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Colour Class Domination equivalence partition and Colourful resolving sets in graphs

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Abstract. Let $H = (V, E)$ be a simple graph. A subset S of $V(H)$ is an equivalence set if the subgraph induced by S is component-wise complete. Let $P = \{V_1, V_2, \cdots, V_k\}$ be a partition of $V(H)$ where each $V_i, 1 \leq i \leq k$ is an equivalence set of H. The partition P is called a colourful domination equivalence partition if the set $D = \{u_1, u_2, \cdots, u_k\}$ where $u_i, 1 \leq i \leq k$ are suitably chosen vertices one each from V_i, $1 \leq i \leq k$ is a dominating set. The trivial partition where every partite set is a singleton is an example of a colourful domination equivalence partition. The minimum cardinality of such a partition is called colourful domination equivalence partition number of G and is denoted by $\chi_{\gamma_{eq}}(H)$. A subset $S = \{u_1, u_2, \cdots, u_r\}$ of $V(H)$ is called a resolving set of H if for any v in $V(H)$, the code of v with respect to S namely $(d(v, u_1), \cdots, d(v, u_r))$ (denoted by $c_S(v)$) is distinct for distinct v where $d(v, u_i)$ denotes the distance between v and u_i . Since $u_1, \cdots, u_r$ are evidently resolved by S, it is enough if vertices in $V - S$ are resolved by S. Given a proper colour partition $P = \{V_1, V_2, \cdots, V_k\}$, a resolving set $S = \{u_1, u_2, \cdots, u_k\}$ is said to be colourful with respect to P if $u_i \in V_i$ for every i, $1 \leq i \leq k$. A proper colour partition P is said to be a colour resolving colour partition if some set of vertices, one each from each colour class is a revolving set. The minimum cardinality of a resolving set of H is called the metric dimension of G [8] and is denoted by $Dim(H)$. The trivial partition consisting of n singletons where n is the order of H is a colourful resolving colour partition of H. The minimum cardinality of a colourful resolving colour partition of H is denoted by $\chi_{re}(H)$. A study of these two new parameters is initiated in this paper.

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Key Words: Colour Class domination, Equivalence set Colour class domination partition number.

1. INTRODUCTION

Domination [15] and colouring are two important areas of graph Theory. We refer [14], for all basic definitions. Let $H = (V, E)$ be a simple graph. A subset T of $V(H)$ is considered a vertex cover if T contains at least one end vertex of every edge of the graph. The minimum cardinality of a such set is the vertex covering several H and is denoted by $\alpha_0(H)$. A subset W of $V(H)$ is said to be an independent set if no two vertices in W are adjacent. The minimum cardinality of an independent set is the independence number of H and is denoted by $\beta_0(H)$. For any graph $H, \alpha_0(H) + \beta_0(H) = |V(H)|$. Any chromatic colouring can be modified so that one of the partite sets is dominating. Many authors introduced different variants of colouring in graphs [2], [1]. In the Dominator colouring of graphs, introduced by Gera et. al. [11, 12], the set formed by arbitrarily selecting one element each from each partite set of a minimum dominator colour partition is a dominating set. But this is not the case in any arbitrary proper colour partition. Inspired by this interesting property of Dominator colouring, S.Hamid et. al., studied those proper colour partitions which have this property. A proper colour partition is the colour class domination partition [25, 26] in which each colour class is dominated by a vertex of G and the minimum cardinality of a $cd-$ partition is $\chi_{cd}(G)$. Following these studies, Praba et. al. [18] introduced colourful domination colour class domination partition. While in dominator colouring by Gera et al., the set formed by arbitrarily selecting one element each from each partite set of a minimum dominator colour partition is a dominating set, in the gamma colouring introduced by S.Hamid et al. and in the colourful domination colour class domination partition introduced by Praba et al., the set formed by suitable selection of one element each from each colour class is a dominating set. Thus we have two different studies of the color class partition. A study of a partition P in which arbitrary selection of one element each from each partite set is a dominating set is one and the other is the study of a partition in which suitable selection of one element each from each partite set is a dominating set. The former is called a perfect colourful domination partition and the latter, is a colourful domination partition. Further, in the minimum dominator colour partition, $\chi_d(H) \leq \chi(H) + \gamma(H)$ and the bounds are sharp. But in gamma colouring and colourful domination colour class partition $\chi_\gamma(H) \leq \chi(H) + \gamma(H) - 1$, $\chi^\gamma_{cd}(H) \leq \chi_{cd}(H) + \gamma(H) - 1$ respectively and the bounds are sharp. In the first section of this paper, equivalence partition is considered for colourful domination. Bound on colour class domination equivalence partition numbers of H ($\chi_{\gamma_{eq}(H)}$) are found in the second section. In the fourth section, colourful resolving sets in a graph is introduced and studied its bounds in the last section. At the end, we listed out the related references.

2. COLOUR CLASS DOMINATION EQUIVALENCE PARTITION NUMBER

Definition 2.1. *Let $H = (V, E)$ be a simple graph. A subset S of $V(H)$ is called an equivalence set of H if $\langle S\rangle$ (the subgraph induced by S) is component-wise complete. Let P be a partition of $V(H)$ into equivalence sets such that the set formed by selecting a suitable one element from each partite set is a dominating set of H. The trivial partition is one such partition. The minimum cardinality of an equivalence colour partition in which the set formed by selecting one suitable element from each colour class is a dominating set of H is called colour class domination equivalence partition number of H and is denoted by $\chi_{\gamma_{eq}(H)}$.*

In the following, the value of $\chi_{\gamma_{eq}(G)}$ is presented for some classes of graphs like complete graphs K_n, stars $K_{1,n}$, complete bipartite graphs $K_{m,n}$, paths P_n, cycles C_n, Wheels W_n (C_{n-1} plus a universal vertex),), multipartite graphs $K_{n_1,n_2,\cdots,n_r}$ and Petersen graph P.

$\chi_{\gamma_{eq}(G)}$ **for some known classes of graphs.**

(1) $\chi_{\gamma_{eq}}(K_n) = 1$.

(2) $\chi_{\gamma_{eq}}(K_{1,n}) = 2$.
(3) $\chi_{\gamma_{eq}}(K_{m,n}) = 2$.
(4) $\chi_{\gamma_{eq}}(P_n) = \left\lceil \frac{n}{3} \right\rceil = \gamma(P_n)$.
(5) $\chi_{\gamma_{eq}}(C_n) = \left\lceil \frac{n}{3} \right\rceil = \gamma(C_n)$.
(6) $\chi_{\gamma_{eq}}(W_n) = \begin{cases} 1, & \text{if } n = 4 \\ 2, & \text{if } n = 5 \\ 3, & \text{if } n \geq 6 \end{cases}$
(7) $\chi_{\gamma_{eq}}(K_{n_1,n_2,\cdots,n_r}) = r$
(8) $\chi_{\gamma_{eq}}(K_m(a_1, a_2, \cdots, a_m)) = m$.
(9) $\chi_{\gamma_{eq}}(P) = 3$ *(since, if $V(P) = \{u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3, v_4, v_5\}$ where the first five vertices are those on the outer five cycle and the next five vertices correspond to the first five vertices in the inner pentagon, the partition $T = \{\{u_1, u_4, v_3, v_4\}, \{u_2, u_3, u_5, v_1\}$, $\{v_2, v_5\}\}$ is a $\chi_{\gamma_{eq}}$ - partition of P since $\{u_1, u_3, v_2\}$ is a colourful dominating set of P).*

3. Bounds on $\chi_{\gamma_{eq}(H)}$

Proposition 3.1. *For any graph H, $\chi_{\gamma_{eq}}(H) \leq \chi_{eq}(H) + \gamma(H) - 1$ and $\chi_{re}(H) \leq \chi_{eq}(H) + \gamma(H) - 1$.*

Proof. □

Since any $\chi_{\gamma_{eq}}$ - partition is a χ_{eq} - partition and since any $\chi_{\gamma_{eq}}$ - partition gives a dominating set of cardinality $\chi_{\gamma_{eq}}(H)$, the lower bound follows. Let $P = \{V_1, V_2, \cdots, V_t\}$ be a $\chi_{eq}(H)$-partition of G where $t = \chi_{eq}(H)$. Let $D = \{u_1, u_2, \cdots, u_k\}$ be a minimum dominating set of H where $k = \gamma(H)$. Give $\chi_{eq}(H)+1, \chi_{eq}(H)+2, \cdots, \chi_{eq}(H)+k-1$ colours to $u_1, u_2, \cdots, u_{k-1}$ respectively retaining the colours of other vertices as before. Let $P_1 = \{V_1 - D_1, V_2 - D_1, \cdots, V_t - D_1, \{u_1\}, \{u_2\}, \cdots, \{u_{k-1}\}\}$ where $D_1 = \{u_1, u_2, \cdots, u_{k-1}\}$. P_1 is an equivalence partition. $u_k \in V - D_1$ and hence $u_k \in V_j - D_1$ for some $j, 1 \leq j \leq t$.

Suppose $V_i - D_1 =$. Then V_i contains $\{u_1, u_2, \cdots, u_{k-1}\}$ and hence u_i for some $i, 1 \leq i \leq k-1$ can be chosen form $V_i - D_1$. In which case, z_i for some i will be u_i. In this case, the partition P_1 shall contain less than $t + k - 1$ elements and hence $\chi_{\gamma_{eq}}(H) \leq |P_1| \langle t + k - 1 = \chi_{eq}(H) + \gamma(H) - 1$.

Let $S = \{z_1, z_2, \cdots, z_t, u_1, u_2, \cdots, u_{k-1}\}$ where $z_i \in V_i - D_1, 1 \leq i \leq t$. Choose $z_j = u_k$. Then S is a colourful dominating set with respect to P_1. Therefore, $\chi_{\gamma_{eq}}(H) \leq |P_1| = t + k - 1 = \chi_{eq}(G) + \gamma(H) - 1$.

Remark 3.1. *When $H = K_n$, $\chi_{\gamma_{eq}}(H) = \chi_{eq}(H) = \gamma(H) = 1$ and hence the lower bound is attained. When $H = K_{1,n}$, $\chi_{eq}(H) = 2, \gamma(H) = 1$ and $\chi_{\gamma_{eq}}(H) = 2$. Hence, $\chi_{\gamma_{eq}}(H) = \chi_{eq}(H) + \gamma(H) - 1$. Thus, the bounds are sharp.*

In the following, the value of $\chi_{\gamma_{eq}}$ is presented for some classes of graphs like complete graphs K_n, stars $K_{1,n}$, complete bipartite graphs $K_{m,n}$, paths P_n, cycles C_n, wheelsW_n, multipartite graphs $K_{n_1,n_2,\cdots,n_r}$, multistar graphs $K_m(a_1, a_2, \cdots, a_m)$ and Petersen graph P.

Observation 3.1.

(1) $\chi_{\gamma_{eq}}(H) = 1$ if and only if $H = K_n$.
(2) $\chi_{\gamma_{eq}}(H) = n$ if and only if $H = \overline{K_n}$.
(3) $\chi_{\gamma_{eq}}(H) = n - 1$ if and only if $H = \overline{K_{n-2}} \cup K_2$.

(4) $\chi_{\gamma_{eq}}(H) = 2$ if and only if (i) $H = K_r \cup K_s$ where there are edges between K_r and K_s (ii) H is a bipartite graph with bipartition V_1, V_2 where $\langle V_1 \rangle$ and $\langle V_2 \rangle$ have more than one complete component and except for one complete component of $\langle V_1 \rangle$, all other complete components of $\langle V_1 \rangle$ are to be dominated by a single vertex of $\langle V_2 \rangle$ and vice-versa.

Proof. □

(1) Suppose $\chi_{\gamma_{eq}}(H) = 1$. Then H is an equivalence graph with $\gamma(H) = 1$. Therefore, H is a connected equivalence graph. That is, $H = K_n$. The converse is obvious.
(2) If $H = \overline{K_n}$, then clearly $\chi_{\gamma_{eq}}(H) = n$. Conversely, let $\chi_{\gamma_{eq}}(H) = n$. Suppose there exists an edge uv in H. Then the partition containing $\{u, v\}$ and other vertices as singletons are an equivalence partition and $\chi_{\gamma_{eq}}(H) \leq n - 1$. Hence $H = \overline{K_n}$.
(3) Suppose $H = \overline{K_{n-2}} \cup K_2$. Then clearly, $\chi_{\gamma_{eq}}(H) = n-1$. Conversely, let $\chi_{\gamma_{eq}}(H) = n-1$. Suppose H contains a maximal independent set of cardinality t. Then in an equivalence partition from which a dominating set can be selected must contain t singletons. If an equivalence set contains a K_3 , then one element from each set can be selected and hence $\chi_{\gamma_{eq}}(H) = n - 2$. This is a contradiction. Hence an equivalence set can contain K_2 and singletons in every equivalence sets are to be taken out to form singletons equivalence sets If there are two ore more than two equivalence sets contain K_2. Then $\chi_{\gamma_{eq}}(H) \leq n - 2$. Therefore, H contains exactly one K_2. Therefore $H == \overline{K_{n-2}} \cup K_2$.
(4) If any one of the conditions (i) or (ii) hold, then $\chi_{\gamma_{eq}}(H) = 2$. Conversely, suppose $\chi_{\gamma_{eq}}(H) = 2$. Then there exist subsets V_1 and V_2 of $V(H)$ such that $V(H) = V_1 \cup V_2$ and $\langle V_1 \rangle$ and $\langle V_2 \rangle$ are equivalence sets.
Case(i): $\langle V_1 \rangle$ and $\langle V_2 \rangle$ have exactly one complete component.
Subcase(i): $\langle V_1 \rangle$ and $\langle V_2 \rangle$ have no edges between them. Then $H = K_r \cup K_s$. *Subcase(ii):* $\langle V_1 \rangle$ and $\langle V_2 \rangle$ have edges between them. Without loss of generality, we can assume that $\langle V_1 \rangle$ and $\langle V_2 \rangle$ are maximal equivalence sets. Then no vertex of V_1 can be adjacent to all the vertices of V_2 and vice-versa. In this case, $H = K_r \cup K_s$ with edges between K_r and K_s.
Case(ii): $\langle V_1 \rangle$ and $\langle V_2 \rangle$ have more than one complete component. Suppose $\langle V_1 \rangle$ has t complete components and $\langle V_2 \rangle$ has s complete components, all other complete components are to be dominated by a single vertex of V_2. The same holds for $\langle V_2 \rangle$. Thus, $\chi_{\gamma_{eq}}(H) = 2$.

Observation 3.2. *Given positive integers $a, b, a \leq b$, there exists a graph H with $\chi_{eq}(H) = a$ and $\chi_{\gamma_{eq}}(H) = b$.*

For: Let $H = K_{n_1,n_2,\cdots,n_a} \cup \overline{K_{b-a}}$. $\chi_{eq}(H) = a$ and $\chi_{\gamma_{eq}}(H) = b$.
$G, \chi_{eq}(G) = \chi(G)$ and $\chi_{\gamma_{eq}}(G)$ and $\chi_{\gamma_{eq}}(G) = \gamma_{eq}(G)$. $G = H_1 \cup H_2$ where H_1 is complete multipartite and H_2 is an independent set of k vertices , then $\chi_{\gamma_{eq}}(G) = k+$ number of partite sets in H_1.

4. Colourful Resolving Sets in a graph

This section is due to inspiration by two seminal papers, one by Gera On Dominator colouring in graphs [11] and the other by S.Hameed et al. on Gamma colouring of graphs[13]. In dominator colouring, a proper colour partition in which every vertex dominates a colour class is considered. A salient feature of a minimum dominator colouring is that, if one vertex from each colour class is selected, then, this set of vertices is a dominating set of the graph. Thus, we get a colourful dominating set for any choice of one element from each colour class. This property inspired the paper titled Gamma colouring in which a proper colour partition is considered in which a suitable vertex from each colour class is chosen so that the set of chosen vertices is a dominating set called a colourful dominating set. The trivial colouring provides a colourful dominating set. A subset $S = \{u_1, u_2, \cdots, u_r\}$ of $V(H)$ is called a resolving set of H if for any v in $V(H)$, the code of v with respect to S namely $(d(v, u_1), \cdots, d(v, u_r))$ (denoted by $c_S(v)$) is distinct for distinct v where $d(v, u_i)$ denotes the distance between v and u_i. Since $u_1, \cdots, u_r$ are evidently resolved by S, it is enough if vertices in $V - S$ are resolved by S. A detailed study of this parameter is made in the literature. A few recents publications are [22], [19],[20],[21]. In this section, colourful resolving set is defined and studied concerning proper colouring. A further study can be made of colourful resolution set from a dominator colouring partition or a colour class domination partition (that is, a proper colour partition in which every colour class is dominated by a vertex).

Definition 4.1. *A subset S of $V(H)$ of a connected graph H is called a resolving set of H if for every vertex $u \in H$, the code of u with respect to S, defined by $(d(u, v_1), d(u, v_2), \cdots, d(u, v_k))$ where $S = \{v_1, v_2, \cdots, v_k\}$ is different for different u. The minimum cardinality of a resolving set of H is called the dimension of H and is denoted by $Dim(H)$.*

Definition 4.2. *A resolving set S of a simple, connected graph H is called a colourful resolving set concerning a proper colouring C, if distinct vertices of S receive distinct colours and all the colours, are represented in S. A proper colouring C of G is called a resolving colouring if there exists a colourful resolving set concerning C. The minimum cardinality of a proper colouring which is also a resolving colouring is called the resolving colouring number of H and is denoted by $\chi_{re}(H)$.*

The trivial partition of $V(H)$ in which all partite sets are singletons is a resolving colouring and hence every connected graph admits a resolving colouring. Thus, the new parameter is meaningful. In the following, the value of $\chi_{re}(H)$ is presented for some classes of graphs like complete graphs K_n, stars $K_{1,n}$, complete bipartite graphs $K_{m,n}$, paths P_n, cycles C_n, wheelsW_n, multipartite graphs $K_{n_1,n_2,\cdots,n_r}$, multistar graphs $K_m(a_1, a_2, \cdots, a_m)$ and Petersen graph P.

(1) $\chi_{re}(K_n) = n$.
(2) $\chi_{re}(K_{1,n}) = n$ (Since, if u is the central vertex and $v_1, v_2, \cdots, v_n$ are the pendent vertices, then $P = \{\{u\}, \{v_1\}, \cdots, \{v_{n-2}\}, \{v_{n-1}, v_n\}\}$ is a minimum colourful resolving proper colour partition of G containing n elements.)
(3) $\chi_{re}(K_{m,n}) = m + n - 2$.
(4) $\chi_{re}(P_n) = 2 = \chi(H)$ (Since, if $V(P_n) = \{v_1, v_2, \cdots, v_n\}$, then $\{v_1, v_3, \cdots, v_n\}$,$\{v_2, v_4,$ $\cdots, v_{n-1}\}$ or $\{v_1, v_3, \cdots, v_{n-1}\}$, $\{v_2, v_4,$ $\cdots, v_n\}$ according as n is odd or even is colourful resolving colouring partition of P_n of minimum cardinality.)
(5) $\chi_{re}(C_n) = \chi(H) = \begin{cases} 2, & \text{if n is even} \\ 3, & \text{if n is odd} \end{cases}$
(6) $\chi_{re}(W_n) = \begin{cases} 4, & \text{if n = 4} \\ \left\lceil \dfrac{n}{3} \right\rceil + 1, & \text{if } n \equiv 1, 2 \ (mod\ 3) \\ \left\lceil \dfrac{n}{3} \right\rceil + 2, & \text{if } n \equiv 0 \ (mod\ 3). \end{cases}$
(7) $\chi_{re}(K_{n_1,n_2,\cdots,n_r}) = n_1 + n_2 + \cdots + n_r - r$.
(8) $\chi_{re}(K_m(a_1, a_2, \cdots, a_m)) = a_1 + a_2 + \cdots + a_m$.
(9) $\chi_{re}(P) = 3 = \chi(P)$ (Since, for $\Pi = \{\{1, 3, 7\}, \{2, 4, 6, 10\}, \{5, 8, 9\}\}$, Π is a χ - partition of P and $S = \{1, 2, 8\}$ is a χ- resolving set of P.)

5. Bounds for $\chi_{re}(H)$

Proposition 5.1. *For a connected graph H, $\max(Dim(H), \chi(H)) \leq \chi_{re}(H) \leq \chi(H) + Dim(H) - 1$.*

Proof. Let $\Pi = \{V_1, V_2, \cdots, V_k\}$ be a $\chi_{re}-$partition of H. Then, there exists a resolving set $S = \{x_1, x_2, \cdots, x_k\}$ for H such that $x_i \in V_i$, $1 \leq i \leq k$. Hence $dim(H) \leq |S| = k = \chi_{re}(H)$. Since $\chi_{re}(H)$ denotes the cardinality of a colourful resolving proper colour partition of $G, \chi(H) \leq \chi_{re}(H)$. Hence the lower bound. Let $\Pi = \{V_1, V_2, \cdots, V_k\}$ be a χ - partition of H where $k = \chi(H)$. Let $D = \{x_1, x_2, \cdots, x_t\}$ be a minimum resolving set of H where $Dim(H) = t$. Assign colours $\chi(H) + 1$, $\chi(H) + 2$, $\cdots$, $\chi(H) + t$ to the vertices $x_1, x_2, \cdots, x_t$ leaving the other vertices coloured as before.

Let $P_1 = \{V_1 - (D - \{x_t\}), V_2 - (D - \{x_t\}), \cdots, V_k - (D - \{x_t\}), \{x_1\}, \{x_2\}, \cdots, \{x_t\}\}$. That is $P_1 = \{V_1 - D_1, V_2 - D_1, \cdots, V_k - D_1, \{x_1\}, \{x_2\}, \cdots, \{x_t\}\}$ where $D_1 = \{x_1, x_2, \cdots, x_{t-1}\}$. Let $z_i \in V_i - D_1 (1 \leq i \leq k)$. $\cup_{1\leq i \leq k}(V_i - D_1) = V - D_1$. $x_t \in V - D_1$. Therefore, $x_t \in V_j - D_1$ for some j, $1 \leq j \leq k$. Take $z_j = x_t$. Then, $D_2 = \{z_1, z_2, \cdots, z_k, x_1, x_2, \cdots, x_{t-1}\}$ is a colourful resolving set with respect to the proper colouring P_1. $\chi_{re}(H) \leq |D_2| = k + t - 1 = \chi(H) + Dim(H) - 1$. $\square$

Observation 5.1. *$\chi_{re}(H) = n$ if and only if $H = K_n$.*

Proof. Suppose $\chi_{re}(H) = n$. Then, any $\chi_{re}-$ partition of H contains n singletons. If any two vertices say u,v are non-adjacent, then the partition $P = \{\{u, v\}, \{x_3\}, \cdots, \{x_n\}\}$ where $V(H) = \{u, v, x_3, \cdots, x_n\}$ is a colourful resolving proper colour partition of H and hence $\chi_{re}(H) \leq n - 1$, a contradiction. Therefore, $H = K_n$. The converse is obvious. $\square$

Remark 5.1. *There is no relation between $\chi_{re}(H)$ and $\beta_0(H)$. When $n \geq 3$, $\beta_0(K_n) = 1$where as $\chi_{re}(K_n) = n\rangle\beta_0(G)$. When $n \geq 3$, $\chi_{re}(K_{1,n}) = n = \beta_0(K_{1,n})$. When $n \geq 6$, $\chi_{re}(C_n)\langle\beta_0(C_n)$.*

Proposition 5.2. *Let H be a simple, connected graph. $\chi_{re}(H) = 2$ if and only if $H = P_n$ or a connected bipartite graph with the degree of any vertex less than or equal to 3.*

Proof. Let $\chi_{re}(H) = 2$. Then G is bipartite with bipartition (V_1, V_2). Since $Dim(H) \leq \chi_{re}(H)$, $Dim(H) =$ 1 or 2. If $Dim(H) = 1$, then $H = P_n$. Suppose $Dim(H) = 2$. Let $S = \{u, v\}$ be a minimum resolving set of G where $u \in V_1$ and $v \in V_2$. Then, by Theorem 6.1 of [24], the distance partition with reference to the vertices u and v, namely $\{U_1, \cdots, U_{k_1}, V_1, \cdots, V_{k_2}\}$ are such that $|U_i \cap V_j| \leq 1$, for all i, j, $1 \leq i \leq k_1$, $1 \leq j \leq k_2$ where k_1 and k_2 are the eccentricities of u and v respectively. This implies that Using Corollary 2.8 of [18] , we get that H is a connected bipartite graph with degree of any vertex less than or equal to 3. The converse is obvious. $\square$

Corollary 5.2. *$\chi_{re}(H) = 2$, where H is an even cycle.*

Proposition 5.3. *Let H be a simple connected graph. Let $2 \leq a \leq b$. Then, there exists a connected graph G such that $\chi_{re}(H) = a$ and order of H equals b.*

Proof. If $a = b \geq 2$, then $H = K_a$ gives the required graph with $\chi_{re}(H) = a$ and order of H equal to a. Let $a\langle b$. When $a = 2$, any connected bipartite graph of order b with degree of any vertex less than or equal to three and which is not a path satisfies the conditions. Suppose, $a \geq 3$ and $b \geq 7$. The cycle C_6 with suitable number of pendent vertices attached at any one vertex of C_6, serves the purpose.

When $a = 2$, $b = 3$, $H = P_3$. When $a = 3$, $b = 4$, $H = K_{1,3}$. When $a = 3$, $b = 5$, $H = C_4$ with a single pendent vertex attached at any one vertex of C_4. When $a = 3$, $b = 6$, $H = C_4$ with two pendent vertices one each at two consecutive vertices of C_4. When $a = 4$, $b = 5$, $H = K_{1,4}$. When $a = 4$, $b = 6$, $H = C_4$ with two pendent vertices both attached at a single vertex of C_4. When $a = 5$, $b = 6$, $H = K_{1,5}$. □

6. Conclusion

A partition of the vertex set of a graph H can be made in different ways. Colouring and domination are the two major types of partitions, known as proper colour partition and domatic partition of $V(H)$. In this research work, we study a new kind of a partition namely colourful domination equivalence partition if the set $D = \{u_1, u_2, \cdots, u_k\}$ where $u_i, 1 \leq i \leq k$ are suitably chosen vertices one each from $V_i,\ 1 \leq i \leq k$ is a dominating set and the minimum cardinality of such a partition is called colourful domination equivalence partition number of G and is denoted by $\chi_{\gamma_{eq}}(H)$. Also, given a proper colour partition $P = \{V_1, V_2, \cdots, V_k\}$, a resolving set $S = \{u_1, u_2, \cdots, u_k\}$ is said to be colourful with respect to P if $u_i \in V_i$ for every i, $1 \leq i \leq k$. A proper colour partition P is said to be a colourful resolving colour partition if some set of vertices, one each from each colour class is a resolving set. The minimum cardinality of a resolving set of H is called the metric dimension of G [8] and is denoted by $Dim(H)$. The minimum cardinality of a colourful resolving colour partition of H is denoted by $\chi_{re}(H)$. A study of these two new parameters is initiated in this paper. The bounds of these new two partition parameters concerning other graph parameters are also determined.

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