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Partial Differential Equations Possessing New Complex Derivatives

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Abstract- The recently proposed new operator for complex differentiation is applied to partial differential equations for the first time. The definition involves complex numbers in the differentiation operators. In this sense, it does not possess any resemblance with the real number differentiation of complex valued functions. First, the operator is defined with its basic properties. Several partial differential equations are considered involving the new complex number derivatives. First, the homogenous constant real coefficient linear partial differential equations are considered. If the real and imaginary components of the derivatives are equal, the imaginary parts can be ignored. For complex valued coefficients, for non-homogenous equations and for derivatives having different real and imaginary parts, the imaginary parts cannot be ignored. Some nonlinear partial differential equations are also considered. In the calculations, the Cauchy-Riemann as well as the Laplace equations appears frequently. Expression of the Hamilton's equations and the Schrödinger equation in terms of the new differential operators are outlined. The new operators have the potential to be applied to mathematical modeling of several physical problems in the future.

AMS (MOS) Subject Classification Codes: 35A08, 35C05, 35E05, 35G05, 35G20
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1. INTRODUCTION

In the usual way of differentiation, the orders of differentiations are positive integers. Fractional numbers of differentiation have been employed recently to model physical problems. Various definitions of non-integer derivatives appeared in the literature, the common ones being the Caputo, the Riemann-Liouville and the Grunwald-Letnikov definitions [10,14,21]. The usual integer first derivative has a clear geometrical meaning: It represents the slope of the tangent line to the function at the point where the derivative is evaluated.

In contrast, a simple explanation of the geometric correspondence of fractional orders of differentiation cannot be given.

Recently, complex numbers were also employed for fractional order derivatives [2,20,22]. Complex fractional orders were employed in modeling the Van der Pol oscillator [20] and the heat conduction equation [2]. Several complex fractional order differential equations with variable coefficients were posed and solved [22]. Fuzzy fractional differential equations were solved by Laplace transforms [23]. Delay-differential equations were also proposed for non-integer derivatives [11]. Nonlinear integro-differential equations were treated with fractional order differentiation [9]. Lie Group theory was applied to the fractional Sawada–Kotera–Ito equation [27]. An extension of the classical fractional derivatives with the aid of Bessels function were also given [16]. Using iterative Laplace transform techniques, fractional order diffusion equations are solved [8].

In this work, a recently defined complex derivative operator in [17] is used for modeling partial differential equations. The new operator is different from the complex fractional derivatives given in the literature. The differential operator is said to be $(m + ni)^{th}$ order, where m is the real component being a natural number and n is the imaginary component being an integer. The definition, basic properties and some applications to calculus are discussed in [17] for the new differential operator together with the geometrical definition of the first $(1 + i)^{th}$ derivative. Compared to the fractional complex derivative definitions, the definition given here is much more simple and straightforward involving only integer differentiation orders for the complex order derivatives. The new differentiation operator should also not be confused with the integer derivatives of complex variables. Textbooks on complex analysis usually discuss the integer derivatives of complex functions under the topic of "derivatives of complex valued functions" [1,15].

The linear partial differential equations with constant real coefficients are treated first. For homogenous such equations, if the real and complex parts of the number of differentiation are equal, the complex derivative model reduces to the usual real derivative model. However, if the coefficients are complex and/or the equation is non-homogenous and/or the number of differentiation is different for real and complex parts, then the imaginary derivative parts cannot be ignored. Reduction of some of the equations to Cauchy-Riemann and Laplace equations are shown. See [7,15,26] for information about Cauchy-Riemann and Laplace equations and their solutions. The nonlinear partial differential equations are treated also. The Hamilton's equation of motion [12] and the Schrödinger equation [24] are re-expressed in terms of the new complex number derivatives. For complex coefficient differential equations and their complex valued solutions with employment of the usual real derivatives, see [13,28] for example.

Travelling wave type solutions frequently appear as class of solutions in the mathematical physics problems in the form of partial differential equations such as the Extended (2+1)-dimensional Boussinesq model [4], (3+1)-dimensional nonlinear evolution equation [26], Ito integro-differential equation [3]. Travelling wave type solutions stem from the spatial and time translational symmetries of the specific partial differential equations [19]. Therefore Lie Group techniques [5,19] are widely used to reduce the problem into an ordinary differential equation. Semi analytical and analytical methods were frequently used to construct travelling wave solutions, such as the complex exponential method [18], the hyperbolic ansatz approach and the Adomian decomposition technique [4], Hirota bilinear method [25] and the (G'/G^2) expansion approach [3], to name a few of them. Implication of the new complex differential operators to advanced mathematical physics problems and the models inheriting travelling type solutions may be addressed in the future as further extensions of this pioneering study. In Section 6, some well-known examples are mentioned to exploit the idea.

In summary, application of the new derivative definition to partial differential equations presented in this work is rather new and may find applications in mathematical modeling of physical systems in the future.

2. DEFINITION OF THE NEW DERIVATIVE

The definition of the new complex derivative is given below [17].

Definition 2.1. The complex derivative operator with $(m\mp ni)^{th}$ order is defined as

$$\frac{d^{m\mp ni}}{dx^{m\mp ni}} = \frac{d^m}{dx^m} \mp i(n)\frac{d^n}{dx^n}, \quad \mu(n) = \begin{cases} 0 & \text{if} \quad n=0 \\ 1 & \text{if} \quad n\neq 0 \end{cases}, \tag{2. 1}$$

where m and n are positive integers or zero with $i=\sqrt{-1}$.

For a real valued function $y=y(x)$

$$\frac{d^{m+ni}(y)}{dx^{m+ni}} = \frac{d^m y}{dx^m} + i\mu(n)\frac{d^n y}{dx^n}. \tag{2. 2}$$

On the contrary, the derivative of the complex valued function $z(x)=p(x)+iq(x)$ is

$$\frac{d^{m+ni}(z)}{dx^{m+ni}} = \left(\frac{d^m}{dx^m} + i\mu(n)\frac{d^n}{dx^n}\right)(p+iq) = \left(\frac{d^m p}{dx^m} - \mu(n)\frac{d^n q}{dx^n}\right) + i\left(\frac{d^m q}{dx^m} + \mu(n)\frac{d^n p}{dx^n}\right). \tag{2. 3}$$

When stating the first complex derivative, one may choose the phrase "$(1+i)^{th}$ derivative", and for the n'th complex derivative "$(n+ni)^{th}$ derivative" for brevity although the longer notation is always preferable to avoid ambiguity.

For the successive operation of the complex number derivatives, the relationship is

$$\frac{d^{m+ni}}{dx^{m+ni}}\frac{d^{k+li}}{dx^{k+li}} = \frac{d^{m+k}}{dx^{m+k}} - \mu(n)\mu(l)\frac{d^{n+l}}{dx^{n+l}} + i\left(\mu(l)\frac{d^{m+l}}{dx^{m+l}} + \mu(n)\frac{d^{n+k}}{dx^{n+k}}\right), \tag{2. 4}$$

with m and k being natural numbers and n and l being integers. The above expression can be deduced from the successive application of the Definition 2.1. Note that successive operation of the complex conjugate derivatives produces real derivatives

$$\frac{d^{m+ni}}{dx^{m+ni}}\frac{d^{m-ni}}{dx^{m-ni}} = \frac{d^{2m}}{dx^{2m}} + \frac{d^{2n}}{dx^{2n}}. \tag{2. 5}$$

Another property of the operators is the below identity

$$\frac{d^{m+ni}}{dx^{m+ni}} = i\frac{d^{n-mi}}{dx^{n-mi}}, \tag{2. 6}$$

where $m>0$ and $n>0$. For more information about the properties of the new derivative definition, as well as its geometrical meaning, see [17].

3. LINEAR PDES WITH IMAGINARY DERIVATIVES IGNORABLE

If the partial differential equation has the following properties

(a) Linear
(b) Homogeneous
(c) Real coefficients
(d) Real and complex number parts of all derivatives being equal,

then, the imaginary number derivative PDE reduces to the corresponding real number derivative PDE.

Theorem 3.1. For a linear homogeneous real coefficient partial differential equation with all derivatives having equal components of real and imaginary parts, the imaginary parts of the derivatives can be ignored

Proof Assume that the complex partial derivative operator $\mathcal{L}^c$ has equivalent real and imaginary derivative parts. It can then be decomposed into the form $\mathcal{L}^c = \mathcal{L} + i\mathcal{L}$ where $\mathcal{L}$ is a real valued partial differential operator. For a complex valued function $z = p + iq$

$$\mathcal{L}^c z = (\mathcal{L} + i\mathcal{L})(p + iq) = \mathcal{L}p - \mathcal{L}q + i(\mathcal{L}p + \mathcal{L}q) = 0.$$

Separating real and imaginary parts

$$\mathcal{L}p - \mathcal{L}q = 0, \mathcal{L}p + \mathcal{L}q = 0,$$

and solving yields

$$\mathcal{L}p = 0, \mathcal{L}q = 0.$$

The first equation added by the multiplication of the second equation by *i* yields

$$\mathcal{L}z = 0,$$

which indicates that the imaginary parts of the derivative operator can be neglected.

Example 3.2. Consider the first order complex derivative partial differential equation,

$$\frac{\partial^{1+i} z}{\partial x^{1+i}} = \frac{\partial^{1+i} z}{\partial t^{1+i}}, \tag{3. 7}$$

with $z = p(x,t) + iq(x,t)$. Applying Theorem 3.1, the equation with imaginary parts ignored is

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial t}, \tag{3. 8}$$

or

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial t}, \frac{\partial q}{\partial x} = \frac{\partial q}{\partial t}, \tag{3. 9}$$

with the solutions $p = p(x + t)$ and $q = q(x + t)$ by the method of characteristics. The solution is $z(x + t) = p(x + t) + iq(x + t)$. Direct application of Definition 2.1 to (3. 7) would yield the same result.

Example 3.3. The complex derivative partial differential equation with $z = p(x,t) + iq(x,t)$,

$$\frac{\partial^{1+i} z}{\partial t^{1+i}} = \frac{\partial^{2+2i} z}{\partial x^{2+2i}}, \tag{3. 10}$$

reduces to the real derivative partial differential equation

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}. \tag{3. 11}$$

To see this, use the basic definition, i.e. Definition 2.1,

$$\left(\frac{\partial}{\partial t} + i\frac{\partial}{\partial t}\right)(p + qi) = \left(\frac{\partial^2}{\partial x^2} + i\frac{\partial^2}{\partial x^2}\right)(p + qi), \tag{3. 12}$$

which separates into real and imaginary parts

$$\frac{\partial p}{\partial t} - \frac{\partial q}{\partial t} = \frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 q}{\partial x^2}, \tag{3. 13}$$

$$\frac{\partial p}{\partial t} + \frac{\partial q}{\partial t} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 q}{\partial x^2}. \tag{3. 14}$$

Adding (3. 13) and (3. 14) and subtracting (3. 13) from (3. 14) leads to

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2}, \frac{\partial q}{\partial t} = \frac{\partial^2 q}{\partial x^2}. \tag{3. 15}$$

If the first equation is added to the i times the second equation side by side, equation (3. 11) is retrieved.

Example 3.4. By direct application of Theorem 3.1, the imaginary parts of the following equations can be ignored for $z = p(x,t) + iq(x,t)$

$$\frac{\partial^{m+mi} z}{\partial x^{m+mi}} = \frac{\partial^{n+ni} z}{\partial x^{n+ni}} \quad \rightarrow \quad \frac{\partial^m z}{\partial x^m} = \frac{\partial^n z}{\partial x^n}\ , \tag{3. 16}$$

$$\frac{\partial^{m+n+(m+n)i} z}{\partial x^{m+mi} \partial t^{n+ni}} = 0 \quad \rightarrow \quad \frac{\partial^{m+n} z}{\partial x^m \partial t^n} = 0\ . \tag{3. 17}$$

Example 3.5. The theorem is applicable for variable coefficient homogenous linear equations also as long as the coefficients are real functions. Consider the equation with $z = p(x,y) + iq(x,y)$

$$x\frac{\partial^{1+i} z}{\partial x^{1+i}} + y\frac{\partial^{1+i} z}{\partial y^{1+i}} = 0. \tag{3. 18}$$

By direct application of the derivative operator given by Definition 2.1

$$x\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial x}\right)(p + qi) + y\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial y}\right)(p + qi), \tag{3. 19}$$

the separated equations are

$$x\left(\frac{\partial p}{\partial x} - \frac{\partial q}{\partial x}\right) + y\left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial y}\right) = 0, \tag{3. 20}$$

$$x\left(\frac{\partial p}{\partial x}+\frac{\partial q}{\partial x}\right)+y\left(\frac{\partial p}{\partial y}+\frac{\partial q}{\partial y}\right)=0. \tag{3. 21}$$

Adding (3. 20) and (3. 21) and subtracting (3. 20) from (3. 21)

$$x\frac{\partial p}{\partial x}+y\frac{\partial p}{\partial y}=0, x\frac{\partial q}{\partial x}+y\frac{\partial q}{\partial y}=0, \tag{3. 22}$$

which is the separated form of

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0. \tag{3. 23}$$

The solution is

$$z=p\left(\frac{y}{x}\right)+iq\left(\frac{y}{x}\right), \tag{3. 24}$$

by the method of characteristics.

4. LINEAR PDES WITH IMAGINARY DERIVATIVES NON-IGNORABLE

In addition to being linear, if at least one of the properties hold for the complex number derivative PDE

(a) Non-Homogeneous function with non-equal real and complex parts
(b) Complex coefficients with at least one of them having non-equal real and imaginary parts
(c) Real and complex number parts of at least one of the derivatives not being equal

then, the imaginary parts of the derivatives cannot be ignored. Sample problems are given below.

Example 4.1. Consider the complex derivative partial differential equation,

$$\frac{\partial z}{\partial x}=\frac{\partial^i z}{\partial y^i}-z, \tag{4. 25}$$

with $z=p\,(x,y)+iq(x,y)$. Applying Definition 2.1

$$\frac{\partial}{\partial x}\,(p+qi)=\left(1+i\frac{\partial}{\partial y}\right)(p+qi)-(p+iq), \tag{4. 26}$$

performing the algebra and separating real and imaginary parts

$$\frac{\partial p}{\partial x}=-\frac{\partial q}{\partial y},\frac{\partial q}{\partial x}=\frac{\partial p}{\partial y}. \tag{4. 27}$$

The above equations are the Cauchy-Rieman equations encountered frequently in complex analysis and determining the exactness of first order differential equations [1,6,26]. The first equation is differentiated with respect to x and the second with respect to y. Upon adding both

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=0. \tag{4. 28}$$

On the other hand, if the first equation is differentiated with respect to y and the second with respect to x and subtracted

$$\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2}=0. \tag{4. 29}$$

Multiplying (4. 29) by *i* and adding to (4. 28)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0, \tag{4. 30}$$

which states that solving the complex number derivative PDE (4. 25) reduces to the well-known Laplace equation.

Example 4.2. The imaginary parts of the derivatives for the non-homogenous complex derivative partial differential equation with $z = p\,(x, y) + iq(x, y)$,

$$f\,(x, y)\,\frac{\partial^{m+mi} z}{\partial x^{m+mi}} + g\,(x, y)\,\frac{\partial^{n+ni} z}{\partial y^{n+ni}} = r\,(x, y) + is(x, y), \tag{4. 31}$$

cannot be ignored, since by using Definition 2.1 for the differential operators

$$f\,(x, y)\left(\frac{\partial^m}{\partial x^m} + i\frac{\partial^m}{\partial x^m}\right)(p + qi) + g\,(x, y)\left(\frac{\partial^n}{\partial x^n} + i\frac{\partial^n}{\partial x^n}\right)(p + qi) = r + is, \tag{4. 32}$$

separates into

$$f\,(x, y)\left(\frac{\partial^m p}{\partial x^m} - \frac{\partial^m q}{\partial x^m}\right) + g\,(x, y)\left(\frac{\partial^n p}{\partial y^n} - \frac{\partial^n q}{\partial y^n}\right) = r, \tag{4. 33}$$

$$f\,(x, y)\left(\frac{\partial^m p}{\partial x^m} + \frac{\partial^m q}{\partial x^m}\right) + g\,(x, y)\left(\frac{\partial^n p}{\partial y^n} + \frac{\partial^n q}{\partial y^n}\right) = s. \tag{4. 34}$$

Adding (4. 33) and (4. 34) and subtracting (4. 33) from (4. 34)

$$f\,(x, y)\,\frac{\partial^m p}{\partial x^m} + g\,(x, y)\,\frac{\partial^n p}{\partial y^n} = \frac{r + s}{2}, \tag{4. 35}$$

$$f\,(x, y)\,\frac{\partial^m q}{\partial x^m} + g\,(x, y)\,\frac{\partial^n q}{\partial y^n} = \frac{s - r}{2}. \tag{4. 36}$$

Multiplying (4. 36) by *i* and adding to (4. 35)

$$f\,(x, y)\,\frac{\partial^m z}{\partial x^m} + g\,(x, y)\,\frac{\partial^n z}{\partial y^n} = \frac{1}{2}\left((r + s) + i(s - r)\right), \tag{4. 37}$$

yields the equivalent real number derivative expression for (4. 31). Note that (4. 37) cannot be obtained from (4. 31) by simply deleting the imaginary parts of the derivatives.

Example 4.3. As a last example, consider the equation having different real and imaginary derivative components

$$\frac{\partial^{2+i} z}{\partial x^{2+i}} = \frac{\partial^{1+2i} z}{\partial t^{1+2i}} \quad . \tag{4. 38}$$

with $z = p\,(x, t) + iq(x, t)$. Using Definition 2.1

$$\left(\frac{\partial^2}{\partial x^2} + i\frac{\partial}{\partial x}\right)(p + qi) = \left(\frac{\partial}{\partial t} + i\frac{\partial^2}{\partial t^2}\right)(p + qi)\,, \tag{4. 39}$$

and separating real and imaginary parts

$$\frac{\partial^2 p}{\partial x^2} - \frac{\partial q}{\partial x} = \frac{\partial p}{\partial t} - \frac{\partial^2 q}{\partial t^2}, \tag{4. 40}$$

$$\frac{\partial^2 q}{\partial x^2}+\frac{\partial p}{\partial x}=\frac{\partial q}{\partial t}+\frac{\partial^2 p}{\partial t^2}, \tag{4. 41}$$

which is a highly coupled equation in terms of the components of $z(x,t)$. Or in terms of the complex function, the equivalent real number derivative equation is

$$\frac{\partial^2 z}{\partial x^2}+i\frac{\partial z}{\partial x}=\frac{\partial z}{\partial t}+i\frac{\partial^2 z}{\partial t^2}\ . \tag{4. 42}$$

5. NON-LINEAR PDES

The non-linear complex number derivative PDEs can be reduced to equivalent real number derivative equations in terms of their components by separating real and imaginary parts.

Example 5.1. Consider the nonlinear complex derivative partial differential equation,

$$\frac{\partial^{2+i} z}{\partial x^{2+i}}+\frac{\partial^{1+i} z}{\partial t^{1+i}}+z^2=0, \tag{5. 43}$$

with $z=p\,(x,t)+iq(x,t)$. Applying Definition 2.1

$$\left(\frac{\partial^2}{\partial x^2}+i\frac{\partial}{\partial x}\right)(p+qi)+\left(\frac{\partial}{\partial t}+i\frac{\partial}{\partial t}\right)(p+qi)+(p+iq)^2=0, \tag{5. 44}$$

performing the algebra and separating real and imaginary parts

$$\frac{\partial^2 p}{\partial x^2}-\frac{\partial q}{\partial x}+\frac{\partial p}{\partial t}-\frac{\partial q}{\partial t}+p^2-q^2=0, \tag{5. 45}$$

$$\frac{\partial^2 q}{\partial x^2}+\frac{\partial p}{\partial x}+\frac{\partial q}{\partial t}+\frac{\partial p}{\partial t}+2pq=0, \tag{5. 46}$$

which are coupled nonlinear equations in terms of the components.

Example 5.2. Consider the nonlinear complex derivative partial differential equation,

$$\frac{\partial^{2-i} z}{\partial t^{2-i}}+\left(\frac{\partial^{1+i} z}{\partial x^{1+i}}\right)^2=0, \tag{5. 47}$$

with $z=p\,(x,t)+iq(x,t)$. Applying Definition 2.1

$$\left(\frac{\partial^2}{\partial t^2}-i\frac{\partial}{\partial t}\right)(p+qi)+\left(\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial x}\right)(p+qi)\right)^2=0, \tag{5. 48}$$

performing the algebra and separating real and imaginary parts

$$\frac{\partial^2 p}{\partial t^2}+\frac{\partial q}{\partial t}-4\frac{\partial p}{\partial x}\frac{\partial q}{\partial x}=0, \tag{5. 49}$$

$$\frac{\partial^2 q}{\partial t^2}-\frac{\partial p}{\partial t}+2\left(\left(\frac{\partial p}{\partial x}\right)^2-\left(\frac{\partial q}{\partial x}\right)^2\right)=0, \tag{5. 50}$$

which are highly coupled nonlinear equations.

6. EQUATIONS FROM MATHEMATICAL PHYSICS

Some of the well-known equations in mathematical physics can be expressed in terms of the new complex number derivative PDEs. Three examples are given in this section.

Example 6.1. Consider the Schrödinger equation [24],

$$ih\frac{\partial}{\partial t}\Psi(x,t)=\left[-\frac{h^2}{2m}\frac{\partial^2}{\partial x^2}+V(x,t)\right]\Psi(x,t), \tag{6. 51}$$

which governs the behavior of wave functions in quantum mechanics where $\Psi(x,t)$ is the complex wave function, m is the mass, $V(x,t)$ represents the potential of the environment, h is the reduced Planck constant and i is the imaginary unit number.

The equation can be reformulated in terms of the newly proposed complex number derivatives

$$h\left(\frac{\partial^i}{\partial t^i}-1\right)\Psi(x,t)=\left[-\frac{h^2}{2m}\frac{\partial^2}{\partial x^2}+V(x,t)\right]\Psi(x,t), \tag{6. 52}$$

where the coefficients are real for the new expression.

Example 6.2. Consider the canonical form of Hamilton's equations of motion which possess applications in quantum mechanics, classical mechanics and astrophysics [12],

$$\frac{\partial\mathcal{H}}{\partial p_k}=\frac{\partial q_k}{\partial t},\frac{\partial\mathcal{H}}{\partial q_k}=-\frac{\partial p_k}{\partial t}, \tag{6. 53}$$

where $\mathcal{H}$ is the Hamiltonian representing the total amount of energy in the system, q_k are the generalized coordinates and p_k are the momenta.

The coupled equations can be expressed as a single equation in terms of the complex number derivatives

$$\left(\frac{\partial}{\partial p_k}+\frac{\partial^i}{\partial q_k^i}-1\right)\mathcal{H}=\frac{\partial}{\partial t}(q_k)-\left(\frac{\partial^i}{\partial t^i}-1\right)(p_k). \tag{6. 54}$$

The separation of the above equation into its real and imaginary parts yields exactly the coupled system (6. 53).

Example 6.3. Define the complex valued function $z=p(x,t)+iq(x,t)$ where $p(x,t)$ represents the positions of the set of particles located at spatial coordinate x and time t and $q(x,t)$ represents the velocity of the set of particles located at spatial coordinate x and time t. Then the condition

$$\frac{\partial^{1+i}z}{\partial t^{1+i}}=v_0+iv_0, \tag{6. 55}$$

represents in a compact way the constant velocity motion with no acceleration. To see this, use Definition 2.1

$$\left(\frac{\partial}{\partial t}+i\frac{\partial}{\partial t}\right)(p+qi)=v_0+iv_0, \tag{6. 56}$$

and after applying the operators and separating real and imaginary parts yields

$$\frac{\partial p}{\partial t}-\frac{\partial q}{\partial t}=v_0, \tag{6. 57}$$

$$\frac{\partial p}{\partial t} + \frac{\partial q}{\partial t} = v_0. \tag{6. 58}$$

Solving the coupled equations

$$\frac{\partial p}{\partial t} = v_0 \ , \quad \frac{\partial q}{\partial t} = 0, \tag{6. 59}$$

reveals that the velocities are constants and the accelerations are zero.

7. CONCLUSION

The complex-number derivative defined recently [17] has been employed for the first time to express new complex partial differential equations. The complex number derivative definition is relatively simple, easy to apply and reduces successfully to the real derivatives when the imaginary parts of the derivatives are zero. The differential equations can be expressed in a more compact form in this new notation. Coupled differential equations can be expressed as a single equation. As stated in [17], two differential conditions can be augmented in a single condition such as the maximum of a function which requires vanishing of the first derivative and second derivative being negative. Velocity and acceleration restrictions of a special motion in dynamics can also be expressed in a compact single equation.

First, the linear equations are considered and the conditions under which the imaginary parts of the derivatives can be ignored are outlined. Among other conditions, if the real and imaginary parts of all derivatives in a linear PDE are the same, the imaginary derivatives can be ignored. Examples where this reduction cannot be done are outlined also. Then the nonlinear equations are treated. For nonlinear equations, the imaginary parts cannot be ignored in any case. The Schrödinger equation, the Hamilton's equation of motion and a sample kinematical problem are expressed in the new notation as examples. The new derivative definition is expected to be employed in modeling a variety of applied mathematics problems. Dealing with reverse problems may also be possible. Starting from the travelling wave type solutions, the mathematical form producing such solutions can be obtained in a systematic way.

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