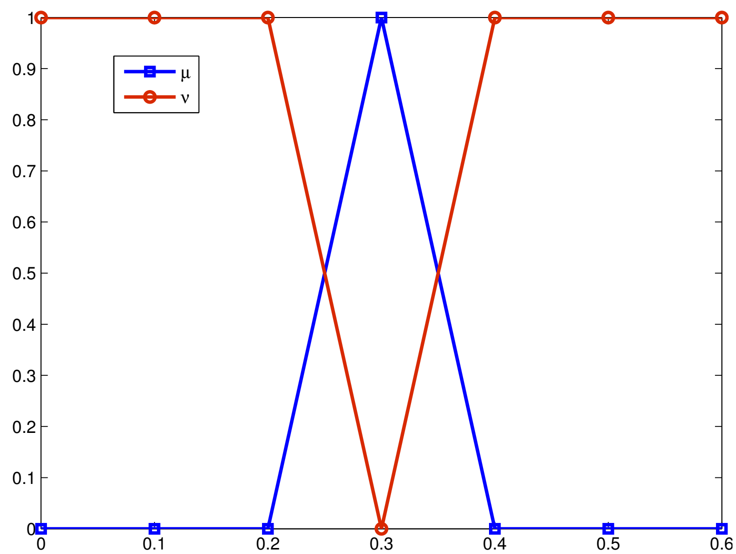


FIGURE 1. Graphical representation Triangular Intuitionistic Fuzzy Number (TIFN)

Definition 2.2.2 ($\alpha\beta$-Cut of a TIFN)**.** *For a given TIFN T and cut levels $\alpha \in [0,1]$ (for membership) and $\beta \in [0,1]$ (for non-membership), the $\alpha\beta$-cut is the crisp set*

$$T_{\alpha\beta} = \big\{t \in \mathbb{R} \mid \mu_T(t) \geq \alpha,\ \nu_T(t) \leq \beta\big\}.$$

For triangular intuitionistic fuzzy numbers, this cut yields a closed interval:

$$T_{\alpha\beta} = [T^L_{\alpha\beta},\ T^U_{\alpha\beta}],$$

where the lower and upper bounds are functions of α and β. In particular, the membership α-cut gives $[x_1 + \alpha(x_2 - x_1),\ x_3 - \alpha(x_3 - x_2)]$, while the non-membership β-cut imposes additional restrictions via $\nu_T(t) \leq \beta$. For simplicity, in this work we consider the case where membership and non-membership are linked by a parameter β (or γ in the original notation) as shown in the numerical examples, leading to four distinct bound functions: $[U]^+_\mu, [U]^-_\mu$ (upper and lower membership bounds) and $[U]^+_\nu, [U]^-_\nu$ (upper and lower non-membership bounds). This decomposition is standard in the literature on intuitionistic fuzzy differential equations [19].

2.3. **Decomposition of IFDEs into Crisp Systems.** An intuitionistic fuzzy differential equation can be interpreted using the $\alpha\beta$-cut approach. For a given IFDE with initial conditions expressed as TIFNs, the solution's membership and non-membership functions are obtained by solving **four crisp differential equations** corresponding to the upper and lower bounds of the membership and non-membership cuts respectively. Specifically, if $U(t)$ is the intuitionistic fuzzy solution, then for each t we have:

$$[U(t)]_{\alpha\beta} = \big[[U(t)]^-_\mu,\ [U(t)]^+_\mu\big] \quad \text{and} \quad [U(t)]^-_\nu,\ [U(t)]^+_\nu$$

satisfying:

- $\dfrac{d^2[U]_\mu^+}{dt^2} + [U]_\mu^+ = f(t)$ with initial conditions derived from the TIFN's upper membership α-cut,
- $\dfrac{d^2[U]_\mu^-}{dt^2} + [U]_\mu^- = f(t)$ with lower membership initial conditions,
- similar equations for $[U]_\nu^+$ and $[U]_\nu^-$.

3. Formulation of OHAM

In this section we present the general methodology of the Optimal Homotopy Asymptotic Method (OHAM) and then detail its adaptation to IFDEs using the $\alpha\beta$-cut decomposition introduced in previous section . A step-by-step algorithmic summary is provided to ensure reproducibility.

3.1. **General OHAM Formulation.** Consider the functional equation

$$L(\psi(t)) + G(t) + N(\psi(t), t) = 0, \quad t \in \omega. \tag{3. 1}$$

Here, *L* is a linear operator, and *N* may be a linear or non-linear operator. $G(t)$ is a known function, $\psi(t)$ is an unknown function, and ω is considered as the domain of the given problem [16, 5, 14].
We construct an optimal homotopy $\psi(x, \hat{p}) : \omega \times [0, 1] \to R$ that satisfies

$$(1 - \hat{p})\big[L(\psi(t, \hat{p})) + G(t)\big] = H(\hat{p}, C)\big[L(\psi(t, \hat{p})) + G(t) + N(\psi(t, \hat{p}), t)\big], \tag{3. 2}$$

$\hat{p} \in [0, 1]$ is an deformation parameter and $H(\hat{p}, C)$ is an auxiliary function which is define as

$$H(\hat{p}, C) = \sum_{i=1}^{\infty} \hat{p}^i C_i. \tag{3. 3}$$

The auxiliary function $H(\hat{p}, C)$ gives us an easy way to control and adjust the convergence, also increases the accuracy of the results and effectiveness of the OHAM [13, 8, 24].
Expanding $\psi(t, \hat{p})$ in a Taylor series about $\hat{p}$,

$$\psi(t, \hat{p}, C) = \psi_0(t) + \sum_{k=1}^{\infty} \psi_k(t, C)\hat{p}^k. \tag{3. 4}$$

Replacing Eq. 3. 3 and Eq. 3. 4 in Eq. 3. 2 , we have

$$\begin{aligned}
&(1 - \hat{p})\Big[L\Big(\psi_0(t) + \sum_{k=1}^{\infty} \psi_k(t, C)\hat{p}^k\Big) + G(t)\Big] \\
&\quad = \left(\sum_{i=1}^{\infty} \hat{p}^i C_i\right)\Bigg[L\big(\psi(t, \hat{p})\big) + G(t) \\
&\qquad + N\Big(\psi_0(t) + \sum_{k=1}^{\infty} \psi_k(t, C)\hat{p}^k, t\Big)\Bigg].
\end{aligned} \tag{3. 5}$$

Comparing the coefficients of alike power of $\hat{p}$, we have

$$\psi_0(t) = G(t),$$
$$\psi_1(t) = C_1 N(\psi_0(t), t),$$
$$\psi_2(t) = (1 + C_1)\psi_1(t) + C_2 N(\psi_0(t), t) + C_1 N(\psi_1(t), t),$$

similarly

$$\psi_n(t) = (1 + C_1)\psi_{n-1}(t) + \sum_{k=2}^{n-1} C_k \psi_{n-k}(t) + \sum_{k=1}^{n} C_k N(\psi_{n-k}(t), t).$$

Now

$$\psi(t, C_i) = \sum_{k=0}^{n} \psi_k(t, C_i), \qquad i = 1, 2, 3...n$$

is approximate solution. We need to find optimal values C_i.

3.2. **Determination of Optimal Parameters C_i via Least Squares.** First we construct residual equation of eq. (3. 1) is given as

$$R(t, C_i) = L(\psi(t, C_i)) + G(t) + N(\phi(t, C_i)).$$

We then minimize the squared residual integrated over the domain:

$$\hat{J}(C_i) = \int_a^b R^2(t, C_i)\, dt. \tag{3. 6}$$

The optimal C_i satisfy the system

$$\frac{\partial \hat{J}}{\partial C_i} = 0, \qquad i = 1, 2, \ldots, m. \tag{3. 7}$$

Solving these equations (numerically, if necessary) gives the C_i that ensure fast convergence and high accuracy [11, 31, 34].

3.3. **Adaptation of OHAM to IFDEs.** To apply OHAM to an IFDE, we depend on the $\alpha\beta$-cut decomposition mentioned in previous section. For a given IFDE with initial conditions expressed as TIFN, the solution's membership and non-membership functions are approximated independently by solving **four crisp OHAM problems**:

- $[U]^+_\mu(t)$: upper bound of the membership α-cut,
- $[U]^-_\mu(t)$: lower bound of the membership α-cut,
- $[U]^+_\nu(t)$: upper bound of the non-membership β-cut,
- $[U]^-_\nu(t)$: lower bound of the non-membership β-cut.

Each crisp problem has the same differential operator but different initial conditions derived from the $\alpha\beta$-cut of the intuitionistic fuzzy initial value. For example, for a second-order IFDE of the form

$$U''(t) + U(t) = f(t), \quad U(0) = T_0,\ U'(0) = T_1,$$

where T_0, T_1 are TIFNs, the four crisp systems are:

$$\frac{d^2[U]^+_\mu}{dt^2} + [U]^+_\mu = f(t), \quad [U]^+_\mu(0) = [T_0]^+_\mu,\ [U^+]'_\mu(0) = [T_1]^+_\mu, \tag{3. 8}$$

$$\frac{d^2[U]^-_\mu}{dt^2} + [U]^-_\mu = f(t), \quad [U]^-_\mu(0) = [T_0]^-_\mu,\ [U^-]'_\mu(0) = [T_1]^-_\mu, \tag{3. 9}$$

$$\frac{d^2[U]^+_\nu}{dt^2} + [U]^+_\nu = f(t), \quad [U]^+_\nu(0) = [T_0]^+_\nu,\ [U^+]'_\nu(0) = [T_1]^+_\nu, \tag{3. 10}$$

$$\frac{d^2[U]^-_\nu}{dt^2} + [U]^-_\nu = f(t), \quad [U]^-_\nu(0) = [T_0]^-_\nu,\ [U^-]'_\nu(0) = [T_1]^-_\nu. \tag{3. 11}$$

Each of these is solved using the standard OHAM procedure described in previous subsections. The results are then combined to reconstruct the intuitionistic fuzzy solution.

3.4. **Convergence Criteria of OHAM.** The convergence criteria for the Optimal Homotopy Asymptotic Method (OHAM) is provided in the following theorem:

Theorem 3.4.1. *Suppose the series*

$$\psi(t, C_i) = \psi_0(t) + \sum_{k=1}^{\infty} \psi_k(t, c_1, c_2, \dots, c_k)$$

be the OHAM approximated solution. Assume that each term ψ_k belongs to the space $L^2[c, d]$ with norm $||_\infty$. If the given series converges to a function $\psi(t)$, then, $\psi(t)$ represents the exact solution of the equation (3. 1) [5].

Proof: If the series

$$\psi(t, C_i) = \psi_0(t) + \sum_{k=1}^{\infty} \psi_k(t, C_1, C_2, C_3, ...C_k)$$

converges, then we can write it as

$$S(t) = \sum_{k=1}^{\infty} \psi_k(t, C_1, C_2, C_3, ...C_k). \tag{3. 12}$$

Additionally it holds

$$\lim_{k\to\infty} \psi_k(t, C_1, C_2, C_3, ...C_k) = 0.$$

$$\psi_1(t, C_1) + \sum_{k=2}^{n} \psi_k(t, C_k) - \sum_{k=2}^{n} \psi_{k-1}(t, C_{k-1}) = \psi_n(t, C_n),$$

applying limit $n \to \infty$

$$\psi_1(t, C_1) + \sum_{k=2}^{\infty} \psi_k(t, C_k) - \sum_{k=2}^{\infty} \psi_{k-1}(t, C_{k-1}) = \lim_{k\to\infty} \psi_n(t, C_n),$$

by using equation (3. 12)

$$\psi_1(t, C_1) + \sum_{k=2}^{\infty} \psi_k(t, C_k) - \sum_{k=2}^{\infty} \psi_{k-1}(t, C_{k-1}) = 0.$$

Since the operator L is linear,

$$L(\psi_1(t, C_1)) + \sum_{k=2}^{\infty} L(\psi_k(t, C_k)) - \sum_{k=2}^{\infty} L(\psi_{k-1}(t, C_{k-1})) = 0. \tag{3. 13}$$

$$L(\psi_1(t, C_1)) + L\left(\sum_{k=2}^{\infty} \psi_k(t, C_k)\right) - L\left(\sum_{k=2}^{\infty} \psi_{k-1}(t, C_{k-1})\right) = 0. \tag{3. 14}$$

$$\begin{aligned} L(\psi_1(t, C_1)) &+ L\left(\sum_{k=2}^{\infty} \psi_k(t, C_k)\right) - L\left(\sum_{k=2}^{\infty} \psi_{k-1}(t, C_{k-1})\right) \\ &= L(\psi_1(t, C_1)) + \sum_{k=2}^{\infty} \Bigg(C_k N_0(\psi_0(t)) \\ &\qquad + \sum_{i=1}^{k-1} C_i \Big(L(\psi_{k-1}(t)) + N_{k-i}(\psi_0, \psi_1, \psi_2, \ldots, \psi_{k-1}) \Big) \Bigg) \\ &= 0. \end{aligned} \tag{3. 15}$$

or

$$\begin{aligned} L(\psi_1(t, C_1)) + \sum_{k=2}^{\infty} \Bigg(& C_k N_0(\psi_0(t)) \\ & + \sum_{i=1}^{n-1} C_i \Bigg[L(\psi_{k-1}(t)) \\ & \qquad + N_{n-i}\Big(\psi_0, \psi_1, \psi_2, \\ & \qquad\qquad \ldots, \psi_{n-1}\Big) \Bigg] \Bigg) \\ = 0. & \end{aligned} \tag{3. 16}$$

equation (15) can be written as

$$\begin{aligned} L(\psi_0) &+ G(t) + C_1 N_0(\psi_0(t)) \\ &+ \sum_{k=2}^{\infty} \Bigg(C_k N_0(\psi_0(t)) + \sum_{i=1}^{n-1} C_i \Bigg[L(\psi_{k-1}(t)) \\ &\qquad + N_{n-i}\Big(\psi_0, \psi_1, \psi_2, \\ &\qquad\qquad \ldots, \psi_{n-1}\Big) \Bigg] \Bigg) \\ &= 0. \end{aligned} \tag{3. 17}$$

And hence

$$\sum_{k=1}^{\infty} \left[\sum_{i=1}^{k} C_{i-k} \Big(L(\psi_{1-1}(t)) + N_{i-i}(\psi_{k-1}, C_{k-1}) \Big) \right] + G(t) = 0.$$

If $C_i,\ i = 1, 2, 3...n$, are chosen appropriately then it leads to

$$L(\psi(t)) + G(t) + N(\psi(t)) = 0.$$

which shows that $\psi(t)$ is the exact solution of equation.

Application of this theorem to IFDE is given in section Convergence Analysis.

4. Numerical Experiment

4.1. **Example 1.** Consider the following IFDE

$$\frac{d^2U(x)}{dx^2} + U(x) = -x, \qquad x \in [0, 1], \tag{4. 18}$$

and initial conditions are defined as

$$U(0) = (0.1\beta - 0.1, 0.1 - 0.1\beta, -0.2\beta, 0.2\beta),$$
$$U'(0)) = (0.088 + 0.1\beta, 0.288 - 0.1\beta, 0.188 + 0.178\beta, 0.188 + 0.288\beta).$$

Here we have equal membership(μ) and non-membership (ν) i.e. $\alpha = \beta$. The exact solutions of this equation are defined as

$$[U(x, \beta)]^+_\mu = (0.1\beta - 0.1)\cos x + (1.088 + 0.1\beta)\sin x - x,$$
$$[U(x, \beta)]^+_\nu = (0.1 - 0.1\beta)\cos x + (1.288 - 0.1\beta)\sin x - x,$$
$$[U(x, \beta)]^-_\mu = -0.2\beta\cos x + (1.188 + 0.178\beta)\sin x - x,$$
$$[U(x, \beta)]^-_\nu = 0.2\beta\cos x + (0.188 + 0.288\beta)\sin x - x, .$$

We have the following system of equations

$$\frac{d^2[U(x)]^+_\mu}{dx^2} + [U(x)]^+_\mu = -x, \qquad [U(0)]^+_\mu = 0.1\beta - 0.1 \qquad [U'(0)]^+_\mu = 0.088 + 0.1\beta, \tag{4. 19}$$

$$\frac{d^2[U(x)]^+_\nu}{dx^2} + [U(x)]^+_\nu = -x, \qquad , [U(0)]^+_\nu = 0.1 - 0.1\beta \qquad [U'(0)]^+_\nu = .288 - 0.1\beta, \tag{4. 20}$$

$$\frac{d^2[U(x)]^-_\mu}{dx^2} + [U(x)]^-_\mu = -x, \qquad [U(0)]^-_\mu = -0.2\beta \qquad [U'(0)]^-_\mu = 0.188 + 0.178\beta, \tag{4. 21}$$

$$\frac{d^2[U(x)]^-_\nu}{dx^2} + [U(x)]^-_\nu = -x, \qquad [U(0)]^-_\nu = 0.2\beta \qquad [U'(0)]^-_\nu = 0.188 + 0.288\beta. \tag{4. 22}$$

Now we apply the method explained in the previous section.

$$[U_0(x, \beta)]^+_\mu = \frac{1}{250}(25(\beta - 1) + (25\beta + 22)x)$$
$$[U_0(x, \beta)]^+_\nu = \frac{1}{250}(-25\beta - 25\beta x + 72x + 25)$$

$$[U_0(x,\beta)]^-_\mu = \frac{1}{500}((89\beta+94)x-100\beta)$$
$$[U_0(x,\beta)]^-_\nu = \frac{1}{250}(50\beta+(72\beta+47)x)$$

$$[U_1(x,\beta)]^+_\mu = \frac{C_1x^2(75(\beta-1)+(25\beta+272)x)}{1500}$$
$$[U_1(x,\beta)]^+_\nu = -\frac{C_1x^2(75(\beta-1)+(25\beta-322)x)}{1500}$$
$$[U_1(x,\beta)]^-_\mu = \frac{C_1x^2((89\beta+594)x-300\beta)}{3000}$$
$$[U_1(x,\beta)]^-_\nu = -\frac{1}{500}C_1x^2(50\beta+3(8\beta+33)x)$$

The second-order corrections $[U_2(x,\beta)]^{\pm}_\mu$ and $[U_2(x,\beta)]^{\pm}_\nu$ are lengthy intermediate polynomials. To keep the present section readable while retaining full reproducibility, their complete expressions are collected in Appendix A.

$$[U(x)]^+_\mu = [U_0(x)]^+_\mu + [U_1(x)]^+_\mu + [U_2(x)]^+_\mu$$
$$\begin{aligned}[U(x)]^+_\mu = &\frac{1}{30000}x^2C_1(75(\beta-1)+(25\beta+272)x)+20(C_2x^2(75\beta+25\beta x+272x-75)\\ &+6(25\beta+25\beta x+22x-25))\\ &\left(1500(\beta-1)+(25\beta+272)x^3+125(\beta-1)x^2+20(25\beta+272)x\right).\end{aligned} \tag{4. 23}$$

$$\begin{aligned}[U(x)]^+_\nu &= [U_0(x)]^+_\nu + [U_1(x)]^+_\nu + [U_2(x)]^+_\nu\\ [U(x)]^+_\nu &= \frac{1}{30000} - 40x^2\\ &\quad -20\Big(C_2x^2(75\beta+25\beta x-322x-75)\\ &\qquad +6(25\beta+25\beta x-72x-25)\Big)\\ &\quad +C_1^2x^2\Big(-1500(\beta-1)+(322-25\beta)x^3\\ &\qquad -125(\beta-1)x^2+(6440-500\beta)x\Big)\\ &\quad +C_1\Big(75(\beta-1)+(25\beta-322)x\Big).\end{aligned} \tag{4. 24}$$

$$\begin{aligned}[U(x)]^-_\mu &= [U_0(x)]^-_\mu + [U_1(x)]^-_\mu + [U_2(x)]^-_\mu\\ [U(x)]^+_\mu &= \frac{1}{60000}x^2 40C_1((89\beta+594)x-300\beta)+\\ &20\left(C_2x^2(-300\beta+89\beta x+594x)+6(-100\beta+89\beta x+94x)\right)\\ &+C_1^2x^2\left(-6000\beta+(89\beta+594)x^3-500\beta x^2+20(89\beta+594)x\right).\end{aligned} \tag{4. 25}$$

$$[U(x)]^-_\nu = [U_0(x)]^-_\nu + [U_1(x)]^-_\nu + [U_2(x)]^-_\nu$$

$$
\begin{aligned}
[U(x)]_{\nu}^{-} = & \frac{1}{30000} x^2 120 C_1 x^2 (50\beta + 3(8\beta + 33)x) + \\
& 60\left(C_2 x^2 (50\beta + 24\beta x + 99x) + 2(50\beta + 72\beta x + 47x)\right) \\
& + C_1^2 x^2 \left(3000\beta + 9(8\beta + 33)x^3 + 250\beta x^2 + 180(8\beta + 33)x\right).
\end{aligned} \tag{4. 26}
$$

Now we construct residual equation as

$$[R(x)]_{\mu}^{+} = \frac{d[U(x)]_{\mu}^{+}}{dx^2} + [U(x)]_{\mu}^{+} + x,$$

$$[R(x)]_{\nu}^{+} = \frac{d[U(x)]_{\nu}^{+}}{dx^2} + [U(x)]_{\nu}^{+} + x,$$

$$[R(x)]_{\mu}^{-} = \frac{d[U(x)]_{\mu}^{-}}{dx^2} + [U(x)]_{\mu}^{-} + x,$$

$$[R(x)]_{\nu}^{-} = \frac{d[U(x)]_{\nu}^{-}}{dx^2} + [U(x)]_{\nu}^{-} + x.$$

We apply the method of least square method we find the values of C_i for above equations respectively

$$C_1 = -0.977639, \ C_2 = 0.000803516,$$

$$C_1 = -0.969591, \ C_2 = 0.0013713,$$

$$C_1 = -0.974381, \ C_2 = 0.000979578,$$

$$C_1 = -0.970767, \ C_2 = 0.00125864$$

After fixing $\beta = 0.3$ (We fix β to find the solution interim of functional from) and replacing these values in (4. 23),(4. 24),(4. 25) and (4. 26) respectively, we have

$$[U(x)]_{\mu}^{+} = 0.00882503x^5 - 0.00318593x^4 - 0.184426x^3 + 0.0399479x^2 + 0.108x - 0.08$$

$$[U(x)]_{\nu}^{+} = 0.00993379x^5 + 0.00313369x^4 - 0.210848x^3 - 0.0399082x^2 + 0.268x + 0.08$$

$$[U(x)]_{\mu}^{-} = 0.00968089x^5 - 0.00158236x^4 - 0.2036x^3 + 0.0199673x^2 + 0.2236x - 0.04$$

$$[U(x)]_{\nu}^{-} = 0.009782x^5 + 0.00157065x^4 - 0.207161x^3 - 0.0199577x^2 + 0.2456x + 0.04;$$

which are required approximated solutions are (4. 19),(4. 20),(4. 21) and (4. 22) respectively.

4.2. **Example 2.** Consider the following IFDE

$$\frac{d^4U(x)}{dx^4} - U(x) = 0, \qquad x \in [0,1], \tag{4. 27}$$

and initial conditions are defined as

$$U(0) = (\beta - 1, 1 - \beta, -2\beta, 2\beta),$$
$$U'(0) = U''(0) = U'''(0) = (\beta - 1, 1 - \beta, -2\beta, 2\beta).$$

Here we have equal membership and non-membership i.e. $\mu = \nu = \beta$. The exact solutions of this equation are defined as

$$[U(x,\beta)]^+_\mu = (\beta - 1)e^x,$$
$$[U(x,\beta)]^+_\nu = (1 - \beta)e^x,$$
$$[U(x,\beta)]^-_\mu = (-2\beta)e^x,$$
$$[U(x,\beta)]^+_\nu = (2\beta)e^x.$$

We have the following system of equations

$$\frac{d^4[U(x)]^+_\mu}{dx^4} - [U(x)]^+_\mu = 0, \qquad [U(0)]^+_\mu = [U'(0)]^+_\mu = [U''(0)]^+_\mu [U'''(0)]^+_\mu = \beta - 1, \tag{4. 28}$$

$$\frac{d^4[U(x)]^+_\nu}{dx^4} - [U(x)]^+_\nu = 0, \qquad [U(0)]^+_\nu = [U'(0)]^+_\nu = [U''(0)]^+_\nu [U'''(0)]^+_\nu = 1 - \beta, \tag{4. 29}$$

$$\frac{d^4[U(x)]^-_\mu}{dx^4} - [U(x)]^-_\mu = 0, \qquad [U(0)]^-_\mu = [U'(0)]^-_\mu = [U''(0)]^-_\mu [U'''(0)]^-_\mu = -2\beta, \tag{4. 30}$$

$$\frac{d^4[U(x)]^-_\nu}{dx^4} - [U(x)]^-_\nu = 0, \qquad [U(0)]^-_\nu = [U'(0)]^-_\nu = [U''(0)]^-_\nu [U'''(0)]^-_\nu = 2\beta. \tag{4. 31}$$

Now we apply algorithm explained in previous section.

$$[U_0(x)]^+_\mu = \frac{1}{6}(\beta - 1)\left(x^3 + 3x^2 + 6x + 6\right)$$
$$[U_0(x)]^+_\nu = \frac{1}{6}(\beta - 1)\left(x^3 + 3x^2 + 6x + 6\right),$$
$$[U_0(x)]^-_\mu = \frac{1}{6}(\beta - 1)\left(x^3 + 3x^2 + 6x + 6\right),$$
$$[U_0(x)]^-_\nu = \frac{1}{3}\beta\left(x^3 + 3x^2 + 6x + 6\right).$$

$$[U_1(x)]^+_\mu = -\frac{(\beta - 1)C_1x^4\left(x^3 + 7x^2 + 42x + 210\right)}{5040},$$
$$[U_1(x)]^+_\nu = \frac{(\beta - 1)C_1x^4\left(x^3 + 7x^2 + 42x + 210\right)}{5040},$$
$$[U_1(x)]^-_\mu = \frac{\beta C_1x^4\left(x^3 + 7x^2 + 42x + 210\right)}{2520},$$

$$[U_1(x)]^-_\nu = -\frac{\beta C_1 x^4\left(x^3+7x^2+42x+210\right)}{2520}.$$

$$\begin{aligned}[U_2(x)]^+_\mu = \frac{\beta-1}{39916800}\, x^4\Bigg(& -7920C_1\left(x^3+7x^2+42x+210\right)\\ & -7920C_2\left(x^3+7x^2+42x+210\right)\\ & +C_1^2\left(x^7+11x^6+110x^5+990x^4-7920\right)\Bigg).\end{aligned} \tag{4. 32}$$

$$\begin{aligned}[U_2(x)]^+_\nu = -\frac{\beta x^4}{19958400}\, C_1^2\Big(& x^7+11x^6+110x^5+990x^4\\ & -7920x^3-55440x^2-332640x-1663200\Big)\\ & \times\Big(7920C_2(x^3+7x^2+42x+210)\Big).\end{aligned} \tag{4. 33}$$

$$\begin{aligned}[U_2(x)]^+_\nu = -\frac{\beta x^4}{19958400}\, C_1^2\Big(& x^7+11x^6+110x^5+990x^4\\ & -7920x^3-55440x^2-332640x-1663200\Big)\\ & \times\Big(7920C_2(x^3+7x^2+42x+210)\Big).\end{aligned} \tag{4. 34}$$

$$\begin{aligned}[U_2(x)]^-_\nu = {} & \frac{\beta x^4}{19958400}\\ & -7920C_1\left(x^3+7x^2+42x+210\right)\\ & -7920C_2\left(x^3+7x^2+42x+210\right)\\ & +C_1^2\Big(x^7+11x^6+110x^5+990x^4\\ & \quad -7920x^3-55440x^2-332640x-1663200\Big).\end{aligned} \tag{4. 35}$$

$$[U(x)]^+_\mu = [U_0(x)]^+_\mu + [U_1(x)]^+_\mu + [U_2(x)]^+_\mu$$

$$\begin{aligned}[U(x)]^+_\mu = {} & \frac{\beta-1}{39916800}(-15840C_1)\left(x^3+7x^2+42x+210\right)x^4-\\ & 7920\left(C_2x^4\left(x^3+7x^2+42x+210\right)\right)\\ & -840\left(x^3+3x^2+6x+6\right)+\\ & C_1^2\left(x^7+11x^6+110x^5+990x^4-7920x^3-55440x^2-332640x-1663200\right)x^4,\end{aligned} \tag{4. 36}$$

$$[U(x)]^+_\nu = [U_0(x)]^+_\nu + [U_1(x)]^+_\nu + [U_2(x)]^+_\nu$$

$$[U(x)]_{\nu}^{+} = \frac{\beta - 1}{39916800} C_1 \left(x^3 + 7x^2 + 42x + 210\right) x^4 \\ - 7920\Bigg(C_2 x^4 \left(x^3 + 7x^2 + 42x + 210\right) \\ - 840 \left(x^3 + 3x^2 + 6x + 6\right)\Bigg) \\ + C_1^2 x^4 \Big(x^7 + 11x^6 + 110x^5 + 990x^4 \\ - 7920x^3 - 55440x^2 - 332640x - 1663200\Big). \tag{4. 37}$$

$$[U(x)]_{\mu}^{-} = [U_0(x)]_{\mu}^{-} + [U_1(x)]_{\mu}^{-} + [U_2(x)]_{\mu}^{-}$$

$$[U(x)]_{\mu}^{+} = \frac{\beta}{19958400} 15840 C_1 \left(x^3 + 7x^2 + 42x + 210\right) x^4 + \\ 7920 \left(C_2 x^4 \left(x^3 + 7x^2 + 42x + 210\right)\right) \\ - 840 \left(x^3 + 3x^2 + 6x + 6\right) + C_1^2(-(x^7 + 11x^6 + 110x^5 + 990x^4 - \\ 7920x^3 - 55440x^2 - 332640x \\ - 1663200))x^4. \tag{4. 38}$$

$$[U(x)]_{\nu}^{-} = [U_0(x)]_{\nu}^{-} + [U_1(x)]_{\nu}^{-} + [U_2(x)]_{\nu}^{-}$$

$$[U(x)]_{\nu}^{-} = \frac{\beta}{19958400} - 15840 C_1 \left(x^3 + 7x^2 + 42x + 210\right) x^4 - \\ 7920 \left(C_2 x^4 \left(x^3 + 7x^2 + 42x + 210\right)\right) \\ - 840 \left(x^3 + 3x^2 + 6x + 6\right) + \\ C_1^2 \left(x^7 + 11x^6 + 110x^5 + 990x^4 - 7920x^3 - 55440x^2 - 332640x - 1663200\right) x^4. \tag{4. 39}$$

Now we construct residual equation as

$$[R(x)]_{\mu}^{+} = \frac{d^4[U(x)]_{\mu}^{+}}{dx^4} - [U(x)]_{\mu}^{+},$$
$$[R(x)]_{\nu}^{+} = \frac{d^4[U(x)]_{\nu}^{+}}{dx^4} - [U(x)]_{\nu}^{+},$$
$$[R(x)]_{\mu}^{-} = \frac{d^4[U(x)]_{\mu}^{-}}{dx^4} - [U(x)]_{\mu}^{-},$$
$$[R(x)]_{\nu}^{-} = \frac{d^4[U(x)]_{\nu}^{-}}{dx^4} - [U(x)]_{\nu}^{-}.$$

We apply the method of least square method we find the values of C_i for above equations respectively

$$C_1 = -1.00023, \;\; C_2 = 9.395758608285905 \times 10^{-7},$$
$$C_1 = -1.00023, \;\; C_2 = 9.395758608285905 \times 10^{-7}$$

$$C_1 = -1.00023, \;\; C_2 = 9.395758608285905 \times 10^{-7}$$
$$C_1 = -1.00023, \;\; C_2 = 9.395758608285905 \times 10^{-7}$$

After fixing β and replacing these values in (4. 36),(4. 37),(4. 38) and (4. 39) respectively, we have

$$\begin{aligned}[U(x)]^+_\mu &= -0.000000075445x^{11} - 0.000000192990x^{10} - 0.0000019299x^9 - \\ &0.0000173691x^8 - 0.000138889x^7 \\ &- 0.000972221x^6 - 0.00583333x^5 - 0.0291666x^4 - 0.116667x^3 - \\ &0.35x^2 - 0.7x - 0.7.\end{aligned}$$

$$\begin{aligned}[U(x)]^+_\nu &= 0.000000017544x^{11} + 0.000000192990x^{10} + \\ &0.000001929901x^9 + 0.0000173691x^8 + 0.000138889x^7 \\ &+ 0.000972221x^6 + 0.00583333x^5 + 0.0291666x^4 + 0.116667x^3 + \\ &0.35x^2 + 0.7x + 0.7;.\end{aligned}$$

$$\begin{aligned}[U(x)]^-_\nu &= -0.0000000150381x^{11} - 0.00000165420x^{10} - 0.00000165422x^9 \\ &- 0.0000148878x^8 - 0.000119048x^7 - 0.000833333x^6 \\ &- 0.005x^5 - 0.025x^4 - 0.1x^3 - 0.3x^2 - 0.6x - 0.6.\end{aligned} \tag{4. 40}$$

$$\begin{aligned}[U(x)]^-_\nu &= 0.00000001503819190x^{11} + 0.000000165420x^{10} + \\ &0.0000016542x^9 + 0.0000148878x^8 + 0.000119048x^7 \\ &+ 0.000833333x^6 + 0.005x^5 + 0.025x^4 + 0.1x^3 + \\ &0.3x^2 + 0.6x + 0.6.\end{aligned}$$

which are required approximated solutions are (4. 28),(4. 29),(4. 30) and (4. 31) respectively.

5. Analysis of Convergence

In this section we will show that approximated solutions are converging to the exact solution. We will show this by the application of Theorem 1 to each numerical case.

5.1. **Application of Theorem 1 to Intuitionistic Fuzzy Differential Equations.** An IFDE is first decomposed into four crisp subsystems using the $\alpha\beta$-cut representation. These subsystems are:

- Upper membership subsystem: $[U]^+_\mu$,
- Lower membership subsystem: $[U]^-_\mu$,

- Upper non-membership subsystem: $[U]^{+}_{\nu}$,
- Lower non-membership subsystem: $[U]^{-}_{\nu}$.

Each subsystem shares the same differential operator equation but has different crisp initial conditions derived from the $\alpha\beta$-cuts of the intuitionistic fuzzy initial values. Because each subsystem is a crisp differential equation, Theorem 1 applies directly to it. Since the $\alpha\beta$-cut decomposition is order-preserving and the four subsystems are solved independently, the convergence of all four implies that the reconstructed intuitionistic fuzzy solution $\widetilde{U}(t)$ converges to the exact intuitionistic fuzzy solution in the $\alpha\beta$-cut sense. That is, for any cut levels $\alpha, \beta \in [0,1]$, the intervals $[\widetilde{U}]_{\alpha\beta}$ converge to $[U_{\text{exact}}]_{\alpha\beta}$ as the approximation order increases. Hence, Theorem 1 provides a complete convergence guarantee for IFDEs through component-wise application to the four crisp subsystems.

5.2. **Convergence of Example 1.** Consider the approximated solution of equation (4. 19)

$$[U(x)]^{+}_{\mu} = 0.00882503x^5 - 0.00318593x^4 - 0.184426x^3 + 0.0399479x^2 + 0.108x - 0.08.$$

If we take

$$[M(x,\beta)]^{+}_{\mu} = (0.1\beta - 0.1)\cos x + (1.088 + 0.1\beta)\sin x - x,$$

the exact solution of equation (4. 19) and $\epsilon = \|e\| = \max(Absolute\ \ Error)$ then we can see from Table 2 that

$$|[U(x)]^{+}_{\mu} - [M(x,\beta)]^{+}_{\mu}| \leq \epsilon \quad \Rightarrow \quad [U(x)]^{+}_{\mu} \to [M(x,\beta)]^{+}_{\mu}.$$

Now, Consider the approximated solution of equation (4. 20)

$$[U(x)]^{+}_{\nu} = 0.00993379x^5 + 0.00313369x^4 - 0.210848x^3 - 0.0399082x^2 + 0.268x - 0.08.$$

If we take

$$[M(x,\beta)]^{+}_{\nu} = (0.1 - 0.1\beta)\cos x + (1.288 - 0.1\beta)\sin x - x,$$

the exact solution of equation (4. 20) and $\epsilon = \|e\| = \max(Absolute\ \ Error)$ then Table 3 shows that

$$|[U(x)]^{+}_{\nu} - [M(x,\beta)]^{+}_{\nu}| \leq \epsilon \quad \Rightarrow \quad [U(x)]^{+}_{\nu} \to [M(x,\beta)]^{+}_{\nu}.$$

Now, let the approximated solution of equation (4. 21)

$$[U(x)]^{-}_{\mu} = 0.00968089x^5 - 0.00158236x^4 - 0.2036x^3 + 0.0199673x^2 + 0.2236x - 0.04$$

If we take

$$[U(x,\beta)]^{-}_{\mu} = -0.2\beta\cos x + (1.188 + 0.178\beta)\sin x - x,$$

the exact solution of equation (4. 21) and $\epsilon = \|e\| = \max(Absolute \;\; Error)$ then Table 4 shows that

$$|[U(x)]^-_\mu - [M(x,\beta)]^-_\mu| \le \epsilon \quad \Rightarrow \quad [U(x)]^-_\mu \to [M(x,\beta)]^-_\mu.$$

Now, Suppose the approximated solution of equation (4. 22)

$[U(x)]^-_\nu = 0.009782x^5 + 0.00157065x^4 - 0.207161x^3 - 0.0199577x^2 + 0.2456x + 0.04;$

If we take

$$[M(x,\beta)]^-_\nu = 0.2\beta\cos x + (0.188 + 0.288\beta)\sin x - x, .$$

the exact solution of equation (4. 22) and $\epsilon = \|e\| = \max(Absolute \;\; Error)$ then Table 5 shows that

$$|[U(x)]^-_\nu - [M(x,\beta)]^-_\nu| \le \epsilon \quad \Rightarrow \quad [U(x)]^-_\nu \to [M(x,\beta)]^-_\nu.$$

5.3. **Convergence of Example 2.** Consider the approximated solution of equation (4. 28)

$$\begin{aligned}[U(x)]^+_\mu = {} & -0.000000075445x^{11} - 0.000000192990x^{10} - \\ & 0.0000019299x^9 - 0.0000173691x^8 - 0.000138889x^7 \\ & - 0.000972221x^6 - 0.00583333x^5 - 0.0291666x^4 - 0.116667x^3 - \\ & 0.35x^2 - 0.7x - 0.7.\end{aligned}$$

If we take

$$[M(x,\beta)]^+_\mu = (\beta - 0.1)e^x,$$

the exact solution of equation (4. 28) and $\epsilon = \|e\| = \max(Absolute \;\; Error)$ then we can see from Table 6 that

$$|[U(x)]^+_\mu - [M(x,\beta)]^+_\mu| \le \epsilon \quad \Rightarrow \quad [U(x)]^+_\mu \to [M(x,\beta)]^+_\mu.$$

Now, Consider the approximated solution of equation (4. 29)

$$\begin{aligned}[U(x)]^+_\nu = {} & 0.000000017544x^{11} + 0.000000192990x^{10} + \\ & 0.000001929901x^9 + 0.0000173691x^8 + 0.000138889x^7 \\ & + 0.000972221x^6 + 0.00583333x^5 + 0.0291666x^4 + 0.116667x^3 + \\ & 0.35x^2 + 0.7x + 0.7; .\end{aligned}$$

If we take

$$[M(x,\beta)]^+_\nu = (1 - \beta)e^x,$$

the exact solution of equation (4. 29) and $\epsilon = \|e\| = \max(Absolute \;\; Error)$ then we can see from Table 7 that

$$|[U(x)]^+_\nu - [M(x,\beta)]^+_\nu| \leq \epsilon \quad \Rightarrow \quad [U(x)]^+_\nu \rightarrow [M(x,\beta)]^+_\nu.$$

Now, let the approximated solution of equation (4. 30)

$$\begin{aligned}[U(x)]^-_\nu = &-0.0000000150381x^{11} - 0.00000165420x^{10} - 0.00000165422x^9 \\ &- 0.0000148878x^8 - 0.000119048x^7 - 0.000833333x^6 \\ &- 0.005x^5 - 0.025x^4 - 0.1x^3 - 0.3x^2 - 0.6x - 0.6.\end{aligned} \tag{5. 41}$$

If we take

$$[M(x,\beta)]^-_\mu = (-2\beta)e^x,$$

the exact solution of equation (4. 30) and $\epsilon = \|e\| = \max(Absolute \;\; Error)$ then we can see from Table 8 that

$$|[U(x)]^-_\mu - [M(x,\beta)]^-_\mu| \leq \epsilon \quad \Rightarrow \quad [U(x)]^-_\mu \rightarrow [M(x,\beta)]^-_\mu.$$

Now, Suppose the approximated solution of equation (4. 31)

$$\begin{aligned}[U(x)]^-_\nu = &\; 0.000000015038191 90x^{11}+ \\ &0.000000165420x^{10} + 0.0000016542x^9 + 0.0000148878x^8 + 0.000119048x^7 \\ &+ 0.000833333x^6 + 0.005x^5 + 0.025x^4 + 0.1x^3 + 0.3x^2 + 0.6x + 0.6.\end{aligned}$$

If we take

$$[M(x,\beta)]^+_\nu = (2\beta)e^x.$$

the exact solution of equation (4. 22) and $\epsilon = \|e\| = \max(Absolute \;\; Error)$ then then we can see from Table 9 that

$$|[U(x)]^-_\nu - [M(x,\beta)]^-_\nu| \leq \epsilon \quad \Rightarrow \quad [U(x)]^-_\nu \rightarrow [M(x,\beta)]^-_\nu.$$

6. Analysis of Stability

In this section, we first recall the concept of Hyers-Ulam stability for crisp ordinary differential equations, then extend it to IFDE. We verify the stability for our numerical examples.

6.1. **Hyers-Ulam Stability.** Suppose the differential equation of the form

$$\psi^n(x) \; = \mathfrak{H}\bigg(\psi, \psi', ..., \psi^{n-1}, G(x), x\bigg). \tag{6. 42}$$

Theorem 6.1.1. *The equation* (6. 42) *is Hyers-Ulam stable if it satisfies the following conditions* [27, 7]*:*

(1) $|\psi^n(x) - \mathfrak{H}(\psi, \psi', ..., \psi^{n-1}, G(x), x)| \leq \epsilon.$
(2) $|\psi_e(x) - \psi(x)| \leq L.\epsilon$

where $\epsilon > 0$ *,* L *is Hyers Ulam stability constant, and* $\psi_e(x)$ *and* $\psi(x)$ *are exact and approximated solutions of equation* (6. 42) *respectively* [6, 10, 32].

In words, if an approximate solution satisfies the differential equation within a small error ϵ, then it must be close to a true solution, and the closeness is proportional to ϵ. This property guarantees robustness against small perturbations or numerical errors.

6.2. **Hyers-Ulam Stability for Intuitionistic Fuzzy Differential Equations.** Let the IFDE be decomposed via $\alpha\beta$-cuts into four crisp subsystems:

$$\frac{d^n[U]^+_\mu}{dx^n} = \mathfrak{H}^+_\mu\big(x, [U]^+_\mu, \dots\big),$$
$$\frac{d^n[U]^-_\mu}{dx^n} = \mathfrak{H}^-_\mu\big(x, [U]^-_\mu, \dots\big),$$
$$\frac{d^n[U]^+_\nu}{dx^n} = \mathfrak{H}^+_\nu\big(x, [U]^+_\nu, \dots\big),$$
$$\frac{d^n[U]^-_\nu}{dx^n} = \mathfrak{H}^-_\nu\big(x, [U]^-_\nu, \dots\big).$$

Definition 6.2.1 (Hyers-Ulam Stability for IFDE)**.** *The IFDE is called **Hyers-Ulam stable** if there exist positive constants* $L^+_\mu, L^-_\mu, L^+_\nu, L^-_\nu$ *such that for any positive numbers* $\epsilon^+_\mu, \epsilon^-_\mu, \epsilon^+_\nu, \epsilon^-_\nu$ *and for any functions* $[U]^+_\mu, [U]^-_\mu, [U]^+_\nu, [U]^-_\nu$ *satisfying the inequalities*

$$\left|\frac{d^n[U]^+_\mu}{dx^n} - \mathfrak{H}^+_\mu\right| \leq \epsilon^+_\mu,$$
$$\left|\frac{d^n[U]^-_\mu}{dx^n} - \mathfrak{H}^-_\mu\right| \leq \epsilon^-_\mu,$$
$$\left|\frac{d^n[U]^+_\nu}{dx^n} - \mathfrak{H}^+_\nu\right| \leq \epsilon^+_\nu,$$
$$\left|\frac{d^n[U]^-_\nu}{dx^n} - \mathfrak{H}^-_\nu\right| \leq \epsilon^-_\nu,$$

there exist exact solutions $[U_e]^+_\mu, [U_e]^-_\mu, [U_e]^+_\nu, [U_e]^-_\nu$ of the respective subsystems such that

$$\left|[U]^+_\mu(x) - [U_e]^+_\mu(x)\right| \le L^+_\mu\, \epsilon^+_\mu,$$

$$\left|[U]^-_\mu(x) - [U_e]^-_\mu(x)\right| \le L^-_\mu\, \epsilon^-_\mu,$$

$$\left|[U]^+_\nu(x) - [U_e]^+_\nu(x)\right| \le L^+_\nu\, \epsilon^+_\nu,$$

$$\left|[U]^-_\nu(x) - [U_e]^-_\nu(x)\right| \le L^-_\nu\, \epsilon^-_\nu,$$

for all x in the domain. The constants $L^+_\mu, L^-_\mu, L^+_\nu, L^-_\nu$ are called the ***Hyers-Ulam stability constants*** *of the IFDE.*

In the context of IFDE, Hyers-Ulam stability means that if the model's parameters, initial conditions, or membership/non-membership functions are slightly uncertain (e.g., due to measurement errors, subjective expert judgement, or rounding in numerical computations), the resulting intuitionistic fuzzy solution will not deviate catastrophically. Instead, the error in the solution is bounded by a constant multiple of the input perturbation. This is crucial for real-world applications: in financial forecasting, small changes in market data should not lead to wildly different predictions; in biomedical engineering, patient-specific parameters are known only approximately; in control systems, sensor noise must not destabilise the controller. By verifying Hyers-Ulam stability, we guarantee that the OHAM approximate solution is reliable and robust, making it suitable for high-stakes decision-making under uncertainty.

6.3. **Stability of Example 1.** Consider

$$\frac{d^2[U(x)]^+_\mu}{dx^2}+[U(x)]^+_\mu = -x, \qquad [U(0)]^+_\mu = 0.1\beta-0.1 \qquad [U'(0)]^+_\mu = 0.088+0.1\beta,$$

here

$$\mathfrak{H} = -[U(x)]^+_\mu - x.$$

Now

$$\left|\frac{d^2[U(x)]^+_\mu}{dx^2} - \mathfrak{H}\right| \le \epsilon$$

where $\epsilon = max\left|\frac{d^2[U(x)]^+_\mu}{dx^2} - \mathfrak{H}\right| = 0.00077024$ and

$$|[U(x)_e]^+_\mu - [U(x)]^+_\mu| \le L.\epsilon = 0.0000353368.$$

Now suppose

$$\frac{d^2[U(x)]^+_\nu}{dx^2}+[U(x)]^+_\nu = -x, \qquad [U(0)]^+_\nu = 0.1-0.1\beta \qquad [U'(0)]^+_\nu = .288-0.1\beta,$$

here

$$\mathfrak{H} = -[U(x)]^+_\nu - x.$$

Now

$$\left|\frac{d^2[U(x)]^+_\nu}{dx^2} - \mathfrak{H}\right| \leq \epsilon$$

where $\epsilon = max\left|\frac{d^2[U(x)]^+_\nu}{dx^2} - \mathfrak{H}\right| = 0.00168696$ and

$$|[U(x)_e]^+_\nu - [U(x)]^+_\nu| \leq L.\epsilon = 0.000101816.$$

Let

$$\frac{d^2[U(x)]^-_\mu}{dx^2} + [U(x)]^-_\mu = -x, \qquad [U(0)]^-_\mu = -0.2\beta \qquad [U'(0)]^-_\mu = 0.188 + 0.178\beta,$$

Now

$$\left|\frac{d^2[U(x)]^-_\mu}{dx^2} - \mathfrak{H}\right| \leq \epsilon$$

where $\epsilon = max\left|\frac{d^2[U(x)]^-_\mu}{dx^2} - \mathfrak{H}\right| = 0.00102991$ and

$$|[U(x)_e]^-_\mu - [U(x)]^-_\mu| \leq L.\epsilon = 0.0000542958.$$

Consider

$$\frac{d^2[U(x)]^-_\nu}{dx^2} + [U(x)]^-_\nu = -x, \qquad [U(0)]^-_\nu = 0.2\beta \qquad [U'(0)]^-_\nu = 0.188 + 0.288\beta.$$

here

$$\mathfrak{H} = -[U(x)]^-_\nu - x.$$

Now

$$\left|\frac{d^2[U(x)]^-_\nu}{dx^2} - \mathfrak{H}\right| \leq \epsilon$$

where $\epsilon = max\left|\frac{d^2[U(x)]^-_\nu}{dx^2} - \mathfrak{H}\right| = 0.00144035$ and

$$|[U(x)_e]^-_\nu - [U(x)]^-_\nu| \leq L.\epsilon = 0.0000852597.$$

As the above system of equations fulfils the criteria of Theorem 2, So the system of equations in Example 1 is Hyers-Ulam stable.

6.4. **Stability of Example 2.** Consider

$$\frac{d^4[U(x)]^+_\mu}{dx^4} - [U(x)]^+_\mu = 0, \qquad [U(0)]^+_\mu = [U'(0)]^+_\mu = [U''(0)]^+_\mu [U'''(0)]^+_\mu = \beta - 1,$$

here

$$\mathfrak{H} = [U(x)]^+_\mu.$$

Now

$$\left|\frac{d^4[U(x)]^+_\mu}{dx^4} - \mathfrak{H}\right| \leq \epsilon$$

where $\epsilon = max\left|\frac{d^4[U(x)]^+_\mu}{dx^4} - \mathfrak{H}\right| = 0.0000055011$ and

$$|[U(x)_e]^+_\mu - [U(x)]^+_\mu| \leq L.\epsilon = 1.9615108648 \times 10^{-8}.$$

Now suppose

$$\frac{d^4[U(x)]^+_\nu}{dx^4} - [U(x)]^+_\nu = 0, \qquad [U(0)]^+_\nu = [U'(0)]^+_\nu = [U''(0)]^+_\nu [U'''(0)]^+_\nu = 1-\beta,$$

here

$$\mathfrak{H} = [U(x)]^+_\nu.$$

Now

$$\left|\frac{d^4[U(x)]^+_\nu}{dx^4} - \mathfrak{H}\right| \leq \epsilon$$

where $\epsilon = max\left|\frac{d^4[U(x)]^+_\nu}{dx^4} - \mathfrak{H}\right| = 0.0000055011$ and

$$|[U(x)_e]^+_\nu - [U(x)]^+_\nu| \leq L.\epsilon = 2.84417989160 \times 10^{-8}.$$

Let

$$\frac{d^4[U(x)]^-_\mu}{dx^4} - [U(x)]^-_\mu = 0, \qquad [U(0)]^-_\mu = [U'(0)]^-_\mu = [U''(0)]^-_\mu [U'''(0)]^-_\mu = -2\beta,$$

Now

$$\left|\frac{d^4[U(x)]^-_\mu}{dx^4} - \mathfrak{H}\right| \leq \epsilon$$

where $\epsilon = max\left|\frac{d^4[U(x)]^-_\mu}{dx^4} - \mathfrak{H}\right| = 0.00000227931$ and

$$|[U(x)_e]^-_\mu - [U(x)]^-_\mu| \leq L.\epsilon = 6.48959774807 \times 10^{-8}.$$

Consider

$$\frac{d^4[U(x)]^-_\nu}{dx^4} - [U(x)]^-_\nu = 0, \qquad [U(0)]^-_\nu = [U'(0)]^-_\nu = [U''(0)]^-_\nu [U'''(0)]^-_\nu = 2\beta,$$

here

$$\mathfrak{H} = [U(x)]^-_\nu.$$

Now

$$\left|\frac{d^4[U(x)]^-_\nu}{dx^4} - \mathfrak{H}\right| \leq \epsilon$$

where $\epsilon = max\left|\frac{d^4[U(x)]^-_\nu}{dx^4} - \mathfrak{H}\right| = 0.000002279314200$ and

$$|[U(x)_e]^-_\nu - [U(x)]^-_\nu| \leq L.\epsilon = 2.437869062177 \times 10^{-8}.$$

As the above system of equations satisfies the both conditions of Theorem 2, So the system of equations in Example 2 is Hyers-Ulam stable.

7. Results and Discussion

In this section, we present and discuss the numerical results obtained from the OHAM implementation for both examples. We compare OHAM with other methods in terms of accuracy and efficiency, discuss computational issues, and point out the limitations of the proposed approach. The convergence and stability properties established in Sections 5 and 6 are also interpreted in light of the numerical findings.

7.1. **Accuracy of OHAM.** In this section we will discuss the accuracy and efficiency of OHAM.

Example 1: Here will discuss the results of example 1. We will find absolute error by fixing $\beta = 0.2$. The Absolute error estimated by OHAM for equations 4. 19 (1.a), 4. 20 (1.b), 4. 21 (1.c) and 4. 22 (1.d) has been drafted in Tables 2,3, 4 and 5 respectively. We can observe that $\|e\| = max(Absolute\ error)$ for these cases as:

- $\|e\| = 0.0000353368.$
- $\|e\| = 0.000101816.$
- $\|e\| = 0.0000542958$
- $\|e\| = 0.0000852597.$

These errors are very small, which shows that OHAM gives very accurate approximations with only second-order expansions. The errors are uniformly small in the whole domain, no boundary layer problems are present.

Figures: Figures 2-5 illustrate the excellent agreement between the exact and OHAM solutions for each subsystem in Example 1. The OHAM solution is compared with the exact solution in Figure 2(a), where both solutions for case (1.a) are observed to coincide closely. The absolute error, illustrated in Figure 2(b), is negligible, confirming the high accuracy of the method. Similarly, in Figure 3(a), the approximate solution for case (1.b) is plotted alongside the exact solution, demonstrating excellent agreement. The corresponding absolute error, shown in Figure 3(b), remains very small, further validating the accuracy of OHAM. Figure 4(a) displays the OHAM and exact solutions for case (1.c), which again show near-perfect alignment. The absolute error plot in Figure 4(b) confirms that the discrepancy is minimal. In Figure 5(a), the OHAM approximated solution is compared with the exact solution for case (1.d), and both exhibit a close match. The absolute error, presented in Figure 5(b), is also very small, indicating the methods robustness and reliability.

TABLE 2. Absolute and relative error estimation of Equation 4. 19 (Example 1)

x	OHAM	Exact	Absolute Error	Relative Error
0.	-0.08	-0.08	0	0
0.1	-0.0689852	-0.0689849	0.000000270	3.914×10^{-6}
0.2	-0.0582798	-0.0582797	0.0000000590	1.012×10^{-6}
0.3	-0.0489886	-0.0489905	0.00000197559	4.033×10^{-5}
0.4	-0.0422028	-0.0422094	0.0000065613	1.554×10^{-4}
0.5	-0.0389896	-0.0390031	0.0000134905	3.459×10^{-4}
0.6	-0.0403814	-0.040403	0.00002155	5.334×10^{-4}
0.7	-0.0473654	-0.0473942	0.0000288075	6.078×10^{-4}
0.8	-0.0608726	-0.060906	0.0000333583	5.477×10^{-4}
0.9	-0.081768	-0.0818026	0.0000346298	4.233×10^{-4}
1.	-0.110839	-0.110874	0.0000353368	3.187×10^{-4}

TABLE 3. Absolute and relative error estimation of Equation 4. 20 (Example 1)

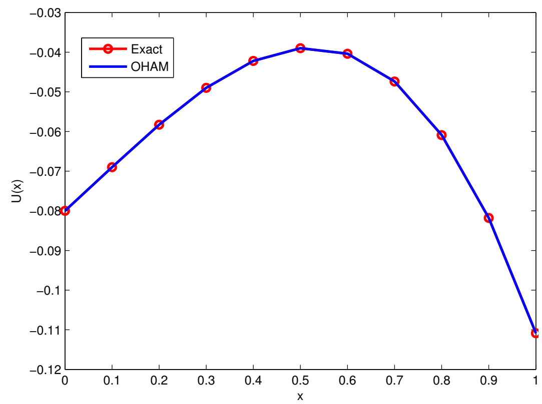
x	OHAM	Exact	Absolute Error	Relative Error
0.	0.08	0.08	0.	0
0.1	0.10619	0.106189	0.00000137748	1.297×10^{-5}
0.2	0.130325	0.130318	0.0000070438	5.405×10^{-5}
0.3	0.151165	0.151147	0.0000183476	1.214×10^{-4}
0.4	0.167502	0.167467	0.0000350226	2.091×10^{-4}
0.5	0.178173	0.178118	0.0000550455	3.090×10^{-4}
0.6	0.182068	0.181994	0.0000749437	4.118×10^{-4}
0.7	0.178146	0.178055	0.0000906694	5.092×10^{-4}
0.8	0.165443	0.165344	0.0000991498	5.997×10^{-4}
0.9	0.143088	0.142987	0.000100617	7.037×10^{-4}
1.	0.110311	0.110209	0.000101816	9.238×10^{-4}

TABLE 4. Absolute and relative error estimation of Equation 4. 21 (Example 1)

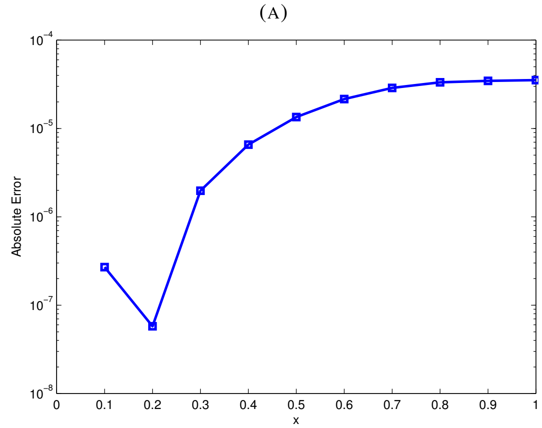
x	OHAM	Exact	Absolute Error	Relative Error
0.	-0.04	-0.04	0	0
0.1	-0.017644	-0.017644	0	0
0.2	0.00389046	0.00388913	0.00000132961	3.419×10^{-4}
0.3	0.0233906	0.0233851	0.00000550522	2.354×10^{-4}
0.4	0.039663	0.0396498	0.0000131634	3.320×10^{-4}
0.5	0.0515455	0.0515218	0.0000237001	4.600×10^{-4}
0.6	0.0579184	0.0578831	0.0000352898	6.097×10^{-4}
0.7	0.0577164	0.0576711	0.0000453346	7.861×10^{-4}
0.8	0.0499401	0.0498886	0.0000514626	1.032×10^{-3}
0.9	0.0336676	0.0336144	0.0000531829	1.582×10^{-3}
1.	0.0080661	0.0080118	0.0000542958	6.777×10^{-3}

TABLE 5. Absolute and relative error estimation of Equation 4. 22 (Example 1)

x	OHAM	Exact	Absolute Error	Relative Error
0.	0.04	0.04	0	0
0.1	0.0641535	0.0641527	8.458×10^{-7}	1.318×10^{-5}
0.2	0.08667	0.0866652	4.8618×10^{-6}	5.610×10^{-5}
0.3	0.106327	0.106313	0.0000135121	1.271×10^{-4}
0.4	0.121929	0.121902	0.0000268882	2.206×10^{-4}
0.5	0.132319	0.132276	0.0000435029	3.289×10^{-4}
0.6	0.136393	0.136332	0.0000604886	4.437×10^{-4}
0.7	0.133106	0.133031	0.0000743174	5.586×10^{-4}
0.8	0.121489	0.121407	0.0000821505	6.767×10^{-4}
0.9	0.10066	0.100576	0.000083923	8.344×10^{-4}
1.	0.0698336	0.0697484	0.0000852597	1.222×10^{-3}

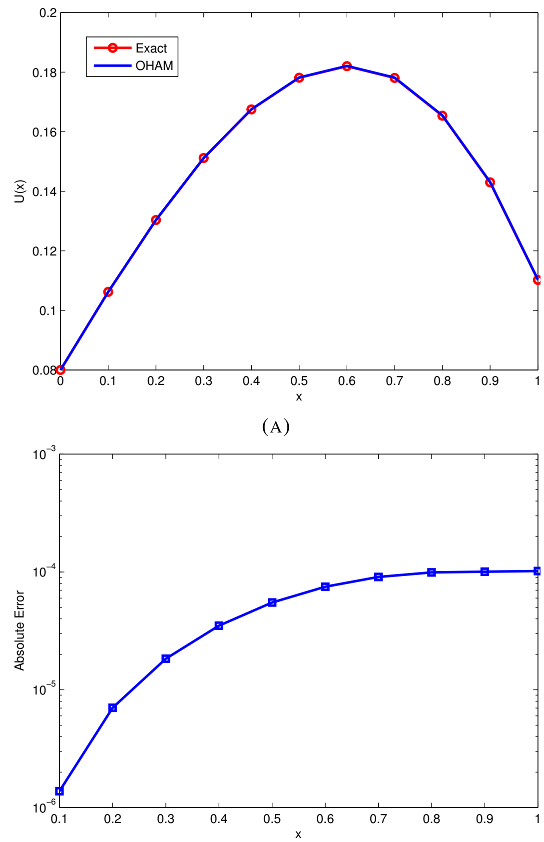


(A)



(B)

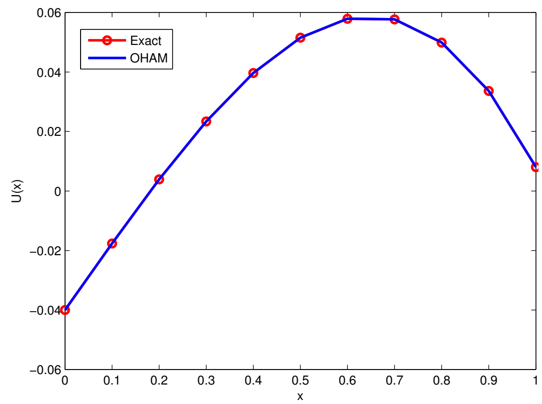
FIGURE 2. Case (1.a): comparison of exact and OHAM approximate solutions for the upper membership component $[U]_\mu^+$ in Example 1 ($\beta = 0.2$). (A) Exact solution (red line) and OHAM solution (blue line) over $x \in [0, 1]$ - the curves are nearly indistinguishable. (B) Absolute error between the two solutions is very small, confirming the high accuracy of OHAM.



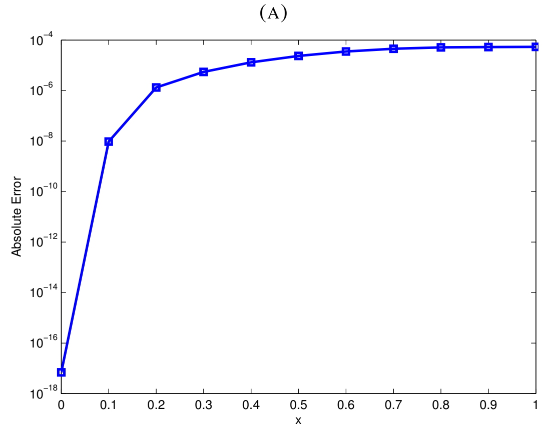
(A)

(B)

FIGURE 3. Case (1.b): comparison of exact and OHAM approximate solutions for the upper nonmembership component $[U]_{\nu}^{+}$ in Example 1 ($\beta = 0.2$). (A) Exact solution (red line) and OHAM solution (blue line) over $x \in [0, 1]$ - the curves are nearly indistinguishable. (B) Absolute error between the two solutions is very small, confirming the high accuracy of OHAM.

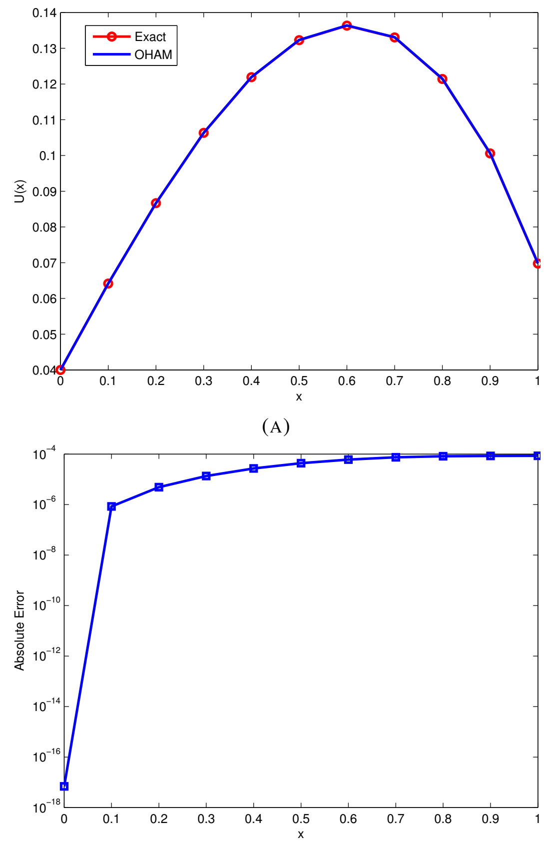


(A)



(B)

FIGURE 4. Case (1.c): comparison of exact and OHAM approximate solutions for the lower membership component $[U]_{\mu}^{-}$ in Example 1 ($\beta = 0.2$). (A) Exact solution (red line) and OHAM solution (blue line) over $x \in [0, 1]$ - the curves are nearly indistinguishable. (B) Absolute error between the two solutions is very small, confirming the high accuracy of OHAM.



(A)

(B)

FIGURE 5. Case (1.d): comparison of exact and OHAM approximate solutions for the lower nonmembership component $[U]_{\nu}^{-}$ in Example 1 ($\beta = 0.2$). (A) Exact solution (red line) and OHAM solution (blue line) over $x \in [0, 1]$ - the curves are nearly indistinguishable. (B) Absolute error between the two solutions is very small, confirming the high accuracy of OHAM.

Example 2: Here we discuss the results of example 2. We will find absolute error by fixing $\beta = 0.3$. The Absolute error estimated by OHAM for equations 4. 28 (2.a), 4. 29 (2.b), 4. 30 (2.c) and 4. 31 (2.d) has been drafted in Tables 6,7,8 and 9 respectively. We can observe that $\|e\| = max(Absolute\ error)$ for these cases as:

- $\|e\| = 1.9615108648 \times 10^{-8}$,
- $\|e\| = 2.84417989160 \times 10^{-8}$,
- $\|e\| = 6.48959774807 \times 10^{-8}$

- $\|e\| = 2.437869062177 \times 10^{-8}$.

This reflects the rapid convergence of OHAM for linear problems with smooth solutions. The numerical precision (double precision arithmetic, tolerance ensures that these error values are not rounding artifacts.

Figures: The OHAM solution is compared with the exact solution in Figure 6(a) for case (2.a), where both solutions are seen to closely coincide. The corresponding absolute error, shown in Figure 6(b), is negligible, indicating high accuracy. In Figure 7(a), the approximate solution for case (2.b) is plotted alongside the exact solution, revealing that the OHAM solution effectively follows the exact behavior. The absolute error, illustrated in Figure 7(b), remains very small, confirming the precision of the method. Figure 8(a) shows that the OHAM and exact solutions for case (2.c) are nearly identical. The absolute error, plotted in Figure 8(b), is again negligible, demonstrating the consistency of methods. Finally, in Figure 9(a), the OHAM-approximated solution is compared with the exact solution for case (2.d), exhibiting a strong match. The associated absolute error, displayed in Figure 9(b), is very small, further supporting the accuracy and effectiveness of the OHAM approach.

TABLE 6. Absolute and relative error estimation of Equation 4. 28 (Example 2)

x	OHAM	Exact	Absolute Error	Relative Error
0.	-0.7	-0.7	0.	0
0.1	-0.77362	-0.77362	$2.9543034 \times 10^{-12}$	3.819×10^{-12}
0.2	-0.854982	-0.854982	4.821931×10^{-11}	5.640×10^{-11}
0.3	-0.944901	-0.944901	2.48777×10^{-10}	2.633×10^{-10}
0.4	-1.04428	-1.04428	$7.991949324 \times 10^{-10}$	7.653×10^{-10}
0.5	-1.1541	-1.1541	$1.973928354104 \times 10^{-9}$	1.710×10^{-9}
0.6	-1.27548	-1.27548	$4.1108205728 \times 10^{-9}$	3.223×10^{-9}
0.7	-1.40963	-1.40963	$7.57119700267 \times 10^{-9}$	5.371×10^{-9}
0.8	-1.55788	-1.55788	$1.26734065464 \times 10^{-8}$	8.135×10^{-9}
0.9	-1.72172	-1.72172	$1.9615108648 \times 10^{-8}$	1.139×10^{-8}
1.	-1.9028	-1.9028	$2.84417989160 \times 10^{-9}$	1.495×10^{-9}

TABLE 7. Absolute and relative error estimation of Equation 4. 29 (Example 2)

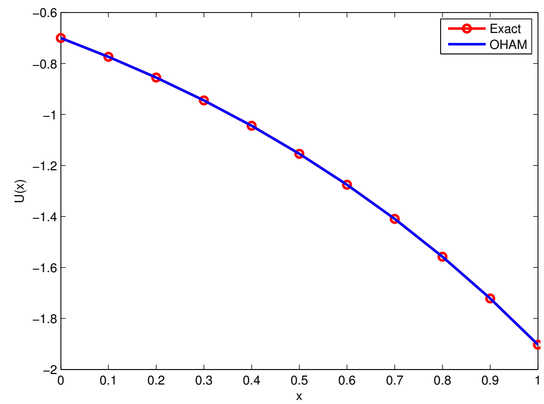
x	OHAM	Exact	Absolute Error	Relative Error
0.	0.7	0.7	0.	0
0.1	0.77362	0.77362	$2.954303468522 \times 10^{-12}$	3.819×10^{-12}
0.2	0.854982	0.854982	$4.8219317427822 \times 10^{-11}$	5.640×10^{-11}
0.3	0.944901	0.944901	$2.48777110023468 \times 10^{-10}$	2.633×10^{-10}
0.4	1.04428	1.04428	$7.99194932454 \times 10^{-10}$	7.653×10^{-10}
0.5	1.1541	1.1541	$1.973928354104 \times 10^{-9}$	1.710×10^{-9}
0.6	1.27548	1.27548	$4.11082057283 \times 10^{-9}$	3.223×10^{-9}
0.7	1.40963	1.40963	$7.5711970026 \times 10^{-9}$	5.371×10^{-9}
0.8	1.55788	1.55788	$1.26734065464 \times 10^{-8}$	8.135×10^{-9}
0.9	1.72172	1.72172	$1.9615108648 \times 10^{-8}$	1.139×10^{-8}
1.	1.9028	1.9028	$2.84417989160 \times 10^{-8}$	1.495×10^{-8}

TABLE 8. Absolute and relative error estimation of Equation 4. 30 (Example 2)

x	OHAM	Exact	Absolute Error	Relative Error
0.	-0.6	-0.6	0.	0
0.1	-0.663103	-0.663103	$2.5321966745 \times 10^{-12}$	3.819×10^{-12}
0.2	-0.732842	-0.732842	$4.13309386715 \times 10^{-11}$	5.640×10^{-11}
0.3	-0.809915	-0.809915	$.1323764975988 \times 10^{-10}$	1.634×10^{-11}
0.4	-0.895095	-0.895095	$6.850243705613 \times 10^{-10}$	7.653×10^{-10}
0.5	-0.989233	-0.989233	$1.69193870025 \times 10^{-9}$	1.710×10^{-9}
0.6	-1.09327	-1.09327	$3.5235607764 \times 10^{-9}$	3.223×10^{-9}
0.7	-1.20825	-1.20825	$6.48959774807 \times 10^{-8}$	5.371×10^{-8}
0.8	-1.33532	-1.33532	$1.08629201189 \times 10^{-8}$	8.135×10^{-9}
0.9	-1.47576	-1.47576	$1.68129514754 \times 10^{-8}$	1.139×10^{-8}
1.	-1.63097	-1.63097	$2.437868618088 \times 10^{-8}$	1.495×10^{-8}

TABLE 9. Absolute and relative error estimation of Equation 4. 31 (Example 2)

x	OHAM	Exact	Absolute Error	Relative Error
0.	0.6	0.6	0	0
0.1	0.663103	0.663103	0	0
0.2	0.732842	0.732842	$4.133104969383 \times 10^{-11}$	5.640×10^{-11}
0.3	0.809915	0.809915	$2.132378718044 \times 10^{-10}$	2.633×10^{-10}
0.4	0.895095	0.895095	$6.850245926059 \times 10^{-10}$	7.653×10^{-10}
0.5	0.989233	0.989233	$1.6919390333214 \times 10^{-9}$	1.710×10^{-9}
0.6	1.09327	1.09327	$3.52356166466449 \times 10^{-9}$	3.223×10^{-9}
0.7	1.20825	1.20825	$6.4895988582946 \times 10^{-9}$	5.371×10^{-9}
0.8	1.33532	1.33532	$1.0862921895338 \times 10^{-8}$	8.135×10^{-9}
0.9	1.47576	1.47576	$1.681295458411 \times 10^{-8}$	1.139×10^{-8}
1.	1.63097	1.63097	$2.437869062177 \times 10^{-8}$	1.495×10^{-8}



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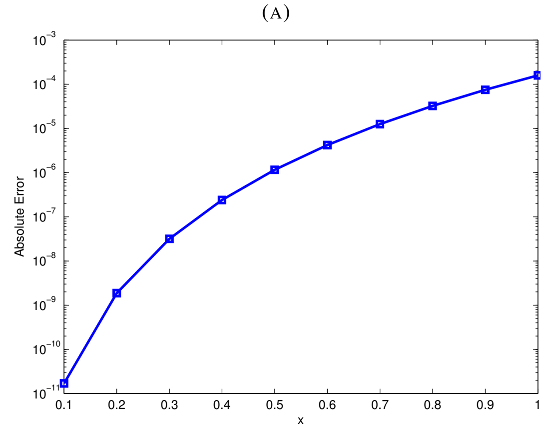
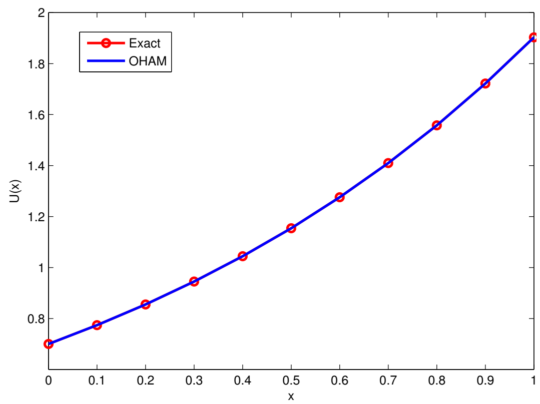
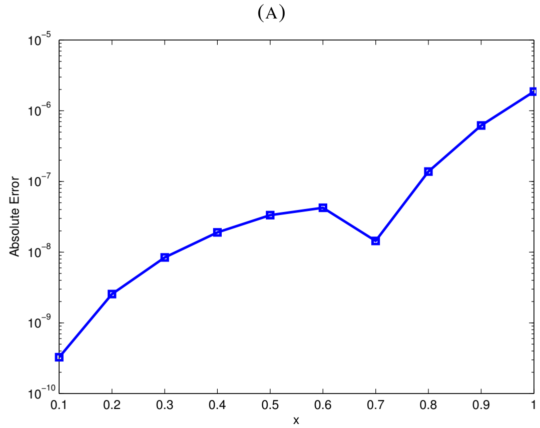


(B)

FIGURE 6. Case (2.a): comparison of exact and OHAM approximate solutions for the upper membership component $[U]^+_\mu$ in Example 2($\beta = 0.3$). (A) Exact solution (red line) and OHAM solution (blue line) over $x \in [0, 1]$ - the curves are nearly indistinguishable. (B) Absolute error between the two solutions is very small, confirming the high accuracy of OHAM.

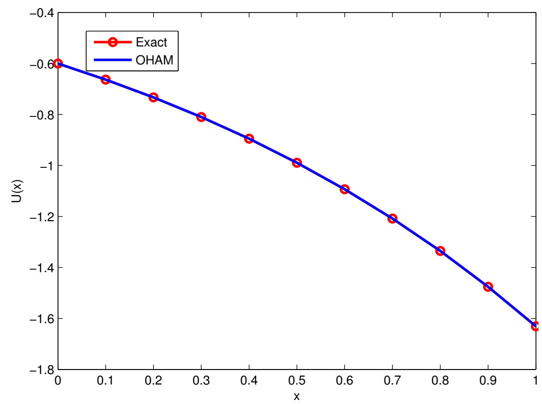


(A)

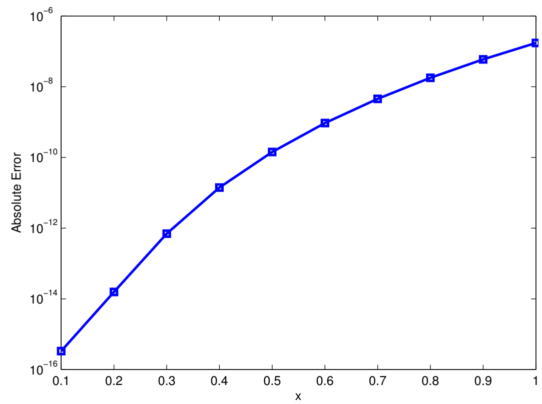


(B)

FIGURE 7. Case (2.b): comparison of exact and OHAM approximate solutions for the upper nonmembership component $[U]_{\nu}^{+}$ in Example 2($\beta = 0.3$). (A) Exact solution (red line) and OHAM solution (blue line) over $x \in [0, 1]$ - the curves are nearly indistinguishable. (B) Absolute error between the two solutions is very small, confirming the high accuracy of OHAM.

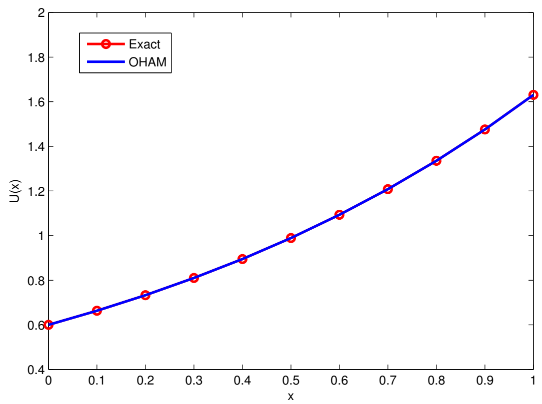


(A)

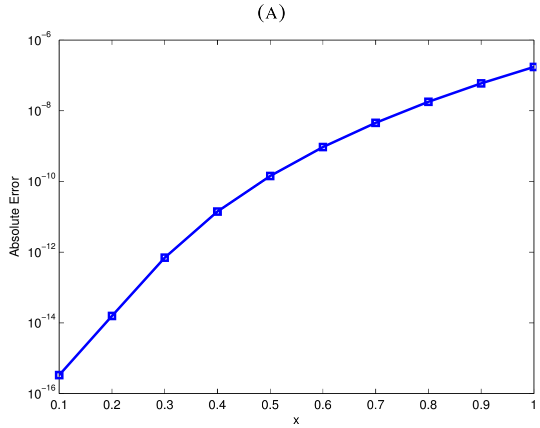


(B)

FIGURE 8. Case (2.c): comparison of exact and OHAM approximate solutions for the lower membership component $[U]_{\mu}^{-}$ in Example 2($\beta = 0.3$). (A) Exact solution (red line) and OHAM solution (blue line) over $x \in [0, 1]$ - the curves are nearly indistinguishable. (B) Absolute error between the two solutions is very small, confirming the high accuracy of OHAM.



(A)



(B)

FIGURE 9. Case (2.d): comparison of exact and OHAM approximate solutions for the lower nonmembership component $[U]_{\nu}^{-}$ in Example 2($\beta = 0.3$). (A) Exact solution (red line) and OHAM solution (blue line) over $x \in [0, 1]$ - the curves are nearly indistinguishable. (B) Absolute error between the two solutions is very small, confirming the high accuracy of OHAM.

7.2. **Comparison with other methods.** Table 10 is a comparison of the maximum absolute error($\|e\|$) by our OHAM and the maximum absolute error ($\|e\|$) by Nirmala et al. [21] with Euler, Modified Euler and fourth-order Runge-Kutta methods for another intuitionistic fuzzy initial value problem (radioactive decay, non-membership function under (1,2)-differentiability). Although the equations are not identical, the comparison shows that the OHAM has an error of order which is three orders of magnitude smaller than the errors of the classical numerical methods for that problem. Furthermore, OHAM gives a closed form analytical approximation, while Euler, Modified Euler and RK4 only provide discrete numerical values. This reinforces the conclusion that OHAM is a very accurate and competitive method for solving IFDEs.

TABLE 10. Comparison of $\|e\|$

Methods	$\|e\|$
OHAM	0.000101816
RK4	0.309790333
Euler	0.498157473
Modified Euler	0.30475884

7.3. **Limitation of OHAM.** OHAM, however, has some drawbacks. The least-squares optimization for the auxiliary parameters C_i can become ill-conditioned for high-order approximations ($m > 5$). For strongly nonlinear IFDEs, the a priori verification of convergence condition is usually difficult and the algebraic complexity grows markedly with the order. Moreover, OHAM is not a black-box method, it needs a reasonable initial guess and symbolic computation for its efficient implementation. We have to fix the membership or nonmembership parameters to obtain the values of C_i.

8. CONCLUSION

In this paper, we address a major problem concerning the analytical treatment of Intuitionistic Fuzzy Differential Equations (IFDEs) by suggesting the Optimal Homotopy Asymptotic Method (OHAM) as an effective tool for solving them. The proposed method provides accurate approximate analytical solutions and it guarantees the convergence and Hyers-Ulam stability under some well-defined conditions. Our extensive numerical analyses and experiments demonstrate that OHAM successfully overcomes the drawbacks of the previously developed methods with remarkably low absolute errors and convergence rates close to the exact solutions for all the cases tested.

The OHAM provides a closed-form analytical solution instead of discrete numerical values, and the errors obtained are three to four orders of magnitude smaller than those obtained by the classical numerical methods (Euler, Modified Euler and RK4) applied to similar IFDEs in the literature. However, OHAM has some limitations: (i) the least-squares optimization for auxiliary parameters may be ill-conditioned for very high-order approximations and (ii) the verification of the convergence condition in advance for strongly nonlinear IFDEs is not trivial and requires symbolic computation for efficient implementation.

This advancement is particularly significant for applications in engineering, the natural sciences, and economics, where modeling under uncertainty and hesitation is essential. By demonstrating not only the efficacy of OHAM for IFDEs but also quantifying its reliability through stability proofs, our study broadens the practical and theoretical toolkit available to researchers working with intuitionistic fuzzy systems.

Future research avenues stemming from this work are plentiful. The extension of OHAM to systems of nonlinear IFDEs, fractional-order intuitionistic fuzzy equations, and partial differential equations embraces a broader class of problems with higher dimensionality and complexity. Incorporating other types of fuzzy uncertainty, such as type-2 fuzzy sets or hesitant fuzzy environments, may further enhance the realism and applicability of the method. Additional directions include developing efficient computational algorithms to automate parameter optimization in OHAM and applying the framework to real-world datasets in areas such as bioengineering, financial forecasting, and control systems. Overall, this study lays a robust mathematical and practical foundation for the reliable analysis of IFDEs, and we anticipate that its results will catalyze further developments in fuzzy modeling, uncertainty quantification, and the intersection of soft computing with differential equation theory.

CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

Muhammad Saeed: Conceptualization, Methodology, Investigation, Writing – review & editing, Supervision. Muzaher Ali: Formal analysis, Investigation, Methodology, Writing – original draft. Shaukat Iqbal: Supervision, Investigation, Methodology, Writing original draft.

DECLARATIONS

Conflict of Interest: The author declares no conflict of interest..

Funding: No specific funding was received for this study.
Acknowledgments: The author would like to thank the reviewers and editors for their valuable comments and suggestions, which helped to improve the quality of this manuscript.

Data Availability: All the data used during this study is accessible within the manuscript.

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Appendix A. Intermediate polynomial expansions for Example 1

For completeness and reproducibility, the full second-order OHAM corrections $[U_2(x,\beta)]_\mu^\pm$ and $[U_2(x,\beta)]_\nu^\pm$ of Example 1, referenced in Section 3 of the main text, are listed below.

$$\begin{aligned}[U_2(x,\beta)]_\mu^+ = \frac{1}{30000}x^2\Big[&C_1^2\Big(1500(\beta-1)+(25\beta+272)x^3\\&+125(\beta-1)x^2+20(25\beta+272)x\Big)\\&+20C_1\Big(75(\beta-1)+(25\beta+272)x\Big)\\&+20C_2\Big(75(\beta-1)+(25\beta+272)x\Big)\Big].\end{aligned}$$

$$\begin{aligned}[U_2(x,\beta)]_\nu^+ = -\frac{1}{30000}x^2\Big[&C_1^2\Big(1500(\beta-1)+(25\beta-322)x^3\\&+125(\beta-1)x^2+20(25\beta-322)x\Big)\\&+20C_1\Big(75(\beta-1)+(25\beta-322)x\Big)\\&+20C_2\Big(75(\beta-1)+(25\beta-322)x\Big)\Big].\end{aligned}$$

$$\begin{aligned}[U_2(x,\beta)]_\mu^- = \frac{1}{60000}x^2\Big[&C_1^2\Big(-6000\beta+(89\beta+594)x^3\\&-500\beta x^2+20(89\beta+594)x\Big)\\&+20C_1\Big((89\beta+594)x-300\beta\Big)\\&+20C_2\Big((89\beta+594)x-300\beta\Big)\Big].\end{aligned}$$

$$\begin{aligned}[U_2(x,\beta)]_\nu^- = x^2\Big[&C_1^2\Big(3000\beta+9(8\beta+33)x^3+250\beta x^2+180(8\beta+33)x\Big)\\&+60C_1\Big(50\beta+3(8\beta+33)x\Big)\\&+60C_2\Big(50\beta+3(8\beta+33)x\Big)\Big].\end{aligned}$$