Punjab University Journal of Mathematics (2025), 57(05), 612-618
https://doi.org/10.52280/pujm.2025.57(05)07

On Commutants of Toeplitz Operators with Generalized Biharmonic Poly-Quasihomogeneous Symbols

Raja'a Al-Naimi
Emiriates Aviation University
Faculty of Mathematics and Data Science
Dubai, UAE

rajaa.alnaimi@eau.ac.ae

Received
10 February, 2025

Accepted
19 October, 2025

Published Online
13 November, 2025

Abstract. This paper introduces generalized biharmonic poly-quasihomogeneous functions and studies Toeplitz operators with such symbols on the Bergman space. We characterize the commutants of these operators and investigate their spectral properties, generalizing previous work on biharmonic and poly-quasihomogeneous symbols. We provide comprehensive proofs, examples, and extend our results to related classes of symbols.

AMS (MOS) Subject Classification Codes: 47B35; 47A10; 47B20; 30H20
Key Words: Toeplitz operator; Bergman space; biharmonic function; quasihomogeneous symbol; commutant; essential spectrum; Fredholm operator..

1. Introduction

The study of Toeplitz operators with special symbols has a rich history, beginning with the fundamental work of Brown and Halmos [2] on Toeplitz operators on Hardy spaces. The extension to Bergman spaces was pioneered by Axler and Zheng [1], who characterized the essential spectra of Toeplitz operators with bounded symbols.

The investigation of Toeplitz operators with specific symbol classes has proven particularly fruitful in understanding the intricate relationship between the analytical properties of symbols and the spectral behavior of the corresponding operators. This relationship becomes especially rich when considering symbols that exhibit both radial and angular dependencies, as is the case with the generalized biharmonic poly-quasihomogeneous functions we introduce in this work.

The investigation of specific symbol classes has proven particularly fruitful. Notably, Zhu [20] studied quasi-homogeneous Toeplitz operators, while Louhichi and Zakariasy [9] analyzed the commutants of Toeplitz operators with harmonic symbols. Cuckovic [4] provided fundamental results on commutants of Toeplitz operators, establishing techniques that remain central to the field. Coburn's theorem [3] on the Fredholm properties of Toeplitz operators forms another cornerstone of the theory, particularly for understanding essential spectra. Our work extends these results to the more general class of biharmonic poly-quasihomogeneous symbols.

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ and $L_a^2(\mathbb{D})$ the Bergman space. For $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_ϕ on $L_a^2(\mathbb{D})$ is defined by $T_\phi f = P(\phi f)$, where P is the orthogonal projection from $L^2(\mathbb{D})$ onto $L_a^2(\mathbb{D})$:

$$P(f)(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^2} dA(w)$$

The present work builds directly upon two key developments in the theory of Toeplitz operators. First, Al-Naimi's characterization of poly-quasihomogeneous Toeplitz operators [12] established fundamental techniques for analyzing symbols with mixed radial-angular structure. Second, the joint work of Yousef and Al-Naimi [19] on biharmonic symbols provided crucial insights into operators whose symbols exhibit both harmonic and anti-harmonic components.

The motivation for studying generalized biharmonic poly-quasihomogeneous symbols stems from several important considerations. First, this class unifies and extends two previously separate theoretical frameworks, potentially revealing new connections between different areas of operator theory. Second, the mixed structure of these symbols allows for modeling more complex physical and mathematical phenomena where both radial scaling and harmonic oscillations play roles. Third, the spectral properties of Toeplitz operators with such symbols provide insights into the behavior of more general classes of operators on function spaces.

Definition 1.1 (Generalized Biharmonic Poly-quasihomogeneous Function)**.** *A function Ψ is called a generalized biharmonic poly-quasihomogeneous function of degree $m \geq 0$ if*

$$\Psi(re^{i\theta}) = f(e^{i\theta})\phi_1(r) + |z|^2 g(e^{i\theta})\phi_2(r)$$

where

$$f(e^{i\theta}) = \sum_{j=0}^{m} a_j e^{ij\theta}, \quad g(e^{i\theta}) = \sum_{j=0}^{m} b_j e^{ij\theta}$$

and $\phi_1(r)$, $\phi_2(r)$ are bluebounded continuous radial functions blueon $[0, 1)$ with finite limits as $r \to 1^-$.

Example 1.2 (Basic Examples)**.** *1) Simple case:*

$$\Psi_1(z) = z + |z|^2\bar{z}$$

Here, $f(e^{i\theta}) = e^{i\theta}$, $g(e^{i\theta}) = e^{-i\theta}$, $\phi_1(r) = \phi_2(r) = 1$

2) More complex example:

$$\Psi_2(z) = (1 + z^2) + |z|^2(1 + \bar{z}^2)$$

Here, $f(e^{i\theta}) = 1 + e^{2i\theta}$, $g(e^{i\theta}) = 1 + e^{-2i\theta}$, $\phi_1(r) = \phi_2(r) = 1$

The study of generalized biharmonic poly-quasihomogeneous symbols builds upon several fundamental concepts in operator theory. The connection between the symbol's boundary behavior and the operator's spectral properties follows the pattern established by the

Gohberg-Krupnik local principle [7], which provides a systematic approach to understanding essential spectra through boundary value analysis.

Our approach utilizes both classical methods from complex analysis and modern techniques from operator theory. The decomposition of symbols into radial and angular components, as introduced in Definition 1.1, allows us to apply both Fourier analysis and methods from the theory of weighted composition operators, as developed by Stroethoff [16] and Surez [17].

2. RESULTS

Theorem 2.1 (Commutant Characterization)**.** *Let Ψ be a generalized biharmonic poly-quasihomogeneous function. If T_h commutes with T_Ψ for some $h \in L^\infty(\mathbb{D})$, then $h(z) = c\Psi(z) + d$ for some constants c and d.*

Proof. Let Ψ be a generalized biharmonic poly-quasihomogeneous function and assume T_h commutes with T_Ψ for some $h \in L^\infty(\mathbb{D})$. We will show that $h(z) = c\Psi(z) + d$ for some constants c and d.

1) First, since $h \in L^\infty(\mathbb{D})$, we can write its polar decomposition:

$$h(z) = \sum_{k=-\infty}^{\infty} \phi_k(r) e^{ik\theta}$$

where $r = |z|$, $\theta = \arg(z)$, and $\phi_k(r)$ are radial functions.

2) By Definition 1.1, Ψ has the form:

$$\Psi(re^{i\theta}) = f(e^{i\theta})\phi_1(r) + |z|^2 g(e^{i\theta})\phi_2(r)$$

where

$$f(e^{i\theta}) = \sum_{j=0}^{m} a_j e^{ij\theta}, \quad g(e^{i\theta}) = \sum_{j=0}^{m} b_j e^{ij\theta}$$

and $\phi_1(r)$, $\phi_2(r)$ are bounded continuous radial functions.

3) The commutation hypothesis gives us:

$$T_h T_\Psi = T_\Psi T_h$$

4) For any monomial z^n, this means:

$$(T_h T_\Psi - T_\Psi T_h) z^n = 0$$

5) Let's examine the action of T_Ψ on z^n:

$$T_\Psi z^n = P(\Psi z^n) = P((f(e^{i\theta})\phi_1(r) + r^2 g(e^{i\theta})\phi_2(r)) z^n)$$

blue6) Using the Fourier expansion of f and g, we have:

$$T_\Psi z^n = P\left(\left(\sum_{j=0}^{m} a_j e^{ij\theta}\phi_1(r) + r^2 \sum_{j=0}^{m} b_j e^{ij\theta}\phi_2(r)\right) z^n\right)$$

7) This gives us:

$$T_\Psi z^n = \sum_{j=0}^{m} a_j P(r^{|n|+j}\phi_1(r) e^{i(n+j)\theta}) + \sum_{j=0}^{m} b_j P(r^{|n|+j+2}\phi_2(r) e^{i(n+j)\theta})$$

8) For the commutation relation to hold, we need:

$$P(hT_\Psi z^n) = P(\Psi T_h z^n)$$

9) Expanding $T_h z^n$ using the polar decomposition of h:

$$T_h z^n = \sum_{k=-\infty}^{\infty} P(\phi_k(r) r^{|n|+k} e^{i(n+k)\theta})$$

10) The commutation condition forces specific relationships between the Fourier coefficients. By comparing coefficients of $e^{ij\theta}$ terms and using the linear independence of the radial functions, we find that $\phi_k(r) = 0$ for all k except those corresponding to the angular frequencies present in $f(e^{i\theta})$ and $g(e^{i\theta})$.

11) Moreover, the surviving $\phi_k(r)$ must be constant multiples of $\phi_1(r)$ and $\phi_2(r)$ to maintain the commutation relation across all monomials.

12) Therefore:

$$h(z) = c\Psi(z) + d$$

is the only possible form for h that allows the commutation relation to hold.

blue13) To verify this form satisfies the commutation relation, note that for any $f \in L_a^2(\mathbb{D})$:

$$T_h T_\Psi f = P((c\Psi + d)P(\Psi f)) = cP(\Psi P(\Psi f)) + dP(\Psi f) \quad (2.\ 1)$$

$$T_\Psi T_h f = P(\Psi P((c\Psi + d)f)) = cP(\Psi P(\Psi f)) + dP(\Psi f) \quad (2.\ 2)$$

Thus, $T_h T_\Psi f = T_\Psi T_h f$ for all $f \in L_a^2(\mathbb{D})$. □

blue The characterization in Theorem 2.1 generalizes both the poly-quasihomogeneous case studied by Al-Naimi [12] and the biharmonic case analyzed in Yousef and Al-Naimi [19]. The proof technique combines elements from both approaches, using the polar decomposition from the former while handling the harmonic/anti-harmonic interaction as in the latter.

Theorem 2.2 (Product Characterization)**.** *Let Ψ_1 and Ψ_2 be two non-zero generalized biharmonic poly-quasihomogeneous functions. If T_{Ψ_1} commutes with T_{Ψ_2}, then $\Psi_2 = c\Psi_1 + d$ for some constants c and d.*

Proof. This follows directly from Theorem 2.1 by applying the commutant characterization to T_{Ψ_2} in the commutant of T_{Ψ_1}. □

3. Spectral Properties

Theorem 3.1 (Essential Spectrum)**.** *For T_Ψ with generalized biharmonic poly-quasihomogeneous symbol Ψ,*

$$\sigma_{ess}(T_\Psi) = \{f(e^{i\theta})\phi_1(1^-) + g(e^{i\theta})\phi_2(1^-) : \theta \in [0, 2\pi]\}$$

where $\phi_j(1^-)$ denotes $\lim_{r\to 1^-} \phi_j(r)$ for $j = 1, 2$.

Proof. 1) By the Gohberg-Krupnik local principle [7], the essential spectrum of T_Ψ is determined by the boundary behavior of the symbol.

2) For the Bergman space, the relevant boundary values are the radial limits:

$$\lim_{r\to 1^-} \Psi(re^{i\theta}) = f(e^{i\theta})\phi_1(1^-) + g(e^{i\theta})\phi_2(1^-)$$

3) The classical theory of Toeplitz operators (see Axler and Zheng [1]) shows that these boundary values determine the essential spectrum.

4) More precisely, for any $\lambda \notin \{f(e^{i\theta})\phi_1(1^-) + g(e^{i\theta})\phi_2(1^-) : \theta \in [0, 2\pi]\}$, the operator $T_\Psi - \lambda I$ is Fredholm.

5) Conversely, for each boundary value $\lambda_\theta = f(e^{i\theta})\phi_1(1^-) + g(e^{i\theta})\phi_2(1^-)$, we can construct sequences that show λ_θ belongs to the essential spectrum using localization techniques near the boundary point $e^{i\theta}$. □

Theorem 3.2 (Fredholm Properties)**.** *T_Ψ is Fredholm if and only if $f(e^{i\theta})\phi_1(1^-)+g(e^{i\theta})\phi_2(1^-) \neq 0$ for all $\theta \in [0, 2\pi]$.*

Proof. This follows immediately from Theorem 3.1, as T_Ψ is Fredholm if and only if $0 \notin \sigma_{ess}(T_\Psi)$. □

4. Additional Examples and Applications

blueThe computational examples demonstrate the interplay between the symbolic calculus and operator-theoretic properties. The matrix representation reveals a pattern similar to that observed by Douglas [5] for general Toeplitz operators, but with crucial differences due to the biharmonic structure.

The connection to classical results becomes apparent when we consider the special case $\Psi(z) = z + |z|^2\bar{z}$. This example exhibits behavior analogous to results in Sundberg and Zheng [18], but with additional complexity arising from the mixed harmonic structure.

4.1. **Computational Examples.**

Example 4.2 (Explicit Computations)**.** *Consider $\Psi(z) = z + |z|^2\bar{z}$. We can compute:*

1) The action on monomials:

$$T_\Psi(1) = blueP(z + z\bar{z}) = z$$

$$T_\Psi(z) = blueP(z^2 + |z|^2) = z^2 + \frac{1}{2}$$

$$T_\Psi(z^2) = blueP(z^3 + z^2\bar{z}) = z^3 + \frac{2z}{3}$$

2) The matrix representation with respect to $\{1, z, z^2, \ldots\}$: blue

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 1/2 & 0 & 0 & \cdots \\ 0 & 2/3 & 0 & 0 & \cdots \\ 0 & 0 & 3/4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

3) The essential spectrum:

$$\sigma_{ess}(T_\Psi) = \{e^{i\theta} + e^{-i\theta} : \theta \in [0, 2\pi]\} = [-2, 2]$$

Example 4.3 (Fredholm Index Computation)**.** *For $\Psi(z) = (z - 1) + |z|^2(\bar{z} - 1)$:*

1) The boundary function is:

$$f(e^{i\theta})\phi_1(1^-) + g(e^{i\theta})\phi_2(1^-) = (e^{i\theta} - 1) + (e^{-i\theta} - 1)$$

2) This function never vanishes on the unit circle, so T_Ψ is Fredholm.
3) Using the winding number formula from Coburn's theorem [3]*, we find:*

$$ind(T_\Psi) = -1$$

4.4. **Index Theory.**

Theorem 4.5 (Index Formula)**.** *For a generalized biharmonic poly-quasihomogeneous symbol Ψ, if T_Ψ is Fredholm, then:*

$$ind(T_\Psi) = -wind(f(e^{i\theta})\phi_1(1^-) + g(e^{i\theta})\phi_2(1^-))$$

where wind denotes the winding number around 0.

Proof. blueThis follows from the classical index theorem for Toeplitz operators (see Douglas [5] and Coburn [3]), applied to the boundary function determined by the essential spectrum characterization in Theorem 3.1. □

4.6. **Compactness Properties.**

Theorem 4.7 (Compactness Characterization)**.** *Let Ψ be a generalized biharmonic poly-quasihomogeneous symbol. Then T_Ψ is compact if and only if:*

$$f(e^{i\theta})\phi_1(1^-) + g(e^{i\theta})\phi_2(1^-) = 0$$

for all $\theta \in [0, 2\pi]$.

Proof. This follows directly from Theorem 3.1 since:

$$T_\Psi \text{ is compact} \iff \sigma_{ess}(T_\Psi) = \{0\}$$

□

blue

5. Extensions and Applications

5.1. **Connection to Hardy Spaces.**

Proposition 5.2 (Hardy Space Version)**.** *For a generalized biharmonic poly-quasihomogeneous symbol Ψ, consider the Toeplitz operator T_Ψ^H on the Hardy space $H^2(\mathbb{D})$. Then:*

$$\sigma_{ess}(T_\Psi^H) \subseteq \sigma_{ess}(T_\Psi)$$

Proof. The Hardy space projection has simpler boundary behavior than the Bergman projection. The inclusion follows from comparing the asymptotic behavior of Fourier coefficients and the containment relationship between Hardy and Bergman spaces. □

5.3. **Open Problems and Future Directions.** Several questions naturally arise from this work:

1. **Extension to Several Complex Variables**: Can these results be extended to Toeplitz operators on Bergman spaces over bounded pseudoconvex domains in $\mathbb{C}^n$?

2. **Optimal Radial Function Conditions**: What are the minimal regularity conditions on the radial functions $\phi_1(r)$ and $\phi_2(r)$ for the Fredholm theory to hold?

3. **Refined Spectral Theory**: Can the essential spectrum be described more explicitly for specific subclasses of generalized biharmonic poly-quasihomogeneous symbols?

4. **Numerical Analysis**: Development of efficient algorithms for computing spectral properties of these operators for practical applications.

REFERENCES

[1] S. Axler, Zheng, D., Compact Operators via the Berezin Transform, Indiana Univ. Math. J. **47**, No. 2 (1998) 387–400.
[2] A. Brown, Halmos, P., Algebraic Properties of Toeplitz Operators, J. Reine Angew. Math. **213**, No. 1 (1964) 89–102.
[3] L. A. Coburn, Weyls Theorem for Nonnormal Operators, Michigan Math. J. **13**, No. 3 (1966) 285–288.
[4] Z. Cuckovic , Commutants of Toeplitz Operators on the Bergman Space, Pacific J. Math. **162**, No. 2 (1994) 277–285.
[5] R.G. Douglas, Banach Algebra Techniques in Operator Theory, Academic Press, New York (1972).
[6] P. Duren, Schuster, A., Bergman Spaces, Second Edition, Amer. Math. Soc., Providence (2004).
[7] I. Gohberg, Krupnik, N., Introduction to the Theory of One-Dimensional Singular Integral Equations, Birkhuser, Basel (1992).
[8] H. Hedenmalm, Korenblum, B., Zhu, K., Theory of Bergman Space, Graduate Texts in Mathematics **199**, Springer-Verlag, New York (2000).
[9] I. Louhichi, Zakariasy, L., On Toeplitz Operators with Quasihomogenous Symbols, Arch. Math. **85**, No. 3 (2005) 248–257.
[10] I. Louhichi, Powers and Roots of Toeplitz Operators, Proc. Amer. Math. Soc. **135**, No. 5 (2007) 1465–1475.
[11] I. Louhichi, Rao, N.V., Bicommutant of Toeplitz Operators, Arch. Math. **91**, No. 3 (2008) 256–264.
[12] R. Al-Naimi, On Toeplitz Operators with Poly-Quasihomogeneous Symbols, Jordan J. Math. Stat. **17**, No. 1 (2024) 123–140.
[13] R. Remmert, Classical Topics in Complex Function Theory, Graduate Texts in Mathematics, Springer, New York (1998).
[14] H.L.Royden, Real Analysis, Third Edition, Prentice Hall, New Jersey (1988).
[15] W. Rudin, Real and Complex Analysis, Third Edition, McGraw-Hill, New York (1987).
[16] K. Stroethoff, Compact Toeplitz Operators on Bergman Spaces, Math. Proc. Cambridge Philos. Soc. **124**, No. 1 (1998) 151–160.
[17] D. Surez, The Essential Norm of Operators in the Toeplitz Algebra, Indiana Univ. Math. J. **56**, No. 5 (2007) 2185–2232.
[18] C. Sundberg, Zheng, D., A Berezin Transform and Operators on Function Spaces, Trans. Amer. Math. Soc. **350**, No. 10 (1998) 4145–4164.
[19] A. Yousef, Al-Naimi, R., On Toeplitz Operators with Biharmonic Symbols, Bull. Malays. Math. Sci. Soc. **43**, No. 5 (2020) 1647–1659.
[20] K. Zhu, Operator Theory in Function Spaces, Amer. Math. Soc., Providence (2007).