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Developable Surfaces from G^2-Quintic Bézier Directrices with G^2 Surface Connection

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Abstract. Developable surfaces are widely used in construction of a range of materials such as leather, metallic sheets and other flexible materials because of their attractive geometric designs. This study presents construction of a ruled surface using G^2-quintic Bézier curves as directrices. The constructed surfaces accurately interpolate tangents, normals, curvatures and data values. Moreover, derivation of sufficient conditions for constructed ruled surfaces as general developable surfaces, cylindrical and canonical developable surfaces is also presented. Constraints are derived for the G^2-connection of developable surfaces through G^2-quintic Bézier geodesics. Simple and directly related sufficient conditions are achieved for tangents, normals, curvatures, data points, and shape parameters which reduce computational complexity many folds. Accurate surface designing of practical applications such as developable, cylindrical, and canonical surfaces is validated by easy implementation of proposed numerical scheme. However, achieving higher-order continuity with existing methods often involves complex computations.

AMS (MOS) Subject Classification Codes: 65D05, 65D07, 65D18, 68U05
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1. Introduction

Developable surfaces are special class of ruled surfaces which are produced by sweeping a line through space [3, 16]. Tangent planes are constant along any given ruling of developable surface. It can be converted into a plane without changing its shape and size. The mathematical representation of developable surfaces is used for constructing sheet or

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plate metal, leather, clothing and other flexible materials [16]. Piecewise continuous developable surfaces are used for more accurate and continuous representation. Cones and cylinders are two different kinds of developable surfaces formed by the tangents of a space curve [7, 16].

Different surface construction techniques were discussed in ([3]-[7], [9], [11]-[23]). A brief review is as follows. Aumann [3] presented an algorithm for the construction of developable Bézier surfaces through a Bézier curve of arbitrary degree. The second boundary curve was constructed using five design parameters. These free parameters controlled the starting points and singular points. The proposed developable surface construction technique [3] has two advantages (i) it does not require the solution of characterizing equation (ii) it is non-singular. Aumann [3] further extended his earlier work on the developable Bézier surfaces. In [3], the number of design parameters were very small. Aumann [4] overcame this restriction using degree elevation technique. Bo and Wang [6] modelled developable surfaces for animation applications. The authors used the representation of developable surface as the envelope of rectifying planes in 3D. The proposed scheme showed the natural continuous isometric deformation of the given surfaces into flat state. It was explained that the proposed scheme is fast, accurate and easy to use. It was further used for the representation of non-convex piecewise smooth developable surface. Chu and Séquin [9] proposed geometric design techniques for quadratic and cubic developable surface patches. Developability conditions were derived geometrically from de Casteljau algorithm. These sufficient conditions were expressed as set of control points of Bézier surfaces. These sufficient conditions were not highly non-linear thus provided a closed form solutions form the control points of the Bézier curves. One of the boundary curves was freely designed and there were five free parameters for the construction of the other boundary curve. Generalized canonical and cylindrical models were also emphasized for higher-order developable surfaces.

In [5, 11] optimization methods were used for the construction of developable surfaces. Azariadis and Aspragathos [5] proposed two methods for a planar development of a three-dimensional surface. The first method is a global optimization method without constraining geodesic curvature. The second method is global optimization method used to control local accuracy of the surface. Both surface construction methods do not depend upon surface parametrization. The accuracy of these methods was dependent upon the triangulation of the objective surface of the optimization problem. The effectiveness of the proposed methods was illustrated through shoe making applications. Hua et al. [11] proposed a numerical scheme for the computation of shape optimized developable surfaces. Generalized developable H-Bézier surfaces (GDHBS) were constructed from H-Bézier curves using the concept of duality between points and planes in 3D projective space. The developable surfaces were optimized through variation of free parameters for minimum values of arc length, bending energy and curvature variation energy of the dual curve. GDHBS shape optimization models were attained by adapting cuckoo search optimization algorithm.

In [14, 15, 16, 19, 21, 22] geodesic based numerical schemes were devised for the construction of developable surfaces. Kasap, Akyildiz and Orbay [14] constructed a surface family from a given spatial geodesic curve. The sufficient conditions were derived from the generalized marching scale functions. These conditions ensured that the curve is an iso-geodesic curve on the given surface. Illustrative examples were presented for iso-geodesic of spherical and cylindrical surface family. Li, Wang and Zhu [15] discussed the developable surface with cubic 3D Bézier curve as geodesic. Necessary and sufficient conditions were derived to construct developable surfaces, cylinder and cones from ruled surfaces. These necessary and sufficient conditions resulted in a large system of equations of the control points of the Bézier curves. No explicit relation between the control points and tangents of the directrix curves was derived. The composite G^1 connection of developable surfaces through abutting cubic Bézier geodesics was also discussed and explained through numerical examples. Li, Wang and Zhu [16] discussed the construction of minimal surfaces using geodesics. A surface pencil was constructed which passes through given curve. Necessary and sufficient conditions for geodesic representation of the given curve of the surface were derived. Finally, the Dirichlet function was minimized to construct minimal surfaces. Surface construction techniques were applied to construct surfaces of revolution, ruled surfaces, bicubic Hermite surfaces. Paluszny [19] presented a generalized surface construction scheme using pre-geodesic. A 3D polynomial curve was used as pre-geodesic. The rectifying and tangent planes of pre-geodesic curves are identical. The ruled cubic patches through pre-geodesics and bicubic patches through pairs of pre-geodesics were computed. The connection of these patches with abutting pre-geodesics was also discussed. Homogeneous parameters were adjusted for the G^1 connection of the composite surfaces using system of linear equations. Sánchez-Reyes and Dorado [21] constructed polynomial surfaces from geodesic curves and used these in textile and garment designs. Degrees of freedom of the constructed surface were identified in terms of control points. Tangent ribbons identified two lines of control points where as remaining control points worked as free parameters. Wang, Tang and Tai [22] studied the problem of constructing a family of surfaces from a given spatial geodesic curve. The authors [22] derived the parametric representation for a surface pencil. Using Frennet trihedron frame for the given geodesic, the authors expressed the surface pencil as a linear combination of the components of local coordinate frame. The method was illustrated by finding exact surface pencil formulations for surfaces of revolution and ruled surfaces. Akram, Batool and Akram [1] computed $(2\mu + 1)$ ordered numerical scheme for solving nonlinear equations. Ashraf and Rasheed [2] introduced a four-point ternary refinement scheme for generating interpolating and approximating curves. Radial basis functions were used in [13] to solve one dimensional partial differential equation. RBF are well established for its efficient approximation properties.

Zhao and Wang [23] represented the surface as a combination of given curve and the three vectors defined in Frennet trihedron. Necessary and sufficient conditions were derived for this surface to be a developable surface. These conditions resulted in a set of constraints relating marching functions as well as curvature and torsion of the directrix curve. Classification of these constraints for the special cases of developable surfaces, cylinders and cones surfaces, was also discussed. Lu [17] proposed G^2-Bézier curve interpolation scheme with minimal strain energy. The Bézier curve was a quintic polynomial interpolating end points, tangents, normals and curvatures. The length of the tangent vector was computed by minimizing strain energy.

Çalışkan, Ünlütürk and Kocabaş [7] recently presented a novel approach for constructing time like developable surfaces with a common line of curvature, offering an alternative differential geometric perspective to geodesic-based methods. Their work demonstrates continued interest in higher-order surface construction techniques. Hussain et al. [12] proposed a high-order, optimization-free method for tangent continuous approximation of conic sections using cubic Bézier curves ensuring computationally efficient geometric designing. In [18], Mirkin et al. used multilayer inflatables for computational fabrication applications. Moreover, a developable surface is known as special class of a ruled surface defined as the set of points obtained by sweeping a straight line and can flatten on a plane without distortion [7, 16].

In [15], cubic Bézier curves were used to generate developable surfaces including canonical and cylindrical surfaces with G^1 continuity. In [17], Lu demonstrated G^2 interpolation for planar curves. Even with all these remarkable contributions, achieving higher-order geometric continuity (G^2) maintaining simple sufficient conditions is still a challenge. In [12], the authors achieved tangent continuity using optimization-free methods making it a viable option for curvature continuity. In this research paper, we are using quintic Bézier curves to generate developable surfaces with G^2 continuity, while ensuring the sufficient conditions remain simple and optimization-free for general developable and cylindrical surfaces. A comparison with the existing techniques is given in Table 1.

The structure of the paper is as follows: In Section 2, first a ruled surface is constructed with G^2-quintic Bézier curves as its directrices. Secondly, sufficient conditions are derived to represent ruled surface as canonical and cylindrical surfaces. Section 3 discusses the method of construction of composite developable surfaces with G^2 continuity. Numerical results and conclusions are discussed in Section 4 and 5 respectively.

Existing surface construction schemes	**Computed surface construction schemes**
The surface construction schemes for developable surfaces, canonical surfaces and cylindrical surfaces proposed in [15] was G^1.	The surface construction schemes for ruled surfaces, canonical and cylindrical surfaces computed in this research paper are G^2, see Section 2, 3, 4.
The directrix curves in [15] were G^1 cubic Bézier curves.	The directrix curves of the surfaces construction schemes presented in this research paper are G^2-quintic Bézier curves.
The existing developable surface construction schemes [15] presented an implicit complex system of equations relating control points of directrix curves and blending functions for surface construction.	Computed surface construction schemes presented in Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 3.1 give explicit relation between the end points, tangents, normals, curvatures and shape parameters of the directrix curves. The surface construction schemes presented in this research paper are more relaxed than [15].
In [15], no shape design parameter is present for the further refinement of the developable surfaces.	Each directrix has four shape design parameters. These shape design parameters can be used to further refine the shape of constructed surface.
The numerical schemes presented in [5, 11] used optimization methods to construct developable surfaces.	Unlike optimization-dependent approaches [5, 11], our methods derive simple, closed-form sufficient conditions that directly relate geometric properties of the directrix curves, following the optimization-free philosophy for ruled surfaces, cylindrical surfaces, and composite surfaces.
In [15], the ruled surface represents a cylinder iff the directrix curve is generalized helix.	In Theorem 2.2, simple sufficient conditions are derived to represent cylindrical surfaces from given data of directrix curve.

TABLE 1. Comparison of existing and proposed surface construction schemes.

2. THE DEVELOPABLE SURFACE CONSTRUCTION SCHEMES

In this section, first ruled surfaces are constructed whose two directrix are G^2-quintic Bézier curves. Then sufficient conditions are computed for the ruled surfaces to represent developable surfaces. In [15], a ruled surface is computed by the following relation.

$$\mathbf{R_1}(u,v) = \mathbf{r}(v) + u\mathbf{p}(v). \tag{2. 1}$$

Here $\mathbf{R_1}(u,v)$ is the ruled surface, $\mathbf{r}(v)$ and $\mathbf{r}_1(v)$ are the directrix of ruled surfaces, and u is the translation parameter. Here $\mathbf{p}(v) = \mathbf{r}_1(v) - \mathbf{r}(v)$. In this research paper, the directrix of the ruled surfaces are taken as G^2-quintic Bézier curves [17]. The G^2-quintic Bézier curves $\mathbf{r}(v)$ and $\mathbf{r}_1(v)$ are defined as follows:

$$\mathbf{r}(v) = \sum_{i=0}^{5} \binom{5}{i} (1-v)^{5-i} v^i b_i, \quad \mathbf{r_1}(v) = \sum_{i=0}^{5} \binom{5}{i} (1-v)^{5-i} v^i c_i, \tag{2. 2}$$

with

$$b_0 = P_0, \quad b_1 = P_0 + \frac{\alpha_0 T_0}{5}, \quad b_2 = P_0 + \left(\frac{2\alpha_0}{5} + \frac{\beta_0}{20}\right) T_0 + \frac{k_0 \alpha_0^2 N_0}{20},$$
$$b_3 = P_1 + \left(\frac{-2\alpha_1}{5} + \frac{\beta_1}{20}\right) T_1 + \frac{k_1 \alpha_1^2 N_1}{20}, \quad b_4 = P_1 - \frac{\alpha_1 T_1}{5}, \quad b_5 = P_1,$$
$$c_0 = Q_0, \quad c_1 = Q_0 + \frac{\widehat{\alpha}_0 \widehat{T}_0}{5}, \quad c_2 = Q_0 + \left(\frac{2\widehat{\alpha}_0}{5} + \frac{\widehat{\beta}_0}{20}\right) \widehat{T}_0 + \frac{\widehat{k}_0 \widehat{\alpha}_0^2 \widehat{N}_0}{20},$$
$$c_3 = Q_1 + \left(\frac{-2\widehat{\alpha}_1}{5} + \frac{\widehat{\beta}_1}{20}\right) \widehat{T}_1 + \frac{\widehat{k}_1 \widehat{\alpha}_1^2 \widehat{N}_1}{20}, \quad c_4 = Q_1 - \frac{\widehat{\alpha}_1 \widehat{T}_1}{5}, \quad c_5 = Q_1.$$

Here P_0, P_1, Q_0, Q_1 are the end points, $T_0, T_1, \widehat{T}_0, \widehat{T}_1$ are the end unit tangents, $N_0, N_1, \widehat{N}_0, \widehat{N}_1$ are the end unit normal of the G^2-quintic Bézier curves. $k_0, k_1, \widehat{k}_0$ and $\widehat{k}_1$ are the curvature of these curves at the end points. Here $\alpha_0, \beta_0, \alpha_1, \beta_1, \widehat{\alpha}_0, \widehat{\beta}_0, \widehat{\alpha}_1$ and $\widehat{\beta}_1$ are the shape design parameters or free parameters.

In the following theorem, sufficient conditions are derived so that $\mathbf{R_1}(u,v)$ to be a developable surface.

Theorem 2.1. *The ruled surface $\mathbf{R_1}(u,v)$ defined in* (2. 1) *is developable if the following sufficient conditions are satisfied:*

(1) *The vectors $T_0, T_1, \widehat{T}_0, \widehat{T}_1, Q_0 - P_0$ and $Q_1 - P_1$ are parallel vectors.*
(2) *The vectors $P_1 - P_0$ and $Q_1 - Q_0$ are parallel vectors.*
(3) *The curvatures $k_0 = 0, k_1 = 0, \widehat{k}_0 = 0,$ and $\widehat{k}_1 = 0$.*

Proof. A ruled surface $\mathbf{R_1}(u,v)$ defined in (2. 1) is developable iff $(\mathbf{r}'(v), \mathbf{p}(v), \mathbf{p}'(v)) = 0$ [11, 16]. Here, $\mathbf{p}(v) = \mathbf{r}_1(v) - \mathbf{r}(v)$ and $\mathbf{p}'(v) = \mathbf{r_1}'(v) - \mathbf{r}'(v)$, $(\mathbf{r}'(v), \mathbf{p}(v), \mathbf{p}'(v))$ stands for scalar triple product of three vectors $\mathbf{r}'(v), \mathbf{p}(v)$ and $\mathbf{p}'(v)$. Substituting the values of $\mathbf{r}(v), \mathbf{p}(v)$ and $\mathbf{p}'(v)$, the relation $(\mathbf{r}'(v), \mathbf{p}(v), \mathbf{p}'(v)) = 0$ reduces to the following relation.

$$\left(\mathbf{r}'(v), \mathbf{r}_1(v) - \mathbf{r}(v), \mathbf{r_1}'(v)\right) = 0. \tag{2. 3}$$

Complete expressions of $\mathbf{r}'(v), \mathbf{r}_1(v) - \mathbf{r}(v)$ and $\mathbf{r_1}'(v)$ are evaluated for the G^2-quintic Bézier curves defined in (2. 2). These values are written in appendix. Substituting the values of $\mathbf{r}'(v), \mathbf{r}_1(v) - \mathbf{r}(v)$ and $\mathbf{r_1}'(v)$, and after simplification we have

$$\left(\mathbf{r}'(v), \mathbf{r}_1(v) - \mathbf{r}(v), \mathbf{r_1}'(v)\right) = \sum_{i=0}^{13} (1-v)^{13-i} v^i A_i = 0. \tag{2. 4}$$

As $v \in [0,1]$, therefore Equation (2. 4) is true only if $A_i = 0, i = 0, 1, 2, \ldots, 13$. Here,

$$\begin{aligned}
A_0 &= B_0 \cdot Z_0 = \alpha_0 \widehat{\alpha}_0 \left(T_0 \cdot (Q_0 - P_0) \times \widehat{T}_0 \right), \\
A_1 &= B_0 \cdot Z_1 + 4B_1 \cdot Z_0 = \left(13\alpha_0 \widehat{\alpha}_0 + \alpha_0 \widehat{\beta}_0 + \widehat{\alpha}_0 \beta_0 \right) \left(T_0 \cdot (Q_0 - P_0) \times \widehat{T}_0 \right) \\
&\quad + \widehat{k}_0 \alpha_0 \widehat{\alpha}_0^2 \left(T_0 \cdot (Q_0 - P_0) \times \widehat{N}_0 \right) + k_0 \widehat{\alpha}_0 \alpha_0^2 \left(N_0 \cdot (Q_0 - P_0) \times \widehat{T}_0 \right), \\
A_2 &= B_0 \cdot Z_2 + 4B_1 \cdot Z_1 + 6B_2 \cdot Z_0, \\
A_3 &= B_0 \cdot Z_3 + 4B_1 \cdot Z_2 + 6B_2 \cdot Z_1 + 4B_3 \cdot Z_0, \\
A_4 &= B_0 \cdot Z_4 + 4B_1 \cdot Z_3 + 6B_2 \cdot Z_2 + 4B_3 \cdot Z_1 + B_4 \cdot Z_0, \\
A_5 &= B_0 \cdot Z_5 + 4B_1 \cdot Z_4 + 6B_2 \cdot Z_3 + 4B_3 \cdot Z_2 + B_4 \cdot Z_1, \\
A_6 &= B_0 \cdot Z_6 + 4B_1 \cdot Z_5 + 6B_2 \cdot Z_4 + 4B_3 \cdot Z_3 + B_4 \cdot Z_2, \\
A_7 &= B_0 \cdot Z_7 + 4B_1 \cdot Z_6 + 6B_2 \cdot Z_5 + 4B_3 \cdot Z_4 + B_4 \cdot Z_3, \\
A_8 &= B_0 \cdot Z_8 + 4B_1 \cdot Z_7 + 6B_2 \cdot Z_6 + 4B_3 \cdot Z_5 + B_4 \cdot Z_4, \\
A_9 &= B_0 \cdot Z_9 + 4B_1 \cdot Z_8 + 6B_2 \cdot Z_7 + 4B_3 \cdot Z_6 + B_4 \cdot Z_5, \\
A_{10} &= 4B_1 \cdot Z_9 + 6B_2 \cdot Z_8 + 4B_3 \cdot Z_7 + B_4 \cdot Z_6, \\
A_{11} &= 6B_2 \cdot Z_9 + 4B_3 \cdot Z_8 + B_4 \cdot Z_7, \\
A_{12} &= 4B_3 \cdot Z_9 + B_4 \cdot Z_8 = \left(13\alpha_1 \widehat{\alpha}_1 - \alpha_1 \widehat{\beta}_1 - \widehat{\alpha}_1 \beta_1 \right) \left(T_1 \cdot (Q_1 - P_1) \times \widehat{T}_1 \right) \\
&\quad - \widehat{k}_1 \alpha_1 \widehat{\alpha}_1^2 \left(T_1 \cdot (Q_1 - P_1) \times \widehat{N}_1 \right) - k_1 \widehat{\alpha}_1 \alpha_1^2 \left(N_1 \cdot (Q_1 - P_1) \times \widehat{T}_1 \right), \\
A_{13} &= B_4 \cdot Z_9 = \alpha_1 \widehat{\alpha}_1 \left(T_1 \cdot (Q_1 - P_1) \times \widehat{T}_1 \right).
\end{aligned}$$

The complete expressions of $B_i, i = 0, 1, 2, 3, 4$ and $Z_i, i = 0, 1, \ldots, 9$ are given in appendix.

The expression $A_i, i = 0, 1, 2, \ldots, 13$, reduces to zero if the following sufficient conditions are satisfied:

(1) The vectors $T_0, T_1, \widehat{T}_0, \widehat{T}_1, Q_0 - P_0$ and $Q_1 - P_1$ are parallel vectors.
(2) The vectors $P_1 - P_0$ and $Q_1 - Q_0$ are parallel vectors.
(3) The curvatures $k_0 = 0, k_1 = 0, \widehat{k}_0 = 0$, and $\widehat{k}_1 = 0$.

It is clear from (2. 2), that the directrix will depend only on the tangent vectors and the end points. Consequently, the developable surfaces will be constructed as a blend of end points and end tangents. The constructed developable surface will be a G^1-continuous surface. □

In the following theorem, sufficient conditions are derived so that $\mathbf{R_1}(u,v)$ represents a cylindrical surface.

Theorem 2.2. *Given the values of curve* (2. 2)*, end points of the curves* (P_0, Q_0, P_1)*, unit tangents vectors* (T_0, T_1)*, unit normal vectors* (N_0, N_1)*, shape design parameters* $(\alpha_0, \widehat{\alpha}_0, \alpha_1, \widehat{\alpha}_1, s_0, s_1 \in \mathbb{R}^+)$*, end curvatures* $(k_0, \widehat{k}_0, k_1, \widehat{k}_1)$*, the ruled surface is cylinder if the following sufficient conditions are satisfied.*

$$\widehat{\beta}_0 = s_0 \widehat{\alpha}_0, \quad \widehat{\beta}_1 = s_1 \widehat{\alpha}_1, \quad \beta_0 = s_0 \alpha_0, \quad \beta_1 = s_1 \alpha_1,$$

$$\widehat{T}_0 = \frac{\alpha_0 T_0}{\widehat{\alpha}_0}, \quad \widehat{T}_1 = \frac{\alpha_1 T_1}{\widehat{\alpha}_1}, \quad \widehat{N}_0 = \frac{k_0\alpha_0^2 N_0}{\widehat{k}_0\widehat{\alpha}_0^2}, \widehat{N}_1 = \frac{k_1\alpha_1^2 N_1}{\widehat{k}_1\widehat{\alpha}_1^2}, \quad Q_1 - P_1 = g(Q_0 - P_0),$$

where $g \in \mathbb{R}^+$.

Proof. A ruled surface $\mathbf{R_1}(u, v)$ defined in (2. 1) is cylindrical iff $\mathbf{p}'(v) \times \mathbf{p}(v) = 0$ [14, 19]. By the definition of $\mathbf{p}(v)$ and $\mathbf{p}^{'}(v)$, $\mathbf{p}^{'}(v) = \mathbf{r_1}^{'}(v) - \mathbf{r}^{'}(v)$ and $\mathbf{p}(v) = \mathbf{r}_1(v) - \mathbf{r}(v)$. Substituting the values of $\mathbf{r}(v), \mathbf{r}_1(v), \mathbf{r}^{'}(v)$ and $\mathbf{r_1}^{'}(v)$ from (2. 2), the following final value of $\mathbf{p}'(v) \times \mathbf{p}(v)$ is obtained for the ruled surface $\mathbf{R_1}(u, v)$ using G^2-quintic Bézier curve (2. 2) as directrix.

$$\mathbf{p}'(v) \times \mathbf{p}(v) = \sum_{i=0}^{9} (1 - v)^{9-i} v^i \widetilde{C}_i,$$

where

$$\begin{aligned}
\widetilde{C}_0 &= Y_0 \times X_0,\\
\widetilde{C}_1 &= 4(Y_1 \times X_0) + 4(Y_0 \times X_1),\\
\widetilde{C}_2 &= 6(Y_2 \times X_0) + 20(Y_1 \times X_1) + 10(Y_0 \times X_2),\\
\widetilde{C}_3 &= 4(Y_3 \times X_0) + 30(Y_2 \times X_1) + 40(Y_1 \times X_2) + 10(Y_0 \times X_3),\\
\widetilde{C}_4 &= (Y_4 \times X_0) + 20(Y_3 \times X_1) + 60(Y_2 \times X_2) + 40(Y_1 \times X_3) + 5(Y_0 \times X_4),\\
\widetilde{C}_5 &= 5(Y_4 \times X_1) + 40(Y_3 \times X_2) + 60(Y_2 \times X_3) + 20(Y_1 \times X_4) + (Y_0 \times X_5),\\
\widetilde{C}_6 &= 10(Y_4 \times X_2) + 40(Y_3 \times X_3) + 30(Y_2 \times X_4) + 4(Y_1 \times X_5),\\
\widetilde{C}_7 &= 10(Y_4 \times X_3) + 20(Y_3 \times X_4) + 6(Y_2 \times X_5),\\
\widetilde{C}_8 &= 5(Y_4 \times X_4) + 4(Y_3 \times X_5),\\
\widetilde{C}_9 &= Y_4 \times X_5.
\end{aligned}$$

The complete expression of $\widetilde{C}_i, i = 0, 1, 2, \ldots, 9$, are written in the appendix. As $v \in [0, 1]$, so $\mathbf{p}'(v) \times \mathbf{p}(v) = 0$ if $\widetilde{C}_i = 0, i = 0, 1, 2, \ldots, 9$.

Thus $\widetilde{C}_0 = 0$ if $\widehat{\alpha}_0\widehat{T}_0 \times (Q_0 - P_0) = T_0 \times (Q_0 - P_0)$. Similarly, $\widetilde{C}_1 = 0$ if

$$\widehat{\beta}_0\widehat{T}_0 \times (Q_0 - P_0) = \beta_0 T_0 \times (Q_0 - P_0), \quad \widehat{k}_0\widehat{\alpha}_0^2\widehat{N}_0 \times (Q_0 - P_0) = k_0\alpha_0^2 N_0 \times (Q_0 - P_0),$$

and $(Q_0 - P_0) \times (Q_1 - P_1) = 0$. The latter holds if the vectors are parallel.

$\widetilde{C}_8 = 0$ if

$$\widehat{\beta}_1\widehat{T}_1 \times (Q_1 - P_1) = \beta_1 T_1 \times (Q_1 - P_1), \quad \widehat{k}_1\widehat{\alpha}_1^2\widehat{N}_1 \times (Q_1 - P_1) = k_1\alpha_1^2 N_1 \times (Q_1 - P_1),$$

and $(Q_0 - P_0) \times (Q_1 - P_1) = 0$.

$\widetilde{C}_9 = 0$ if $\widehat{\alpha}_1\widehat{T}_1 \times (Q_1 - P_1) = \alpha_1 T_1 \times (Q_1 - P_1)$.

The $\widetilde{C}_i = 0, i = 2, 3, 4, 5, 6, 7$, for the above calculated values of $\widehat{\alpha}_0, \widehat{\beta}_0, \widehat{\alpha}_1$ and $\widehat{\beta}_1$, and also for the condition of coplanarity of $(Q_0 - P_0)$ and $(Q_1 - P_1)$. The above sufficient conditions for $\widetilde{C}_i = 0, i = 0, 1, 2, \ldots, 9$, leads to the following equivalent sufficient conditions.

$$\widehat{T}_0 = \frac{\alpha_0 T_0}{\widehat{\alpha}_0}, \quad \widehat{T}_0 = \frac{\beta_0 T_0}{\widehat{\beta}_0}, \quad \widehat{T}_1 = \frac{\alpha_1 T_1}{\widehat{\alpha}_1}, \quad \widehat{T}_1 = \frac{\beta_1 T_1}{\widehat{\beta}_1}, \quad \widehat{N}_0 = \frac{k_0 \alpha_0^2 N_0}{\widehat{k}_0 \widehat{\alpha}_0^2},$$

$$\widehat{N}_1 = \frac{k_1 \alpha_1^2 N_1}{\widehat{k}_1 \widehat{\alpha}_1^2}, \quad Q_1 - P_1 = g(Q_0 - P_0),$$

where $g \in \mathbb{R}^+$. From the above expression it is clear that for the construction of cylindrical surface by the curve (2. 2), we need only points of the curve (P_0, Q_0, P_1), unit tangents vectors at the initial and final points of the curve (T_0, T_1), unit normal vectors (N_0, N_1), and the shape design parameters $\alpha_0, \alpha_1, \widehat{\alpha}_0, \widehat{\alpha}_1$. The values of point Q_1, the unit tangent vectors $(\widehat{T}_0, \widehat{T}_1)$, the unit normal vectors $(\widehat{N}_0, \widehat{N}_1)$, and the shape design parameters $\beta_0, \beta_1, \widehat{\beta}_0$ and $\widehat{\beta}_1$ are calculated from the above formulae. Here $\widehat{\beta}_0 = s_0 \widehat{\alpha}_0, \widehat{\beta}_1 = s_1 \widehat{\alpha}_1, \beta_0 = s_0 \alpha_0, \beta_1 = s_1 \alpha_1$, with $s_0, s_1 \in \mathbb{R}^+$. □

In the following theorem, sufficient conditions are derived so that $\mathbf{R_1}(u, v)$ represents a canonical surface.

Theorem 2.3. *Given the values of curve* (2. 2)*, end points of the curves* (P_0, P_1)*, unit tangents vectors at the end point of the curve* (T_0, T_1)*, unit normal vectors at the end points* (N_0, N_1)*, the ruled surface* $\mathbf{R_1}(u, v)$ *is canonical for the values of shape design parameters* $\alpha_i, \beta_i, i = 0, 1,$ *for which* $\|\mathbf{r}'(v)\| \to 0$.

Proof. A ruled surface $\mathbf{R_1}(u, v)$ defined in (2. 1) is conical if $\mathbf{r}'(v) = 0$ [14, 19]. The value of $\mathbf{r}'(v)$ for the quintic Bézier curve (2. 2) is given in appendix. As the polynomials $\binom{4}{i}(1 - v)^{4-i} v^i, i = 0, 1, 2, 3, 4$ cannot be zero all together for $v \in [0, 1]$. Therefore, $\mathbf{r}'(v) = 0$ if all the coefficients $B_i, i = 0, 1, 2, 3, 4$ are zero.

$$\begin{aligned}
B_0 &= \frac{\alpha_0 T_0}{5}, \\
B_1 &= \left(\frac{\alpha_0}{5} + \frac{\beta_0}{20}\right) T_0 + \frac{k_0 \alpha_0^2 N_0}{20}, \\
B_2 &= (P_1 - P_0) + \left(\frac{-2\alpha_1}{5} + \frac{\beta_1}{20}\right) T_1 + \frac{k_1 \alpha_1^2 N_1}{20} - \left(\frac{2\alpha_0}{5} + \frac{\beta_0}{20}\right) T_0 - \frac{k_0 \alpha_0^2 N_0}{20}, \\
B_3 &= \left(\frac{\alpha_1}{5} - \frac{\beta_1}{20}\right) T_1 - \frac{k_1 \alpha_1^2 N_1}{20}, \\
B_4 &= \frac{\alpha_1 T_1}{5}.
\end{aligned}$$

The above values of $B_i, i = 0, 1, 2, 3, 4$ are zero if the parameters $\alpha_i, \beta_i, i = 0, 1$ are zero and $P_1 = P_0$. It is not possible because in this case the ruled surface reduces to a surface interpolating the end points (P_0, P_1) of the curve. This ruled surface does not ensure tangent and curvature continuity. Therefore, to overcome this situation, the values of $P_0, P_1, T_0, T_1, N_0, N_1$, are calculated by the given data. The values of shape design parameters $\alpha_i, \beta_i, i = 0, 1$ are calculated by minimizing the value of $\|\mathbf{r}'(v)\| \to 0$. Thus an approximation to canonical surface is obtained by the ruled surface (2. 1). □

Values of the shape design parameters $\alpha_i, \beta_i, i = 0, 1$, in Theorem 2.3, are calculated by the following minimizing problem with objective function $\|\mathbf{r}'(v)\|$.

Optimization problem-I.

Minimize $\|\mathbf{r}'(v)\| \to 0$
subject to $\alpha_i > 0,\ \beta_i > 0,\ i = 0, 1.$

The quasi-Newton's method [8] is used to solve the minimization problem using MATLAB built in function fminimax. The Quasi-Newton's method improves the Newton's method by replacing the inverse of Hessian by its successive approximation. For the objective function $f(\mathbf{x}) : \mathbf{R}^n \to R$, on the feasible domain $\Omega \subseteq \mathbf{R^n}$, minimization by the quasi-Newton's method is completed by the following Algorithm 1 [8].

Algorithm 1.

(1) Initial step: Select initial guess $x^{(0)}$ for unknown vector x and compute inverse of Hessian H_0.
(2) If the gradient $g^{(k)}$ is zero stop; else, the search direction $d^{(k)}$ is $d^{(k)} = -H_k g^{(k)}$.
(3) Compute the step size α_k where $\alpha_k = \arg\min_{\alpha \geq 0} f(x^{(k)} + \alpha_k d^{(k)})$ and the next approximation to the unknown vector $x^{(k+1)} = x^{(k)} + \alpha_k d^{(k)}$.
(4) Update the inverse of Hessian H_{k+1} as follows:

$$\begin{aligned} H_{k+1} = H_k &+ \left(1 + \frac{\Delta g^{(k)T} H_k \Delta g^{(k)}}{\Delta g^{(k)T} \Delta x^{(k)}}\right) \frac{\Delta x^{(k)} \Delta x^{(k)T}}{\Delta x^{(k)T} \Delta g^{(k)}} \\ &- \frac{H_k \Delta g^{(k)} \Delta x^{(k)T} + \left(H_k \Delta g^{(k)} \Delta x^{(k)T}\right)^T}{\Delta g^{(k)T} \Delta x^{(k)}}. \end{aligned}$$

(H_{k+1} is the approximation of inverse of Hessian at kth step).
(5) Set $k = k + 1$, go to the Step 2.

Stopping Criterion. In Algorithm 1, the objective function evaluation tolerance and step size evaluation tolerance is 10^{-6}. The Algorithm 1 stops when the absolute value of objective function is less than 10^{-6} or the step size is less than 10^{-6}.

Algorithm for the canonical surface construction scheme presented in Theorem 2.3 is summarized as follows.

Algorithm 2.

(1) Input the initial guess $\mathbf{x} = (\alpha_0, \alpha_1, \beta_0, \beta_1),\ \alpha_i > 0, \beta_i > 0, i = 0, 1$ and the interpolation data $P_0, P_1, T_0, T_1, N_0, N_1, k_0, k_1$.
(2) Calculate $\|\mathbf{r}'(v)\|$ using (2. 2).
(3) Solve the optimization problem-I by Algorithm 1 using MATLAB built in function fminimax and compute the optimal values of $\alpha_0, \alpha_1, \beta_0$ and β_1.
(4) Put the values obtained in Step 3 to (2. 2) to compute the directrix curve $\mathbf{r}(v)$.

(5) Put $\mathbf{r}(v)$ computed in Step 4 in (2. 1) to get the canonical developable surface $\mathbf{R_1}(u,v)$, where $\mathbf{r_1}(v) = Q(x,y,z)$ is the vertex of the canonical developable surface.
(6) Output: Canonical developable surface.

Here $\|\mathbf{r}'(v)\|$ is the Euclidean norm in 2D space.

3. The two G^2 developable surfaces with G^2 Bézier geodesics

In this section, sufficient conditions are derived for the G^2 connection of two developable surfaces, containing G^2 abutting Bézier geodesics. The detail of G^2 connection between any two parametric surfaces can be found in [10].

Let $(\mathbf{r}(v), \mathbf{r_1}(v))$ and $(\mathbf{y}(t), \mathbf{y_1}(t))$ are the directrices of the two ruled surfaces $\mathbf{R_1}(u,v)$ and $\mathbf{R_2}(s,t)$. Explicit expression of these ruled surfaces are the following:

$$\mathbf{R_1}(u,v) = \mathbf{r}(v) + u\mathbf{p}(v), \tag{3. 5}$$

$$\mathbf{R_2}(s,t) = \mathbf{y}(t) + s\widetilde{\mathbf{p}}(t). \tag{3. 6}$$

Here $\mathbf{p}(v) = \mathbf{r}_1(v) - \mathbf{r}(v)$ and $\widetilde{\mathbf{p}}(t) = \mathbf{y}_1(t) - \mathbf{y}(t)$. The G^2-quintic Bézier curves $\mathbf{r}(v), \mathbf{r}_1(v), \mathbf{y}(t)$ and $\mathbf{y}_1(t)$, have control points $b_i, c_i, \widetilde{b}_i$ and $\widetilde{c}_i, i = 0,1,2,3,4,5$. The G^2-quintic Bézier curves $\mathbf{r}(v)$ and $\mathbf{r}_1(v)$, have been already defined in (2. 2).

$$\mathbf{y}(t) = \sum_{i=0}^{5}\binom{5}{i}(1-t)^{5-i}t^i\widetilde{b}_i, \quad \mathbf{y_1}(t) = \sum_{i=0}^{5}\binom{5}{i}(1-t)^{5-i}t^i\widetilde{c}_i,$$

with

$$\begin{aligned}
&\widetilde{b}_0 = f_0, \quad \widetilde{b}_1 = f_0 + \frac{\gamma_0 t_0}{5}, \quad \widetilde{b}_2 = f_0 + \left(\frac{2\gamma_0}{5} + \frac{\mu_0}{20}\right)t_0 + \frac{k_2\gamma_0^2 n_0}{20},\\
&\widetilde{b}_3 = f_1 + \left(\frac{-2\gamma_1}{5} + \frac{\mu_1}{20}\right)t_1 + \frac{k_3\gamma_1^2 n_1}{20}, \quad \widetilde{b}_4 = f_1 - \frac{\gamma_1 t_1}{5}, \quad \widetilde{b}_5 = f_1,\\
&\widetilde{c}_0 = g_0, \quad \widetilde{c}_1 = g_0 + \frac{\widehat{\gamma}_0\widehat{t}_0}{5}, \quad \widetilde{c}_2 = g_0 + \left(\frac{2\widehat{\gamma}_0}{5} + \frac{\widehat{\mu}_0}{20}\right)\widehat{t}_0 + \frac{\widehat{k}_2\widehat{\gamma}_0^2\widehat{n}_0}{20},\\
&\widetilde{c}_3 = g_1 + \left(\frac{-2\widehat{\gamma}_1}{5} + \frac{\widehat{\mu}_1}{20}\right)\widehat{t}_1 + \frac{\widehat{k}_3\widehat{\gamma}_1^2\widehat{n}_1}{20}, \quad \widetilde{c}_4 = g_1 - \frac{\widehat{\gamma}_1\widehat{t}_1}{5}, \quad \widetilde{c}_5 = g_1.
\end{aligned}$$

Here the parameters u, v, s and t are related as follows: (i) $u = s$, and along the common boundaries of two ruled surfaces $\mathbf{R_1}(u,v)$ and $\mathbf{R_2}(s,t)$ $v = t+1$. All the parameters are normalized that is $0 \le u, v, s, t \le 1$.

(i) First condition for the smooth G^2 connection of two adjacent ruled surfaces $\mathbf{R_1}(u,v)$ and $\mathbf{R_2}(s,t)$ is that these surfaces have common generator $\mathbf{r}(1) = \mathbf{y}(0)$ and $\mathbf{r_1}(1) = \mathbf{y_1}(0)$. It yields that the final and initial control points of these curves coincide $P_1 = f_0$ and $Q_1 = g_0$.

(ii) Second condition for the G^2-connection or G^2-continuity of the two developable surfaces along a common ruling is that $\frac{\partial \mathbf{R_1}}{\partial u} \times \frac{\partial \mathbf{R_1}}{\partial v} = \frac{\partial \mathbf{R_2}}{\partial s} \times \frac{\partial \mathbf{R_2}}{\partial t}$. Taking the derivative of ruled surfaces $\mathbf{R_1}(u,v)$ and $\mathbf{R_2}(s,t)$, w.r.t u, v, s and t, and then taking the cross product we get the following equation.

$$\mathbf{p}(v) \times (\mathbf{r}'(v) + u\mathbf{p}'(v))\big|_{v=1} = \widetilde{\mathbf{p}}(t) \times (\mathbf{y}^{'}(t) + s\widetilde{\mathbf{p}}'(t))\Big|_{t=0}$$

$$\mathbf{p}(v)\times\mathbf{r}'(v)+u\mathbf{p}(v)\times\mathbf{p}'(v)|_{v=1}=\left.\widetilde{\mathbf{p}}(t)\times\mathbf{y}^{'}(t)+s\widetilde{\mathbf{p}}(t)\times\widetilde{\mathbf{p}}'(t))\right|_{t=0}$$

The above relation is true if $\mathbf{p}(1)=\widetilde{\mathbf{p}}(0),\mathbf{r}'(1)=\mathbf{y}^{'}(0),\mathbf{p}'(1)=\widetilde{\mathbf{p}}'(0)$. The $\mathbf{p}(1)=\widetilde{\mathbf{p}}(0)$ if $c_5-b_5=\widetilde{c}_0-\widetilde{b}_0$ or $Q_1-P_1=g_0-f_0$. The $\mathbf{r}'(1)=\mathbf{y}^{'}(0)$ if $B_4=D_0$. Substituting the values of B_4 and D_0, the following relation between end unit tangents is obtained $\alpha_1T_1=\gamma_0t_0$. As $\mathbf{r}'(1)=\mathbf{y}^{'}(0)$, so $\mathbf{p}'(1)=\widetilde{\mathbf{p}}'(0)$ if $\mathbf{r}_{\mathbf{1}}^{'}(1)=\mathbf{y}_{\mathbf{1}}^{'}(0)$. We have $\mathbf{r}_{\mathbf{1}}^{'}(1)=\mathbf{y}_{\mathbf{1}}^{'}(0)$ if $\widehat{\alpha}_1\widehat{T}_1=\widehat{\gamma}_0\widehat{t}_0$.

(iii) Third condition for the G^2-continuity along a ruling of the two developable surfaces $\mathbf{R_1}(u,v)$ and $\mathbf{R_2}(s,t)$ is the following.

$$\left.\frac{\partial^2\mathbf{R_2}(s,t)}{\partial s\partial t}\right|_{t=0}=a_{11}\left.\frac{\partial\mathbf{R_1}(u,v)}{\partial u}\right|_{v=1}+b_{11}\left.\frac{\partial\mathbf{R_1}(u,v)}{\partial v}\right|_{v=1}+a_{10}a_{01}\left.\frac{\partial^2\mathbf{R_1}(u,v)}{\partial u^2}\right|_{v=1}$$

$$+(a_{10}b_{01}+a_{01}b_{10})\left.\frac{\partial^2\mathbf{R_1}(u,v)}{\partial u\partial v}\right|_{v=1}+b_{10}b_{01}\left.\frac{\partial^2\mathbf{R_1}(u,v)}{\partial v^2}\right|_{v=1}$$

$$\left.\frac{\partial^2\mathbf{R_2}(s,t)}{\partial t^2}\right|_{t=0}=a_{02}\left.\frac{\partial\mathbf{R_1}(u,v)}{\partial u}\right|_{v=1}+b_{02}\left.\frac{\partial\mathbf{R_1}(u,v)}{\partial v}\right|_{v=1}+a_{01}^2\left.\frac{\partial^2\mathbf{R_1}(u,v)}{\partial u^2}\right|_{v=1}$$

$$+2a_{01}b_{01}\left.\frac{\partial^2\mathbf{R_1}(u,v)}{\partial u\partial v}\right|_{v=1}+b_{01}^2\left.\frac{\partial^2\mathbf{R_1}(u,v)}{\partial v^2}\right|_{v=1}$$

where

$$a_{\mu\lambda}=\left.\frac{\partial^r}{\partial s^\mu\partial t^\lambda}u(s,t)\right|_{(s,t)=(\widetilde{s},\widetilde{t})},\quad b_{\mu\lambda}=\left.\frac{\partial^r}{\partial s^\mu\partial t^\lambda}v(s,t)\right|_{(s,t)=(\widetilde{s},\widetilde{t})},\quad r=\mu+\lambda,$$

and $a_{00}=v(\widetilde{s},\widetilde{t}),b_{00}=u(\widetilde{s},\widetilde{t})$.

The parameters u, v, s and t of the two ruled surfaces are related along the common rulling of $R_1(u,v)$ and $R_2(s,t)$ as $u=s$ and $v=t+1$. Substituting the values of $a_{10},a_{01},a_{11},a_{20},a_{02},b_{10},b_{01},b_{11},b_{20}$ and b_{02} from the above, the equations reduce to the following.

$$\left.\frac{\partial^2\mathbf{R_2}(s,t)}{\partial s\partial t}\right|_{t=0}=\left.\frac{\partial^2\mathbf{R_1}(u,v)}{\partial u\partial v}\right|_{v=1},\quad\left.\frac{\partial^2\mathbf{R_2}(s,t)}{\partial t^2}\right|_{t=0}=\left.\frac{\partial^2\mathbf{R_1}(u,v)}{\partial v^2}\right|_{v=1}.$$

Substituting the values of partial derivatives of $\mathbf{R_1}(u,v)$ and $\mathbf{R_2}(s,t)$ in the above, the following conditions are obtained.

$$\left.\mathbf{y}_{\mathbf{1}}^{'}(t)-\mathbf{y}^{'}(t)\right|_{t=0}=\left.\mathbf{r}_{\mathbf{1}}^{'}(v)-\mathbf{r}^{'}(v)\right|_{v=1},\tag{3. 7}$$

$$\left.(1-s)\mathbf{y}^{''}(t)+s\mathbf{y}_{\mathbf{1}}^{''}(t)\right|_{t=0}=\left.(1-u)\mathbf{r}^{''}(v)+u\mathbf{r}_{\mathbf{1}}^{''}(v)\right|_{v=1}.\tag{3. 8}$$

As $0\le u\le 1$ and $0\le s\le 1$, and we have taken $u=s$. Therefore (3. 7) and (3. 8) are satisfied if the following relations between the directrix curves are true.

$$\mathbf{y}_{\mathbf{1}}^{'}(0)=\mathbf{r}_{\mathbf{1}}^{'}(1),\quad\mathbf{y}^{'}(0)=\mathbf{r}^{'}(1),\quad\mathbf{y}_{\mathbf{1}}^{''}(0)=\mathbf{r}_{\mathbf{1}}^{''}(1),\quad\mathbf{y}^{''}(0)=\mathbf{r}^{''}(1).\tag{3. 9}$$

The set of relations (3. 9) is true of the following relations between end tangents and curvatures of the directrix curves of the two ruled surfaces are obtain.

$$\text{(i) } \alpha_1 T_1 = \gamma_0 t_0, \tag{3. 10}$$

$$\text{(ii) } \widehat{\alpha}_1 \widehat{T}_1 = \widehat{\gamma}_0 \widehat{t}_0, \tag{3. 11}$$

$$\text{(iii) } \widehat{\mu}_0 \widehat{t}_0 + \widehat{k}_2 \widehat{\gamma}_0^2 \widehat{n}_0 = \widehat{\beta}_1 \widehat{T}_1 + \widehat{k}_1 \widehat{\alpha}_1^2 \widehat{N}_1, \tag{3. 12}$$

$$\text{(iv) } \mu_0 t_0 + k_2 \gamma_0^2 n_0 = \beta_1 T_1 + k_1 \alpha_1^2 N_1. \tag{3. 13}$$

The above discussion yields the following equivalent sufficient conditions for the G^2 connection between two ruled surfaces $\mathbf{R_1}(u,v)$ and $\mathbf{R_2}(s,t)$. Given the end points, unit tangents, normals and curvatures of the directrix of the surfaces $\mathbf{R_1}(u,v)$, the above set of conditions yields the following values of the entities of the ruled surface $\mathbf{R_2}(s,t)$.

$$\text{(A)} \quad t_0 = \frac{\alpha_1 T_1}{\gamma_0}, \tag{3. 14}$$

$$\text{(B)} \quad \widehat{t}_0 = \frac{\widehat{\alpha}_1 \widehat{T}_1}{\widehat{\gamma}_0}, \tag{3. 15}$$

$$\text{(C)} \quad f_0 = P_1, \tag{3. 16}$$

$$\text{(D)} \quad g_0 = Q_1, \tag{3. 17}$$

$$\text{(E)} \quad n_0 = \frac{\beta_1 T_1 + k_1 \alpha_1^2 N_1 - \mu_0 t_0}{k_2 \gamma_0^2}, \tag{3. 18}$$

$$\text{(F)} \quad \widehat{n}_0 = \frac{\widehat{\beta}_1 \widehat{T}_1 + \widehat{k}_1 \widehat{\alpha}_1^2 \widehat{N}_1 - \widehat{\mu}_0 \widehat{t}_0}{\widehat{k}_2 \widehat{\gamma}_0^2}. \tag{3. 19}$$

Here, $\alpha_i, \beta_i, \widehat{\alpha}_i, \widehat{\beta}_i, \gamma_i, \mu_i, \widehat{\gamma}_i, \widehat{\mu}_i, i = 0, 1$, are the shape design parameters and can take any real value.

All the above computation provides the result for the G^2 connection of two ruled surfaces $\mathbf{R_1}(u,v)$ and $\mathbf{R_2}(s,t)$ in the form of following theorem.

Theorem 3.1. *The ruled surfaces, defined in* (2. 1) *and* (3. 6)*, are G^2-connected at the boundary geodesic if the sufficient conditions from (A)–(F) are satisfied.*

Remark 3.2. $T_i, \widehat{T}_i, N_i, \widehat{N}_i, t_i, \widehat{t}_i, n_i, \widehat{n}_i, i = 0, 1,$ *can be either unit vectors or zero vectors. The more extensive detail of it can found in* [17].

4. Numerical Examples

To validate the proposed surface construction schemes and to demonstrate their practical applicability, four distinct numerical examples are presented. These examples correspond to the theoretical results of surfaces established in Theorems 2.1, 2.2, 2.3 and 3.1, covering a general developable surface, a cylindrical surface, a canonical surface and composite developable surfaces. Moreover, the input parameters such as tangents, normals, curvatures and data points are selected to satisfy the sufficient conditions derived in Section 2.

Example 1: Developable Surface. The developable surface under consideration has control points
$P_0 = [1, 3, 2], Q_0 = [1, 3, 6], P_1 = [3, 81, 2], Q_1 = [3, 81, 6]$.
Theorem 2.1 is used to compute end unit tangents vectors
$T_0 = [0, 0, 1], T_1 = [0, 0, 1], \widehat{T_0} = [0, 0, 1], \widehat{T_1} = [0, 0, 1]$
for the surface under consideration. The values of end curvatures are $k_0 = 0, k_1 = 0, \widehat{k}_0 = 0, \widehat{k}_1 = 0$ using Theorem 2.1. Figure 1 presents the developable surface obtained using these end points, end unit tangents and curvature values substituted into Equations (2. 1) and (2. 2), confirming that derived sufficient conditions produce developable surfaces from the given data. Moreover, Figure 1 shows constant tangent planes along rulings. Elapsed time is 0.292043 seconds.

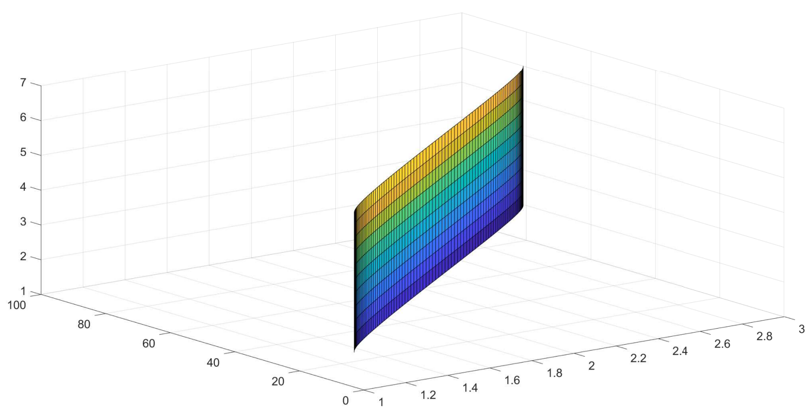

FIGURE 1. G^2-quintic developable surface constructed using Theorem 2.1 with $\alpha_0 = 2, \alpha_1 = 3, \beta_0 = 4, \beta_1 = 1, \widehat{\alpha}_0 = 2, \widehat{\alpha}_1 = 4, \widehat{\beta}_0 = 1, \widehat{\beta}_1 = 1$.

Example 2: Cylindrical Surface. Here
$P_0 = [1.0000, 0.8415, 2.0000], Q_0 = [1.0000, 0.8415, 6.0000]$,
$P_1 = [5.2832, -0.8415, 2.0000], Q_1 = [5.2832, -0.8415, 10.0000]$
are the control points of the cylindrical surfaces. Unit tangents and normal vectors obtained from Theorem 2.2 for the cylindrical surface are
$T_0 = [0.8798, 0.4754, 0], T_1 = [0.8798, 0.4754, 0]$,
$N_0 = [0, -1, 0], N_1 = [0, 1, 0]$,
curvatures are $k_0 = \frac{1}{2}, k_1 = \frac{1}{2}, \widehat{k}_0 = \frac{1}{2}, \widehat{k}_1 = \frac{1}{2}$.

The resulting cylindrical surface is shown in Figure 2. Hence, the proposed method is highly suitable for the construction of cylindrical surfaces. Elapsed time is 0.303874 seconds.

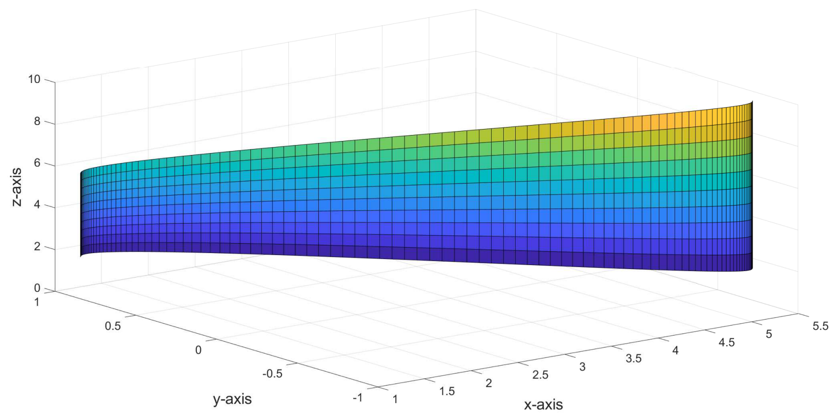

FIGURE 2. Cylindrical surface constructed using Theorem 2.2 with $\alpha_0 = 2, \alpha_1 = 3, \beta_0 = 4, \beta_1 = 1, \widehat{\alpha}_0 = 2, \widehat{\alpha}_1 = 4, \widehat{\beta}_0 = 1, \widehat{\beta}_1 = 1$.

Example 3: Canonical Surface. Figure 3 shows the canonical surface obtained with proposed method highlighting the potential feature for designers as adjusting free parameters of the Bézier representation without additional conditions on unit tangents or unit normals. Here the interpolation data is
$P_0 = [0.5000, 0.4794, 2.0000], P_1 = [1.0708, 0.8776, 2.0000]$,
$T_0 = [0.7516, 0.6596, 0], T_1 = [0.9017, 0.4323, 0]$,
$N_0 = [0, -1, 0], N_1 = [0, -1, 0], k_0 = \frac{1}{2}, k_1 = \frac{1}{2}$.
The optimized values of shape parameters obtained by the optimization problem-I is $\alpha_0 = 0.6983, \alpha_1 = 0.6968, \beta_0 = 0.1541, \beta_1 = 0.0790$. By the interpolating this data by Algorithm 2 using Theorem 2.3 a canonical surface is obtained which represented in Figure 3. Elapsed time is 3.594379 seconds.

Example 4: Composite Developable Surface. The G^2 connecting composite cylindrical surface is constructed using Theorem 2.2 and Theorem 3.1. The graph of composite cylindrical surface is shown in Figure 4. It also contains different views of the surface. Elapsed time is 2.061979 seconds.

Construction of surfaces with G^2-continuity is achieved in above examples with proposed method revealing further two key advantages by comparison with cubic-based methods [15] making them more suitable for practical design applications:

(1) Quintic Bézier curves offer enhanced flexibility for designing by offering two additional shape parameters for each directrix.
(2) Sufficient conditions are obtained in terms of geometric properties such as tangents, normals, curvatures and data points. Therefore, complex differential expressions are not required for efficient implementation.

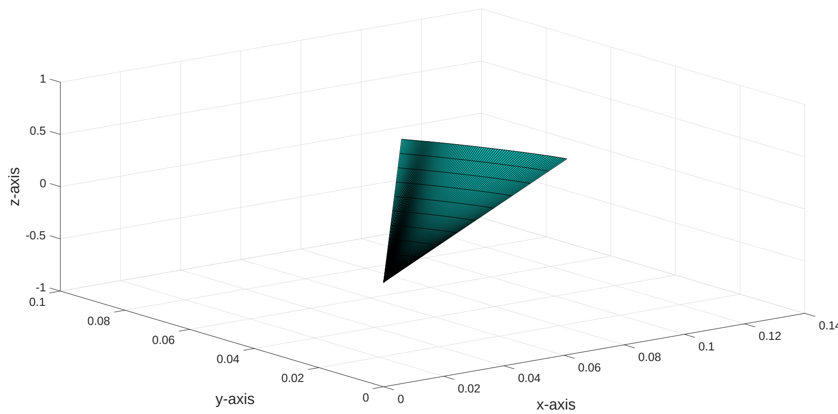

FIGURE 3. Canonical surface constructed using Theorem 2.3, achieved through shape parameter selection alone with $\alpha_0 = 0.6983, \alpha_1 = 0.6968, \beta_0 = 0.1541, \beta_1 = 0.0790.$

5. CONCLUSION

In this paper, a novel scheme is proposed for construction of developable surfaces using G^2-quintic Bézier curves with salient contributions as follows:

- Sufficient conditions for developability are derived in simple, closed form, directly relating the geometric properties of the directrix curves.
- G^2-continuous cylindrical and canonical surfaces are constructed as special cases, with conditions expressed in terms of tangents, normals and shape parameters.
- G^2-connection constraints for composite developable surfaces are established through abutting quintic Bézier geodesics.

The numerical examples validate the theoretical results of proposed scheme confirming its implementation for practical problems.

CREDIT AUTHORSHIP CONTRIBUTION'S STATEMENT

Maria Hussain: Conceptualization, Methodology, Investigation, Supervision, Software, Validation & Visualization, Writing – review & editing. Misbah Jabeen: Formal analysis, Investigation, Methodology, Software, Validation & Visualization, Writing – original draft. Hafiz Abdul Wajid: Investigation, Supervision, Validation & Visualization, Writing – review & editing.

DECLARATIONS

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APPENDIX

The following expressions are used in the proofs of Theorems 2.1, 2.2, 2.3 and 3.1. They are provided exactly as in the manuscript. $\mathbf{r}(v) = \sum_{i=0}^{5}\binom{5}{i}(1-v)^{5-i}v^i b_i, \quad \mathbf{r_1}(v) = \sum_{i=0}^{5}\binom{5}{i}(1-v)^{5-i}v^i c_i$, with

$$b_0 = P_0, \quad b_1 = P_0 + \frac{\alpha_0 T_0}{5}, \quad b_2 = P_0 + \left(\frac{2\alpha_0}{5} + \frac{\beta_0}{20}\right) T_0 + \frac{k_0\alpha_0^2 N_0}{20},$$
$$b_3 = P_1 + \left(\frac{-2\alpha_1}{5} + \frac{\beta_1}{20}\right) T_1 + \frac{k_1\alpha_1^2 N_1}{20}, \quad b_4 = P_1 - \frac{\alpha_1 T_1}{5}, \quad b_5 = P_1,$$
$$c_0 = Q_0, \quad c_1 = Q_0 + \frac{\widehat{\alpha}_0 \widehat{T}_0}{5}, \quad c_2 = Q_0 + \left(\frac{2\widehat{\alpha}_0}{5} + \frac{\widehat{\beta}_0}{20}\right) \widehat{T}_0 + \frac{\widehat{k}_0\widehat{\alpha}_0^2 \widehat{N}_0}{20},$$
$$c_3 = Q_1 + \left(\frac{-2\widehat{\alpha}_1}{5} + \frac{\widehat{\beta}_1}{20}\right) \widehat{T}_1 + \frac{\widehat{k}_1\widehat{\alpha}_1^2 \widehat{N}_1}{20}, \quad c_4 = Q_1 - \frac{\widehat{\alpha}_1 \widehat{T}_1}{5}, \quad c_5 = Q_1.$$

$\mathbf{r}'(v) = 5\sum_{i=0}^{4}\binom{4}{i}(1-v)^{4-i}v^i B_i, \quad \mathbf{r_1}'(v) = 5\sum_{i=0}^{4}\binom{4}{i}(1-v)^{4-i}v^i \widetilde{B}_i$, with

$$B_0 = \frac{\alpha_0 T_0}{5},$$
$$B_1 = \left(\frac{\alpha_0}{5} + \frac{\beta_0}{20}\right) T_0 + \frac{k_0\alpha_0^2 N_0}{20},$$
$$B_2 = (P_1 - P_0) + \left(\frac{-2\alpha_1}{5} + \frac{\beta_1}{20}\right) T_1 + \frac{k_1\alpha_1^2 N_1}{20} - \left(\frac{2\alpha_0}{5} + \frac{\beta_0}{20}\right) T_0 - \frac{k_0\alpha_0^2 N_0}{20},$$
$$B_3 = \left(\frac{\alpha_1}{5} - \frac{\beta_1}{20}\right) T_1 - \frac{k_1\alpha_1^2 N_1}{20},$$
$$B_4 = \frac{\alpha_1 T_1}{5},$$
$$\widetilde{B}_0 = \frac{\widehat{\alpha}_0 \widehat{T}_0}{5},$$
$$\widetilde{B}_1 = \left(\frac{\widehat{\alpha}_0}{5} + \frac{\widehat{\beta}_0}{20}\right) \widehat{T}_0 + \frac{\widehat{k}_0\widehat{\alpha}_0^2 \widehat{N}_0}{20},$$
$$\widetilde{B}_2 = (Q_1 - Q_0) + \left(\frac{-2\widehat{\alpha}_1}{5} + \frac{\widehat{\beta}_1}{20}\right) \widehat{T}_1 + \frac{\widehat{k}_1\widehat{\alpha}_1^2 \widehat{N}_1}{20} - \left(\frac{2\widehat{\alpha}_0}{5} + \frac{\widehat{\beta}_0}{20}\right) \widehat{T}_0 - \frac{\widehat{k}_0\widehat{\alpha}_0^2 \widehat{N}_0}{20},$$
$$\widetilde{B}_3 = \left(\frac{\widehat{\alpha}_1}{5} - \frac{\widehat{\beta}_1}{20}\right) \widehat{T}_1 - \frac{\widehat{k}_1\widehat{\alpha}_1^2 \widehat{N}_1}{20},$$
$$\widetilde{B}_4 = \frac{\widehat{\alpha}_1 \widehat{T}_1}{5}.$$

$\mathbf{y}'(t) = 5\sum_{i=0}^{4}\binom{4}{i}(1-t)^{4-i}t^i \widetilde{\widetilde{b}}_i, \quad \mathbf{y_1}'(t) = 5\sum_{i=0}^{4}\binom{4}{i}(1-t)^{4-i}t^i \widetilde{\widetilde{c}}_i,$

with

$$
\begin{aligned}
\widetilde{\widetilde{b}}_0 &= \frac{\gamma_0 t_0}{5}, \\
\widetilde{\widetilde{b}}_1 &= \left(\frac{\gamma_0}{5} + \frac{\mu_0}{20}\right) t_0 + \frac{k_2 \gamma_0^2 n_0}{20}, \\
\widetilde{\widetilde{b}}_2 &= (f_1 - f_0) + \left(\frac{-2\gamma_1}{5} + \frac{\mu_1}{20}\right) t_1 + \frac{k_3 \gamma_1^2 n_1}{20} - \left(\frac{2\gamma_0}{5} + \frac{\mu_0}{20}\right) t_0 - \frac{k_2 \gamma_0^2 n_0}{20}, \\
\widetilde{\widetilde{b}}_3 &= \left(\frac{\gamma_1}{5} - \frac{\mu_1}{20}\right) t_1 - \frac{k_3 \gamma_1^2 n_1}{20}, \\
\widetilde{\widetilde{b}}_4 &= \frac{\gamma_1 t_1}{5}, \\
\widetilde{\widetilde{c}}_0 &= \frac{\widehat{\gamma}_0 \widehat{t}_0}{5}, \\
\widetilde{\widetilde{c}}_1 &= \left(\frac{\widehat{\gamma}_0}{5} + \frac{\widehat{\mu}_0}{20}\right) \widehat{t}_0 + \frac{\widehat{k}_2 \widehat{\gamma}_0^2 \widehat{n}_0}{20}, \\
\widetilde{\widetilde{c}}_2 &= (g_1 - g_0) + \left(\frac{-2\widehat{\gamma}_1}{5} + \frac{\widehat{\mu}_1}{20}\right) \widehat{t}_1 + \frac{\widehat{k}_3 \widehat{\gamma}_1^2 \widehat{n}_1}{20} - \left(\frac{2\widehat{\gamma}_0}{5} + \frac{\widehat{\mu}_0}{20}\right) \widehat{t}_0 - \frac{\widehat{k}_2 \widehat{\gamma}_0^2 \widehat{n}_0}{20}, \\
\widetilde{\widetilde{c}}_3 &= \left(\frac{\widehat{\gamma}_1}{5} - \frac{\widehat{\mu}_1}{20}\right) \widehat{t}_1 - \frac{\widehat{k}_3 \widehat{\gamma}_1^2 \widehat{n}_1}{20}, \\
\widetilde{\widetilde{c}}_4 &= \frac{\widehat{\gamma}_1 \widehat{t}_1}{5}.
\end{aligned}
$$

Also,

$$
\mathbf{r}_1(v) - \mathbf{r}(v) = \sum_{i=0}^{5} \binom{5}{i} (1-v)^{5-i} v^i X_i,
$$

with

$$
\begin{aligned}
X_0 &= Q_0 - P_0, \\
X_1 &= (Q_0 - P_0) + \frac{\widehat{\alpha}_0 \widehat{T}_0}{5} - \frac{\alpha_0 T_0}{5}, \\
X_2 &= (Q_0 - P_0) + \left(\frac{2\widehat{\alpha}_0}{5} + \frac{\widehat{\beta}_0}{20}\right) \widehat{T}_0 + \frac{\widehat{k}_0 \widehat{\alpha}_0^2 \widehat{N}_0}{20} - \left(\frac{2\alpha_0}{5} + \frac{\beta_0}{20}\right) T_0 - \frac{k_0 \alpha_0^2 N_0}{20}, \\
X_3 &= (Q_1 - P_1) + \left(\frac{-2\widehat{\alpha}_1}{5} + \frac{\widehat{\beta}_1}{20}\right) \widehat{T}_1 + \frac{\widehat{k}_1 \widehat{\alpha}_1^2 \widehat{N}_1}{20} - \left(\frac{-2\alpha_1}{5} + \frac{\beta_1}{20}\right) T_1 - \frac{k_1 \alpha_1^2 N_1}{20}, \\
X_4 &= (Q_1 - P_1) - \frac{\widehat{\alpha}_1 \widehat{T}_1}{5} + \frac{\alpha_1 T_1}{5}, \\
X_5 &= Q_1 - P_1.
\end{aligned}
$$

And

$$Z_0 = (X_0 \times \widetilde{B}_0),$$

$$Z_1 = 4(X_0 \times \widetilde{B}_1) + 5(X_1 \times \widetilde{B}_0),$$

$$Z_2 = 6(X_0 \times \widetilde{B}_2) + 20(X_1 \times \widetilde{B}_1) + 10(X_2 \times \widetilde{B}_0).$$

$$Z_3 = 4(X_0 \times \widetilde{B}_3) + 30(X_1 \times \widetilde{B}_2) + 40(X_2 \times \widetilde{B}_1) + 10(X_3 \times \widetilde{B}_0),$$
$$Z_4 = (X_0 \times \widetilde{B}_4) + 20(X_1 \times \widetilde{B}_3) + 60(X_2 \times \widetilde{B}_2) + 40(X_3 \times \widetilde{B}_1) + 5(X_4 \times \widetilde{B}_0),$$
$$Z_5 = 5(X_1 \times \widetilde{B}_4) + 40(X_2 \times \widetilde{B}_3) + 60(X_3 \times \widetilde{B}_2) + 20(X_4 \times \widetilde{B}_1) + (X_5 \times \widetilde{B}_0),$$
$$Z_6 = 10(X_2 \times \widetilde{B}_4) + 40(X_3 \times \widetilde{B}_3) + 30(X_4 \times \widetilde{B}_2) + 4(X_5 \times \widetilde{B}_1),$$
$$Z_7 = 10(X_3 \times \widetilde{B}_4) + 20(X_4 \times \widetilde{B}_3) + 6(X_5 \times \widetilde{B}_2),$$
$$Z_8 = 4(X_5 \times \widetilde{B}_3) + 5(X_4 \times \widetilde{B}_4), \quad Z_9 = (X_5 \times \widetilde{B}_4).$$

And

$$Y_0 = \frac{\alpha_0 T_0}{5} - \frac{\widehat{\alpha}_0 \widehat{T}_0}{5}, \quad Y_1 = \left(\frac{\alpha_0}{5} + \frac{\beta_0}{20}\right) T_0 + \frac{k_0 \alpha_0^2 N_0}{20} - \left(\frac{\widehat{\alpha}_0}{5} + \frac{\widehat{\beta}_0}{20}\right) \widehat{T}_0 - \frac{\widehat{k}_0 \widehat{\alpha}_0^2 \widehat{N}_0}{20},$$

$$Y_2 = (P_1 - P_0) + \left(\frac{-2\alpha_1}{5} + \frac{\beta_1}{20}\right) T_1 + \frac{k_1 \alpha_1^2 N_1}{20} - \left(\frac{2\alpha_0}{5} + \frac{\beta_0}{20}\right) T_0 - \frac{k_0 \alpha_0^2 N_0}{20} - (Q_1 - Q_0)$$
$$- \left(\frac{-2\widehat{\alpha}_1}{5} + \frac{\widehat{\beta}_1}{20}\right) \widehat{T}_1 - \frac{\widehat{k}_1 \widehat{\alpha}_1^2 \widehat{N}_1}{20} + \left(\frac{2\widehat{\alpha}_0}{5} + \frac{\widehat{\beta}_0}{20}\right) \widehat{T}_0 + \frac{\widehat{k}_0 \widehat{\alpha}_0^2 \widehat{N}_0}{20},$$

$$Y_3 = \left(\frac{\alpha_1}{5} - \frac{\beta_1}{20}\right) T_1 - \frac{k_1 \alpha_1^2 N_1}{20} - \left(\frac{\widehat{\alpha}_1}{5} - \frac{\widehat{\beta}_1}{20}\right) \widehat{T}_1 + \frac{\widehat{k}_1 \widehat{\alpha}_1^2 \widehat{N}_1}{20},$$

$$Y_4 = \frac{\alpha_1 T_1}{5} - \frac{\widehat{\alpha}_1 \widehat{T}_1}{5}.$$

The expressions for $\widetilde{C}_i$ are lengthy and are omitted for brevity; they are exactly as given in the manuscript. They are not needed for compilation but are included in the original document.

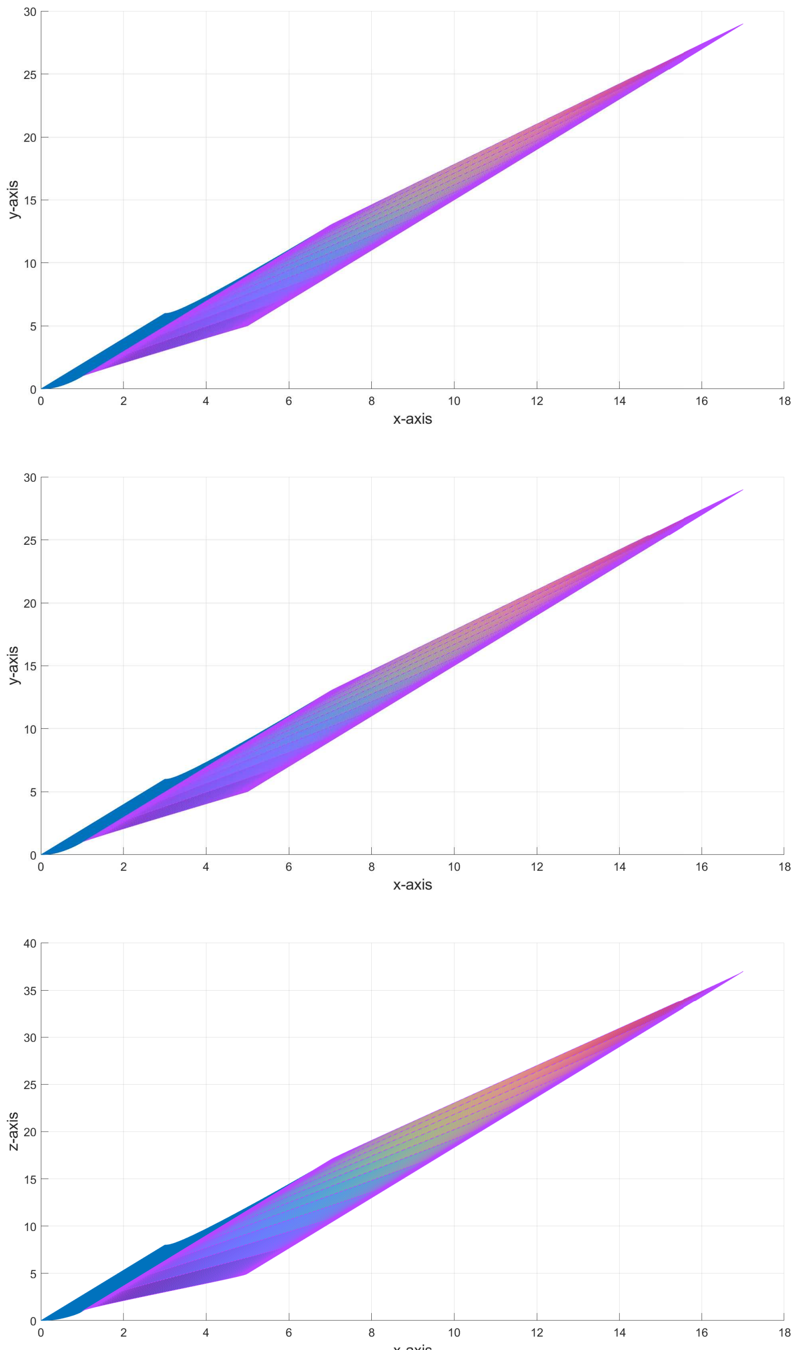

FIGURE 4. Composite developable surface constructed using Theorem 2.2 and Theorem 3.1 and its different views.