Punjab University Journal of Mathematics (2024), 56(7), 334-347
https://doi.org/10.52280/pujm.2024.56(7)01

On Nonlocal Iterative Fractional Terminal Value Problem via Generalized Fractional Derivative

Mansoorali A. Kazi
Department of Mathematics,
Shree Siddheshwar Women's College of Engineering, Solapur-413002, (M. S.), India.
Email: mansooratmsc@gmail.com
Vinod V. Kharat
Department of Mathematics,
N. B. Navale Sinhgad College of Engineering, Kegaon-Solapur-413255, (M. S.), India.
Email: vvkvinod9@gmail.com
Anand R. Reshimkar
Department of Mathematics,
D. B. F. Dayanand College of Arts & Science, Solapur-413002, (M. S.), India.
Machchhindra T. Gophane
Department of Mathematics,
Shivaji University, Kolhapur-416004, (M.S.), India.
Email: *gmachchhindra@gmail.com*

Received: 23 June, 2023 / Accepted: 09 February, 2025 / Published online:12 March, 2025

Abstract. Terminal value problems for fractional differential equation with iterative argument are studied in one dimension. For the proposed problem, we have used generalized Katugampola fractional derivative operator considering its generality. Existence and uniqueness results have been established by using fixed point techniques. A couple of examples are discussed at the last to support our theoretical results.

AMS (MOS) Subject Classification Codes: 26A33; 34A08; 34A12
Key Words: Generalized Katugampola fractional derivative, Iterative, Nonlocal Conditions, Terminal Value Problem, Fixed Point theorems.

1. Introduction

Fractional differential equations have been studied by many researchers and explored different varieties of the problems via initial conditions, boundary conditions etc. Iterative fractional differential equations have been widely studied due to its direct applications in different branches of science. E. Eder studied the theory of iterative differential equations for the first time, see [15]. M. A. Anwar and et al. explored numerical iterative scheme for solving nonlinear boundary value problems of fractional order in [2]. Ibrahim Rabha have studied numerous problems in iterative fractional differential equations for initial conditions, see [17], [18], [23]. L. Byszewski was the first to study of functional differential equation with non-local conditions and studied existence and uniqueness results for the problems, see [9], [10], [11], [37].

The terminal value problems with fractional derivative operator do have many challenges such as non-linearity, vector order, weakly singularity, high-dimensional systems. Some of these challenges have been addressed by the various authors through numerical methods such as collocation method in [3], [4], [32], [33], [38] and [39]. Authors in these cited papers dealt with the terminal value problems in higher dimensions. In these articles, authors studied terminal value problems for systems of fractional differential equations in which discretized piece-wise polynomial collocation methods are used for approximating the exact solution and existence of a unique solution proved by using fixed point theorem. Also, in [4] well-posedness of the terminal value problem for nonlinear systems of generalized fractional differential equations for terminal value problem studied by transforming the problem into weakly singular Fredholm and Volterra integral equations with delay and supported the theory with illustrative examples. For more details on terminal value problems, one may also refer to [27] and [34].

There are no methods available to find the exact solutions to the complex fractional differential equations. Therefore, many a times, researchers choose to discuss the solutions analytically in which the corresponding integral equation to the proposed fractional differential equation is obtained and fixed point technique [1] is employed. Many fixed point theorems such as Banach fixed point theorem, Schauder fixed point theorem, Schafer, Lerray-Schauder fixed point theorems are used according to the space of the differential equation and existence, uniqueness, stability results are established, see [6, 22, 26, 30, 35, 36]. In [5], authors established existence of solutions using Schauder fixed point theorem and explored local attractiveness of solutions, stability of solutions. Some new inequalities of Hermite Hadamard type via Katugampola fractional integral discussed in [8] and comprehensive notes on various effects of some operators of fractional-order derivatives to certain functions for complex domains in [19]. Analysis of multi term fractional differential equations using variational iteration method discussed in [31]. In [6], authors obtained existence result with the aid of Kransnoselski, Schauder and Schafer fixed point theorem.

Terminal value problems are used to describe in particular behaviour of models of viscoelastic materials and models of financial market dynamics [12], [40]. Recent developments in fractional differential equation with terminal value condition can be seen in [7], [13], [14], [24]. In this article, we intend to address following issues:

(1) Our problem deals with the terminal value problem in one dimension.
(2) Although, a good number of articles have been published with terminal conditions, terminal value problems with fractional derivative operator having iterative argument are not yet addressed.
(3) Further, the generalized Katugampola fractional derivative operator used in our problem produces different fractional derivatives for different values of parameters that increases applicability of the problem.
(4) We employed the methodology of fixed point technique to discuss existence and uniqueness of solutions.

Taking motivation from the all cited work above, in this present article, we propose iterative generalized fractional differential equation with terminal condition of the type

$$\begin{cases} ({}^{\alpha}D^{\nu,\sigma}_{a+}f)(t) &= h(t, f(f(t))), \quad 0 < \nu < 1,\ 0 \leq \sigma \leq 1,\ t \in (a, T] \\ ({}^{\alpha}I^{1-\xi}f)(T) &= \sum_{i=1}^{n} \gamma_i f(\zeta_i), \quad \nu \leq \xi = \nu + \sigma(1-\nu),\ \zeta_i \in (a, T] \end{cases} \tag{1. 1}$$

where $\nu \in (0, 1),\ \sigma \in [0, 1],\ \alpha > 0,\ \nu \leq \xi = \nu + \sigma(1-\nu)$. Generalized Katugampola fractional derivative ${}^{\alpha}D^{\nu,\sigma}_{a+}$ is of order ν and generalized Katugampola fractional integral ${}^{\alpha}I^{1-\xi}$ is of order $1-\xi$. Let $h : (a, T] \times \mathbb{R} \to \mathbb{R}$ is a function, ζ_i are pre-defined points which holds $0 < a < \zeta_1 \leq \zeta_2 \leq \cdots \leq \zeta_n < T$ and $\gamma_i,\ i = 1, 2, \cdots, n$ are real numbers.

2. PRELIMINARIES

Consider the following definitions and lemmas for this paper. Gamma and Beta functions are given by

$$\Gamma(\nu) = \int_0^{\infty} t^{\nu-1} e^{-t} dt,\ B(\nu, \sigma) = \int_0^1 (1-t)^{\nu-1} t^{\sigma-1} dt,\ \nu,\ \sigma > 0.$$

Definition 2.1. [25] *Let $X^p_c(a, T)$ is space with $p \geq 1, c \in \mathbb{R}$ contains those real valued Lebesgue measurable functions h on (a, T) such as $||h||_{X^p_c} < \infty$,*

$$||h||_{X^p_c} = \Big(\int_a^b |t^c h(t)|^p \frac{dt}{t} \Big)^{1/p},\ ||h||_{X^{\infty}_c} = \sup_{a \leq t \leq T} |t^c h(t)|.$$

In specifically, for $c = \frac{1}{p}$ we get $L_p(a, T) = X^c_{1/p}(a, T)$.

Definition 2.2. [28] *Consider the space $C[a, T]$ of continuous functions h on $(a, T]$ defined by*

$$||h||_C = \max_{t \in [a,T]} |h(t)|.$$

The weighted space

$$C_{\xi,\alpha}[a, T] = \Big\{ h : (a, T] \to \mathbb{R} : \Big(\frac{t^{\alpha} - a^{\alpha}}{\alpha} \Big)^{\xi} h(t) \in C[a, T] \Big\}, \tag{2. 2}$$

equipped as

$$||h||_{C_{\xi,\alpha}} = \Big\| \Big(\frac{t^{\alpha} - a^{\alpha}}{\alpha} \Big)^{\xi} h(t) \Big\| = \max_{t \in [a,T]} \Big| \Big(\frac{t^{\alpha} - a^{\alpha}}{\alpha} \Big)^{\xi} h(t) \Big| \text{ and } C_{0,\alpha}[a, T] = C[a, T].$$

Definition 2.3. [28] *Let $\Delta_{\alpha} = \Big(t^{\alpha-1} \frac{d}{dt} \Big)$ on $[a, T]$ upto order $(m-1),\ 0 \leq \xi < 1$. If h are continuously differentiable functions on Banach space $C^m[a, T]$, and derivative $\Delta^k_{\alpha} h$ on $(a, T]$ so as $\Delta^m_{\alpha} h \in C_{\xi,\alpha}[a, T]$.*

$$C^m_{\Delta_{\alpha},\xi}[a, T] = \Big\{ \Delta^k_{\alpha} h \in C[a, T], k = 0, 1, \cdots, m-1, \Delta^m_{\alpha} h \in C_{\xi,\alpha}[a, T] \Big\}, m \in \mathbb{N},$$

with the norm

$$||h||_{C^m_{\Delta_{\alpha},\xi}} = \sum_{k=0}^{m-1} ||\Delta^k_{\alpha} h||_C + ||\Delta^m_{\alpha} h||_{C_{\xi,\alpha}},\ ||h||_{C^m_{\Delta_{\alpha},\xi}} = \sum_{k=0}^{m} \max_{t \in (a,T]} |\Delta^k_{\alpha} h(t)|.$$

In specifically, for $m = 0$, it holds $C^0_{\Delta_{\alpha},\xi}[a, T] = C_{\xi,\alpha}[a, T]$.

Definition 2.4. [20] *The left-sided Katugampola fractional integral* ${}^{\alpha}I^{\nu}_{a+}$ *of order* ν, *for* $\nu > 0$ *and* $h \in X^p_c(a,T)$, *from Definition 2.1 is given by*

$${}^{\alpha}I^{\nu}_{a+}h(t) = \frac{1}{\Gamma(\nu)}\int_a^t p^{\alpha-1}\Big(\frac{t^\alpha - a^\alpha}{\alpha}\Big)^{\nu-1}h(p)dp,\ t > a.$$

Definition 2.5. [21] ${}^{\alpha}D^{\nu}_{a+}$ *is left sided Katugampola fractional derivative given by*

$${}^{\alpha}D^{\nu}_{a+}h(t) = \Delta^m_\alpha({}^{\alpha}I^{m-\nu}_{a+}h(p))(t) \tag{2. 3}$$

$$= \Big(t^{\nu-1}\frac{d}{dt}\Big)^m \frac{1}{\Gamma(m-\nu)}\int_a^t p^{\alpha-1}\Big(\frac{t^\alpha - a^\alpha}{\alpha}\Big)^{m-\nu-1}h(p)dp, \tag{2. 4}$$

for $\nu \in \mathbb{R}^+\backslash\mathbb{N}$ *with* $m = [\nu]+1$, *in this* $[\nu]$ *is the integer part of* ν.

Definition 2.6. [28] *Let* $0 < \nu < 1$ *and* $0 \leq \sigma \leq 1$, ${}^{\alpha}D^{\nu,\sigma}_{a+}$ *is left-sided generalized Katugampola fractional derivative given by*

$$({}^{\alpha}D^{\nu,\sigma}_{a+}h)(t) = ({}^{\alpha}I^{\sigma(1-\nu)}_{a+}\Delta^{\alpha}_{\alpha}I^{(1-\sigma)(1-\nu)}_{a+}h)(t), \tag{2. 5}$$

where right-hand side expression is present for those functions.

Lemma 2.7. [29] *Let* $\nu > 0,\ \sigma > 0$ *and* $\alpha,\ c \in \mathbb{R}$ *with* $\alpha \geq c$. *Therefore,* $h \in X^p_c(a,T),\ p \geq 1$, *Katugampola integral holds semi-group condition*

$$i.e. \quad ({}^{\alpha}I^{\nu}_{a+})({}^{\alpha}I^{\sigma}_{a+})h(t) = {}^{\alpha}I^{\nu+\sigma}_{a+}h(t). \tag{2. 6}$$

Lemma 2.8. [21] *Let* $h \in C_{\xi,\alpha}[a,T]$, $t \in (a,T]$ *with* $\nu > 0,\ 0 \leq \xi < 1$. *Then,*

$$({}^{\alpha}D^{\nu}_{a+})({}^{\alpha}I^{\nu}_{a+})h(t) = h(t). \tag{2. 7}$$

Lemma 2.9. [21] *Let* $h \in C_{\xi,\alpha}[a,T]$ *and* ${}^{\alpha}I^{1-\nu}_{a+}h \in C^1_{\xi,\alpha}[a,T]$ *with* $\nu > 0,\ 0 \leq \xi < 1$. *Then,*

$$({}^{\alpha}I^{\nu}_{a+})({}^{\alpha}D^{\nu}_{a+})h(t) = h(t) - \frac{{}^{\alpha}I^{1-\nu}_{a+}h(a)}{\Gamma(\nu)}\Big(\frac{t^\alpha - a^\alpha}{\alpha}\Big)^{\nu-1}. \tag{2. 8}$$

Lemma 2.10. [5] *From Definition 2.4 and 2.5, we have the following*

$${}^{\alpha}I^{\nu}_{a+}\Big(\frac{t^\alpha - a^\alpha}{\alpha}\Big)^{\psi-1} = \frac{\Gamma(\psi)}{\Gamma(\psi+1)}\Big(\frac{t^\alpha - a^\alpha}{\alpha}\Big)^{\psi+\nu-1},\ \nu \geq 0,\ \psi > 0,\ t > a$$

$${}^{\alpha}D^{\nu}_{a+}\Big(\frac{t^\alpha - a^\alpha}{\alpha}\Big)^{\nu-1} = 0,\ 0 < \nu < 1.$$

Remark 2.11. [5] *Let* $0 < \nu < 1,\ 0 \leq \sigma \leq 1$, ${}^{\alpha}D^{\nu,\sigma}_{a+}$ *can be described as in the form of Katugampola fractional derivative given by*

$${}^{\alpha}D^{\nu,\sigma}_{a+} = ({}^{\alpha}I^{\sigma(1-\nu)}_{a+})\Delta_\alpha({}^{\alpha}I^{1-\nu}_{a+}) = ({}^{\alpha}I^{\sigma(1-\nu)}_{a+})({}^{\alpha}D^{\xi}_{a+}),\ \xi = \nu + \sigma(1-\nu).$$

Lemma 2.12. [28] *Let* $h \in C_{1-\xi,\alpha}[a,b]$ *and* $\nu > 0,\ 0 < \xi \leq 1$ *and If* $\nu > \xi$, *then*

$${}^{\alpha}I^{\nu}_{a+}h(a) = \lim_{t\to a+}({}^{\alpha}I^{\nu}_{a+}h)(t) = 0.$$

To deliberate the main results, we require subsequent spaces:

$$C^{\nu,\sigma}_{1-\xi,\alpha}[a,T] = \Big\{h \in C_{1-\xi,\alpha}[a,T] :\ {}^{\alpha}D^{\nu,\sigma}_{a+}h \in C_{1-\xi,\alpha}[a,T]\Big\} \tag{2. 9}$$

and

$$C^{\xi}_{1-\xi,\alpha}[a,T] = \Big\{h \in C_{1-\xi,\alpha}[a,T] : \ {}^{\alpha}D^{\xi}_{a+}h \in C_{1-\xi,\alpha}[a,T]\Big\}, \ 0 < \xi \leq 1,$$

as ${}^{\alpha}D^{\nu,\sigma}_{a+}h = ({}^{\alpha}I^{\sigma(1-\nu)}_{a+})({}^{\alpha}D^{\xi}_{a+})h$, then $C^{\xi}_{1-\xi,\alpha}[a,T] \subseteq C^{\nu,\sigma}_{1-\xi,\alpha}[a,T]$.

Lemma 2.13. [21] *Let* $\nu > 0, \ \sigma > 0$ *and* $\xi = \nu + \sigma - \nu\sigma$. *If* $h \in C^{\xi}_{1-\xi,\alpha}[a,T]$, *then*

$$({}^{\alpha}I^{\xi}_{a+})(({}^{\alpha}D^{\xi}_{a+})h(t) = ({}^{\alpha}I^{\nu}_{a+})({}^{\alpha}D^{\nu;\sigma}_{a+})h(t) =^{\alpha} D^{\sigma(1-\nu)}_{a+}h(t).$$

Lemma 2.14. [27] *With conditions* $0 < \nu < 1, \ 0 \leq \sigma \leq 1, \ \xi = \nu + \eta - \nu\sigma$. *Let* $h : (a,T] \times \mathbb{R} \to \mathbb{R}$ *is a function such that* $h(\cdot, \ f(\cdot)) \in C_{1-\xi,\alpha}[a,T]$, *for any* $f \in C_{1-\xi,\alpha}[a,T]$ *therefore* $f \in C^{\xi}_{1-\xi,\alpha}[a,T]$ *satisfies Terminal value Problem*

$$\begin{cases} ({}^{\alpha}D^{\nu,\sigma}_{a+}f)(t) & = h(t, f(t)), \qquad 0 < \nu < 1, \ 0 \leq \sigma \leq 1, \ t \in (a,T] \\ ({}^{\alpha}I^{1-\xi}f)(T) & = \sum_{i=1}^{n} \gamma_i f(\zeta_i), \quad \nu \leq \xi = \nu + \sigma(1-\nu), \ \zeta_i \in (a,T] \end{cases} \tag{2. 10}$$

if and only if f *satisfies the mixed-type nonlinear Volterra integral equation*

$$f(t) = \frac{\kappa}{\Gamma(\nu)}\Big(\frac{t^{\alpha} - a^{\alpha}}{\alpha}\Big)^{\xi-1} \sum_{i=0}^{n} \gamma_i \int_a^{\zeta_i} p^{\alpha-1}\Big(\frac{\zeta_i^{\alpha} - p^{\alpha}}{\alpha}\Big)^{\nu-1} h(p, f(p))dp$$
$$+ \frac{1}{\Gamma(\nu)} \int_a^t p^{\alpha-1}\Big(\frac{t^{\alpha} - p^{\alpha}}{\alpha}\Big)^{\nu-1} h(p, f(p))dp, \tag{2. 11}$$

where

$$\kappa = \bigg(\Gamma(\xi) - \sum_{i=1}^{n} \gamma_i \Big(\frac{\zeta_i^{\alpha} - a^{\alpha}}{\alpha}\Big)^{\xi-1}\bigg)^{-1}. \tag{2. 12}$$

Theorem 2.15. *Let* $0 < \nu < 1, \ 0 \leq \sigma \leq 1, \ \xi = \nu + \sigma - \nu\sigma$. *If* $h : (a,T] \times \mathbb{R} \to \mathbb{R}$ *such that* $h(\cdot, \ f(\cdot)) \in C_{1-\xi,\alpha}[a,T]$, *for any* $f \in C_{1-\xi,\alpha}[a,T]$ *therefore* $f \in C^{\xi}_{1-\xi,\alpha}[a,T]$ *satisfies the problem (1. 1) if and only if* f *satisfies the mixed-type nonlinear Volterra integral equation*

$$f(t) = \frac{\kappa}{\Gamma(\nu)}\Big(\frac{t^{\alpha} - a^{\alpha}}{\alpha}\Big)^{\xi-1} \sum_{i=0}^{n} \gamma_i \int_a^{\zeta_i} p^{\alpha-1}\Big(\frac{\zeta_i^{\alpha} - p^{\alpha}}{\alpha}\Big)^{\nu-1} h(p, f(f(p)))dp$$
$$+ \frac{1}{\Gamma(\nu)} \int_a^t p^{\alpha-1}\Big(\frac{t^{\alpha} - p^{\alpha}}{\alpha}\Big)^{\nu-1} h(p, f(f(p)))dp, \tag{2. 13}$$

where

$$\kappa = \bigg(\Gamma(\xi) - \sum_{i=1}^{n} \gamma_i \Big(\frac{\zeta_i^{\alpha} - a^{\alpha}}{\alpha}\Big)^{\xi-1}\bigg)^{-1}. \tag{2. 14}$$

Proof. The proof comes from the Lemma 2.14. □

3. RESULTS

This section illustrates existence and uniqueness results for the stated problem. Let some of the hypotheses:

(H_1) Let $h:(a,T]\times\mathbb{R}\to\mathbb{R}$ is a function, $h(\cdot,f(\cdot))\in C^{\sigma(1-\nu)}_{1-\xi,\alpha}[a,T]$ for any $f\in C_{1-\xi,\alpha}$, $\exists$ constant $L>0$ such that $\forall\, f,g\in\mathbb{R}$,

$$|h(t,f(t))-h(t,g(t))|\le L|f-g|. \tag{3. 15}$$

(H_2) There exists constants $\Omega_1,\ \Omega_2$ given by

$$\Omega_1=\frac{\Gamma(\xi)}{\Gamma(\xi+\nu)}\bigg(|\kappa|\sum_{i=0}^{n}\gamma_i\Big(\frac{\zeta_i^\alpha-a^\alpha}{\alpha}\Big)^{\nu+\xi-1}+\Big(\frac{T^\alpha-a^\alpha}{\alpha}\Big)^{\nu}\bigg) \tag{3. 16}$$

and

$$\Omega_2=\frac{M}{\Gamma(\nu+1)}\bigg(|\kappa|\sum_{i=0}^{n}\gamma_i\Big(\frac{\zeta_i^\alpha-a^\alpha}{\alpha}\Big)^{\nu}+\Big(\frac{T^\alpha-a^\alpha}{\alpha}\Big)^{\nu-\xi+1}\bigg). \tag{3. 17}$$

(H_3) $C^{P}_{1-\xi,\alpha}=\{f\in C_{1-\xi,\alpha}:|f(t_1)-f(t_2)|\le P|t_1-t_2|,\ \forall\, t_1,\ t_2\in(a,T]\}$.

(H_4) Let $h:(a,T]\times\mathbb{R}\to\mathbb{R}$, $h(\cdot,f(\cdot))\in C^{\sigma(1-\nu)}_{1-\xi,\alpha}$, for any $f\in C_{1-\xi,\alpha}$ and $\forall\, t\in(a,T]$, $\exists$ constants $M>0$ and $L\ge 0$,

$$|h(t,f)|\le L|f|+M. \tag{3. 18}$$

(H_5) Let $h:(a,T]\times\mathbb{R}\to\mathbb{R}$ such that $h(\cdot,f(\cdot))\in C^{\sigma(1-\nu)}_{1-\xi,\alpha}[a,T]$, for any $f\in C_{1-\xi,\alpha}[a,T]$ with $\forall\, t\in(a,T]$, $\exists$ function $r(t)\in C^{\sigma(1-\nu)}_{1-\xi,\alpha}[a,T]$,

$$|h(t,f)|\le r(t). \tag{3. 19}$$

Theorem 3.1. *Let the hypotheses $(H_1)-(\ H_5)$ holds then the problem (1. 1) has at least one solution in $C^{\xi}_{1-\xi,\alpha}[a,T]\subset C^{\nu,\sigma}_{1-\xi,\alpha}[a,T]$.*

Proof. Now, we show that integral equation (2. 13) has a solution.
Let $Q:C_{1-\xi,\alpha}[a,T]\to C_{1-\xi,\alpha}[a,T]$ defined as

$$(Qf)(t)=\frac{\kappa}{\Gamma(\nu)}\Big(\frac{t^\alpha-a^\alpha}{\alpha}\Big)^{\xi-1}\sum_{i=0}^{n}\gamma_i\int_a^{\zeta_i}p^{\alpha-1}\Big(\frac{\zeta_i^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}h(p,f(f(p)))dp$$
$$+\frac{1}{\Gamma(\nu)}\int_a^t p^{\alpha-1}\Big(\frac{t^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}h(p,f(f(p)))dp \tag{3. 20}$$

Then, Q is well defined.
Define $B_\delta=\{f\in C_{1-\xi,\alpha}[a,T]:||f||_{C_{1-\xi,\alpha}}\le\delta\}$ where $\delta\ge\dfrac{\Omega_2}{1-L\Omega_1}$.
Claim 1: $Q(B_\delta)\subset B_\delta$.

Consider

$$\begin{aligned}
&\left|\Big(\frac{t^\alpha-a^\alpha}{\alpha}\Big)^{1-\xi}(Qf)(t)\right| \leq \frac{|\kappa|}{\Gamma(\nu)}\sum_{i=1}^{n}\gamma_i\int_a^{\zeta_i}p^{\alpha-1}\Big(\frac{\zeta_i^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}\Big|h(p,f(f(p)))\Big|dp\\
&+\Big(\frac{t^\alpha-a^\alpha}{\alpha}\Big)^{1-\xi}\frac{1}{\Gamma(\nu)}\int_a^t p^{\alpha-1}\Big(\frac{t^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}\Big|h(p,f(f(p)))\Big|dp\\
&\leq \frac{|\kappa|}{\Gamma(\nu)}\sum_{i=1}^{n}\gamma_i\int_a^{\zeta_i}p^{\alpha-1}\Big(\frac{\zeta_i^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}\Big[L|f(f(p))|+M\Big]dp\\
&+\Big(\frac{t^\alpha-a^\alpha}{\alpha}\Big)^{1-\xi}\frac{1}{\Gamma(\nu)}\int_a^t p^{\alpha-1}\Big(\frac{t^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}\Big[L|f(f(p))|+M\Big]dp\\
&\leq L\frac{|\kappa|}{\Gamma(\nu)}\sum_{i=1}^{n}\gamma_i\int_a^{\zeta_i}p^{\alpha-1}\Big(\frac{\zeta_i^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}\Big(\frac{p^\alpha-a^\alpha}{\alpha}\Big)^{\xi-1}\Big(\frac{p^\alpha-a^\alpha}{\alpha}\Big)^{1-\xi}|f(f(p))|dp\\
&+M\frac{|\kappa|}{\Gamma(\nu)}\sum_{i=1}^{n}\gamma_i\int_a^{\zeta_i}p^{\alpha-1}\Big(\frac{\zeta_i^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}dp\\
&+L\Big(\frac{t^\alpha-a^\alpha}{\alpha}\Big)^{1-\xi}\frac{1}{\Gamma(\nu)}\int_a^t p^{\alpha-1}\Big(\frac{t^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}\\
&\times\Big(\frac{p^\alpha-a^\alpha}{\alpha}\Big)^{\xi-1}\Big(\frac{p^\alpha-a^\alpha}{\alpha}\Big)^{1-\xi}|f(f(p))|dp\\
&+M\Big(\frac{t^\alpha-a^\alpha}{\alpha}\Big)^{1-\xi}\frac{1}{\Gamma(\nu)}\int_a^t p^{\alpha-1}\Big(\frac{t^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}dp\\
&\leq L||f||_{C_{1-\xi,\alpha}}\frac{|\kappa|}{\Gamma(\nu)}\sum_{i=1}^{n}\gamma_i\int_a^{\zeta_i}p^{\alpha-1}\Big(\frac{\zeta_i^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}\Big(\frac{p^\alpha-a^\alpha}{\alpha}\Big)^{\xi-1}dp\\
&+M\frac{|\kappa|}{\Gamma(\nu)}\sum_{i=1}^{n}\gamma_i\int_a^{\xi_i}p^{\alpha-1}\Big(\frac{\zeta_i^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}dp\\
&+L||f||_{C_{1-\xi,\alpha}}\Big(\frac{t^\alpha-a^\alpha}{\alpha}\Big)^{1-\xi}\frac{1}{\Gamma(\nu)}\int_a^t p^{\alpha-1}\Big(\frac{t^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}\Big(\frac{p^\alpha-a^\alpha}{\alpha}\Big)^{\xi-1}dp\\
&+M\Big(\frac{t^\alpha-a^\alpha}{\alpha}\Big)^{1-\xi}\frac{1}{\Gamma(\nu)}\int_a^t p^{\alpha-1}\Big(\frac{t^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}dp\\
&\leq L||f||_{C_{1-\xi,\alpha}}\frac{|\kappa|\Gamma(\xi)}{\Gamma(\xi+\nu)}\sum_{i=1}^{n}\gamma_i\Big(\frac{\zeta_i^\alpha-a^\alpha}{\alpha}\Big)^{\nu+\xi-1}+\frac{M|\kappa|}{\Gamma(\nu+1)}\sum_{i=1}^{n}\gamma_i\Big(\frac{\zeta_i^\alpha-a^\alpha}{\alpha}\Big)^{\nu}\\
&+L||f||_{C_{1-\xi,\alpha}}\frac{\Gamma(\xi)}{\Gamma(\xi+\nu)}\Big(\frac{T^\alpha-a^\alpha}{\alpha}\Big)^{\nu}+\frac{M}{\Gamma(\nu+1)}\Big(\frac{T^\alpha-a^\alpha}{\alpha}\Big)^{\nu-\xi+1}\\
&\leq L\frac{\Gamma(\xi)}{\Gamma(\xi+\nu)}\Bigg[|\kappa|\sum_{i=1}^{n}\gamma_i\Big(\frac{\zeta_i^\alpha-a^\alpha}{\alpha}\Big)^{\nu+\xi-1}+\Big(\frac{T^\alpha-a^\alpha}{\alpha}\Big)^{\nu}\Bigg]||f||_{C_{1-\xi,\alpha}}\\
&+M\frac{1}{\Gamma(\nu+1)}\Bigg[|\kappa|\sum_{i=1}^{n}\gamma_i\Big(\frac{\zeta_i^\alpha-a^\alpha}{\alpha}\Big)^{\nu}+\Big(\frac{T^\alpha-a^\alpha}{\alpha}\Big)^{\nu-\xi+1}\Bigg]\\
&\leq L\Omega_1\delta+\Omega_2\\
&\leq \delta.
\end{aligned}$$

That is,

$$||Qf||_{C_{1-\xi,\alpha}} \leq \delta.$$

Which gives $Q(B_\delta) \subset B_\delta$.

Claim 2: Q is completely continuous. The sequence $\{f_n\}$ so as $f_n \to f$ in B_δ. Then for every $t \in (a, T]$,
consider

$$\begin{aligned}
&\left|\Big(\frac{t^\alpha - a^\alpha}{\alpha}\Big)^{1-\xi}\Big((Qf_n)(t) - (Qf)(t)\Big)\right| \\
&\leq \frac{|\kappa|}{\Gamma(\nu)}\sum_{i=1}^{n}\gamma_i \int_a^{\zeta_i} p^{\alpha-1}\Big(\frac{\zeta_i^\alpha - p^\alpha}{\alpha}\Big)^{\nu-1}|h(p, f_n(f_n(p))) - h(p, f(f(p)))|dp \\
&+ \Big(\frac{t^\alpha - a^\alpha}{\alpha}\Big)^{1-\xi}\frac{1}{\Gamma(\nu)}\int_a^t p^{\alpha-1}\Big(\frac{t^\alpha - p^\alpha}{\alpha}\Big)^{\nu-1}|h(p, f_n(f_n(p))) - h(p, f(f(p)))|dp \\
&\leq \frac{|\kappa|}{\Gamma(\nu)}\sum_{i=1}^{n}\gamma_i \int_a^{\zeta_i} p^{\alpha-1}\Big(\frac{\zeta_i^\alpha - p^\alpha}{\alpha}\Big)^{\nu-1}\Big(\frac{p^\alpha - a^\alpha}{\alpha}\Big)^{1-\xi} \\
&\times ||h(\cdot, f_n(f_n(\cdot))) - h(\cdot, f(f(\cdot)))||_{C_{1-\xi,\alpha}}dp \\
&+ \Big(\frac{t^\alpha - a^\alpha}{\alpha}\Big)^{1-\xi}\frac{1}{\Gamma(\nu)}\int_a^t p^{\alpha-1}\Big(\frac{t^\alpha - p^\alpha}{\alpha}\Big)^{\nu-1}\Big(\frac{p^\alpha - a^\alpha}{\alpha}\Big)^{1-\xi} \\
&\times ||h(\cdot, f_n(f_n(\cdot))) - h(\cdot, f(f(\cdot)))||_{C_{1-\xi,\alpha}}dp \\
&\leq \frac{\Gamma(\xi)}{\Gamma(\nu+\xi)}\left[|\kappa|\sum_{i=1}^{n}\gamma_i\Big(\frac{\zeta_i^\alpha - a^\alpha}{\alpha}\Big)^{\nu+\xi-1} + \Big(\frac{T^\alpha - a^\alpha}{\alpha}\Big)^{\nu}\right] \\
&\times ||h(\cdot, f_n(f_n(\cdot))) - h(\cdot, f(f(\cdot)))||_{C_{1-\xi,\alpha}} \\
&\leq \Omega_1||h(\cdot, f_n(f_n(\cdot))) - h(\cdot, f(f(\cdot)))||_{C_{1-\xi,\alpha}}.
\end{aligned}$$

Which shows that Q is completely continuous.

Claim 3: $Q(B_\delta)$ is relatively compact. $Q(B_\delta) \subset B_\delta$, it implies that Q is uniformly bounded and equicontinuous with $0 < a < t_1 < t_2 \leq T$.
Consider

$$\begin{aligned}
&|(Qf)(t_1) - (Qf)(t_2)| \\
&\leq \frac{|\kappa|}{\Gamma(\nu)}\left[\Big(\frac{t_1^\alpha - a^\alpha}{\alpha}\Big)^{\xi-1} - \Big(\frac{t_2^\alpha - a^\alpha}{\alpha}\Big)^{\xi-1}\right] \\
&\times \sum_{i=1}^{n}\gamma_i \int_a^{\zeta_i} p^{\alpha-1}\Big(\frac{\zeta_i^\alpha - p^\alpha}{\alpha}\Big)^{\nu-1}|h(p, f(f(p))|dp \\
&+ \frac{1}{\Gamma(\nu)}\left[\int_a^{t_1} p^{\alpha-1}\Big(\frac{t_1^\alpha - p^\alpha}{\alpha}\Big)^{\nu-1} - \int_a^{t_2} p^{\alpha-1}\Big(\frac{t_2^\alpha - p^\alpha}{\alpha}\Big)^{\nu-1}\right]|h(p, f(f(p))|dp
\end{aligned}$$

$$\leq \frac{|\kappa|}{\Gamma(\nu)}\Big[\Big(\frac{t_1^\alpha - a^\alpha}{\alpha}\Big)^{\xi-1} - \Big(\frac{t_2^\alpha - a^\alpha}{\alpha}\Big)^{\xi-1}\Big]$$
$$\times \sum_{i=1}^{n}\gamma_i \int_a^{\zeta_i} p^{\alpha-1}\Big(\frac{\zeta_i^\alpha - p^\alpha}{\alpha}\Big)^{\nu-1}\Big(\frac{p^\alpha - a^\alpha}{\alpha}\Big)^{\xi-1}||h||_{C_{1-\xi,\alpha}}dp$$
$$+ \frac{1}{\Gamma(\nu)}\Big[\int_a^{t_1} p^{\alpha-1}\Big(\frac{t_1^\alpha - p^\alpha}{\alpha}\Big)^{\nu-1} - \int_a^{t_2} p^{\alpha-1}\Big(\frac{t_2^\alpha - p^\alpha}{\alpha}\Big)^{\nu-1}\Big]$$
$$\times \Big(\frac{p^\alpha - a^\alpha}{\alpha}\Big)^{\xi-1}||h||_{C_{1-\xi,\alpha}}dp$$
$$\leq ||h||_{C_{1-\xi,\alpha}}\frac{|\kappa|\Gamma(\xi)}{\Gamma(\nu+\xi)}\Big[\Big(\frac{t_1^\alpha - a^\alpha}{\alpha}\Big)^{\xi-1} - \Big(\frac{t_2^\alpha - a^\alpha}{\alpha}\Big)^{\xi-1}\Big]\sum_{i=1}^{n}\gamma_i\Big(\frac{\zeta_i^\alpha - a^\alpha}{\alpha}\Big)^{\nu+\xi-1}$$
$$+ ||h||_{C_{1-\xi,\alpha}}\frac{\Gamma(\xi)}{\Gamma(\nu+\xi)}\Big[\Big(\frac{t_1^\alpha - a^\alpha}{\alpha}\Big)^{\nu+\xi-1} - \Big(\frac{t_2^\alpha - a^\alpha}{\alpha}\Big)^{\nu+\xi-1}\Big] \to 0.$$

When $t_2 \to t_1$. Therefore, Q is equicontinuous. Hence, the set $Q(B_\delta)$ of equicontinuous implies that $Q(B_\delta)$ is relatively compact.

Claim 4: $S = \{f \in C_{1-\xi,\alpha}[a,T] : f = \mu Qf$ for some $\mu \in (0,1)\}$ is a bounded set. Consider

$$\Big|\Big(\frac{t^\alpha - a^\alpha}{\alpha}\Big)^{1-\xi} f(t)\Big| \leq \Big\|\Big(\frac{t^\alpha - a^\alpha}{\alpha}\Big)^{1-\xi}(Qf)(t)\Big\|$$
$$\leq \frac{\Gamma(\xi)}{\Gamma(\xi+\nu)}\Big(|\kappa|\sum_{i=0}^{n}\gamma_i\Big(\frac{\zeta_i^\alpha - a^\alpha}{\alpha}\Big)^{\nu+\xi-1} + \Big(\frac{T^\alpha - a^\alpha}{\alpha}\Big)^{\nu}\Big)\|r\|_{C_{1-\xi,\alpha}} = C.$$

Hence, we get

$$\|f\|_{C_{1-\xi,\alpha}} \leq C.$$

Which shows that S is bounded set. Therefore, by Schaefer's fixed point theorem the stated problem (1. 1) has at least one solution. □

In the next part, we show the uniqueness of solutions to the problem.

Theorem 3.2. *Let the hypotheses* $(H_1) - (H_3)$ *holds with if* $L(P+1)\Omega_1 < 1$*, then the problem (1. 1) has unique solution in* $C^{\xi}_{1-\xi,\alpha}[a,T] \subset C^{\nu,\sigma}_{1-\xi,\alpha}[a,T]$.

Proof. From Theorem 3.1. Define an operator $Q : C_{1-\xi,\alpha}[a,T] \to C_{1-\xi,\alpha}[a,T]$ by

$$(Qf)(t) = \frac{\kappa}{\Gamma(\nu)}\Big(\frac{t^\alpha - a^\alpha}{\alpha}\Big)^{\xi-1}\sum_{i=0}^{n}\gamma_i\int_a^{\zeta_i} p^{\alpha-1}\Big(\frac{\zeta_i^\alpha - p^\alpha}{\alpha}\Big)^{\nu-1} h(p, f(f(p)))dp$$
$$+ \frac{1}{\Gamma(\nu)}\int_a^t p^{\alpha-1}\Big(\frac{t^\alpha - p^\alpha}{\alpha}\Big)^{\nu-1} h(p, f(f(p)))dp, \quad (3.\ 21)$$

from the Claim 1 of the Theorem 3.1., we can see that for $f \in B_\delta$,
$||Qf||_{C_{1-\xi,\alpha}} \leq \delta$.
Now, we prove Q is a contraction.

Consider

$$\begin{aligned}
&\Big|((Qf)-(Qg))(t)\Big(\frac{t^\alpha-a^\alpha}{\alpha}\Big)^{1-\xi}\Big| \\
&\leq \frac{|\kappa|}{\Gamma(\nu)}\sum_{i=0}^{n}\gamma_i\int_a^{\zeta_i}p^{\alpha-1}\Big(\frac{\zeta_i^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}|h(p,f(f(p)))-h(p,g(g(p)))|dp \\
&+\Big(\frac{t^\alpha-a^\alpha}{\alpha}\Big)^{1-\xi}\frac{1}{\Gamma(\nu)}\int_a^{t}p^{\alpha-1}\Big(\frac{t^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}|h(p,f(f(p)))-h(p,g(g(p)))|dp \\
&\leq \frac{|\kappa|}{\Gamma(\nu)}\sum_{i=0}^{n}\gamma_i\int_a^{\zeta_i}p^{\alpha-1}\Big(\frac{\zeta_i^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}L|f(f(p)))-g(g(p))|dp \\
&+\Big(\frac{t^\alpha-a^\alpha}{\alpha}\Big)^{1-\xi}\frac{1}{\Gamma(\nu)}\int_a^{t}p^{\alpha-1}\Big(\frac{t^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}L|f(f(p)))-g(g(p))|dp \\
&\leq \frac{|\kappa|}{\Gamma(\nu)}\sum_{i=0}^{n}\gamma_i\int_a^{\zeta_i}p^{\alpha-1}\Big(\frac{\zeta_i^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1} \\
&\times L\Big(|f(f(p))-f(g(p))|+|f(g(p))-g(g(p))|\Big)dp \\
&+\Big(\frac{t^\alpha-a^\alpha}{\alpha}\Big)^{1-\xi}\frac{1}{\Gamma(\nu)}\int_a^{t}p^{\alpha-1}\Big(\frac{t^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1} \\
&\times L\Big(|f(f(p))-f(g(p))|+|f(g(p))-g(g(p))|\Big)dp \\
&\leq \frac{|\kappa|}{\Gamma(\nu)}\sum_{i=0}^{n}\gamma_i\int_a^{\zeta_i}p^{\alpha-1}\Big(\frac{\zeta_i^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}\Big(\frac{p^\alpha-a^\alpha}{\alpha}\Big)^{\xi-1}L(P+1)||f-g||_{C_{1-\xi,\alpha}} \\
&+\Big(\frac{t^\alpha-a^\alpha}{\alpha}\Big)^{1-\xi}\frac{1}{\Gamma(\nu)}\int_a^{t}p^{\alpha-1}\Big(\frac{t^\alpha-p^\alpha}{\alpha}\Big)^{\nu-1}\Big(\frac{p^\alpha-a^\alpha}{\alpha}\Big)^{\xi-1} \\
&\times L(P+1)||f-g||_{C_{1-\xi,\alpha}}dp \\
&\leq \frac{\Gamma(\xi)|\kappa|}{\Gamma(\nu+\xi)}\sum_{i=0}^{n}\gamma_i\Big(\frac{\zeta_i^\alpha-a^\alpha}{\alpha}\Big)^{\nu+\xi-1}L(P+1)||f-g||_{C_{1-\xi,\alpha}} \\
&+\frac{\Gamma(\xi)}{\Gamma(\nu+\xi)}\Big(\frac{T^\alpha-a^\alpha}{\alpha}\Big)^{\nu}L(P+1)||f-g||_{C_{1-\xi,\alpha}} \\
&\leq L(P+1)||f-g||_{C_{1-\xi,\alpha}}\frac{\Gamma(\xi)}{\Gamma(\nu+\zeta)}\Big[|\kappa|\sum_{i=0}^{n}\gamma_i\Big(\frac{\zeta_i^\alpha-a^\alpha}{\alpha}\Big)^{\nu+\xi-1}+\Big(\frac{T^\alpha-a^\alpha}{\alpha}\Big)^{\nu}\Big] \\
&\leq L(P+1)\Omega_1||f-g||_{C_{1-\xi,\alpha}}.
\end{aligned}$$

i.e.

$$||Qf-Qg||_{C_{1-\xi,\alpha}} \leq L(P+1)\Omega_1||f-g||_{C_{1-\xi,\alpha}}, \forall f,g\in B_\delta. \qquad (3.\ 22)$$

Thus, operator Q satisfies contraction therefore by fixed point theorem of Banach, the problem (1. 1) has unique solution in $C^{\xi}_{1-\xi,\alpha}[a,T]\subset C^{\nu,\sigma}_{1-\xi,\alpha}[a,T]$. □

4. EXAMPLES

Example 4.1. *Consider the nonlocal problem*

$$\begin{cases} ({}^{\alpha}D_{a+}^{\nu,\sigma}f)(t) &= h(t,f(f(t))), \quad t\in(1,2] \\ ({}^{\alpha}I_{a+}^{1-\xi}f)(2) &= 2f(\frac{5}{4}), \quad \xi=\nu+\sigma(1-\nu) \end{cases} \tag{4. 23}$$

Solution:
Set $\nu=\frac{1}{4},\ \sigma=\frac{1}{3}$ *then* $\xi=\frac{1}{2}$. *Also, let* $\alpha=\frac{1}{2}$ *and*

$$h(t,f(f(t)))=\frac{1}{4}\Big(\frac{t^{\alpha}-1}{\alpha}\Big)^{-1/16}+\frac{1}{5}\Big(\frac{t^{\alpha}-1}{\alpha}\Big)^{15/16}sin(f(f(t))),\ then$$
$$\Big(\frac{t^{\alpha}-1}{\alpha}\Big)^{7/16}h(t,f(f(t)))=\frac{1}{4}\Big(\frac{t^{\alpha}-1}{\alpha}\Big)^{3/8}+\frac{1}{5}\Big(\frac{t^{\alpha}-1}{\alpha}\Big)^{11/8}sin(f(f(t)))\in C[1,2]$$
$$\therefore h(t,f(f(t)))\in C_{1/2,1/2}[1,2].$$

Also, $|h(t,f)-h(t,g)|\le\frac{1}{5}|f-g|,\ \forall\ f,\ g$ *so the Lipschitz constant* $L=\frac{1}{5}$.
We want solution in $C^{P}_{1-\xi,\alpha}=\{f\in C_{1-\xi,\alpha}:|f(t_1)-f(t_2)|\le|t_1-t_2|,\forall t_1,t_2\in[1,2].\}$ *so that* $P=1$.

$$|\kappa|=\left|\left(\Gamma(\frac{1}{2})-2\Big(\frac{(5/4)^{\frac{1}{2}}-1}{\frac{1}{2}}\Big)^{-1/2}\right)^{-1}\right|\approx 0.42664150<1.$$

and

$$\Omega_1=\frac{1.772453}{1.2254167}\left[0.42664150\left(2\Big(\frac{(5/4)^{0.5}-1}{0.5}\Big)^{-1/4}\right)+\Big(\frac{2^{0.5}-1}{0.5}\Big)^{1/4}\right]$$
$$\approx 2.2653272.$$

Then $L(P+1)\Omega_1=\frac{1}{5}(1+1)\times 2.2653272=0.9061309<1$. *Thus, by the Theorem (3.2), the problem (4. 23) possesses a solution in* $C_{1/2,1/2}[1,2]$.

Example 4.2. *Consider the nonlocal problem*

$$\begin{cases} ({}^{\alpha}D_{a+}^{\nu,\sigma}f)(t) &= h(t,f(f(t))), \quad t\in(2,3] \\ ({}^{\alpha}I_{a+}^{1-\xi}f)(3) &= 2f(2.3)+2.5f(2.4), \quad \xi=\nu+\sigma(1-\nu) \end{cases} \tag{4. 24}$$

Solution:
$\nu=\frac{3}{4},\ \sigma=\frac{1}{3}$ *which gives* $\xi=\frac{5}{6},\ \alpha=\frac{1}{3}$ *and*

$$h(t,f(f(t)))=\frac{3}{8}t+\frac{1}{7}f(f(t)).$$

Also, $|h(t,f)-h(t,g)|\le\frac{1}{7}|f-g|$ *hence,* $L=\frac{1}{7}$.
We want solution in $C^{P}_{1-\xi,\alpha}=\{f\in C_{1-\xi,\alpha}:|f(t_1)-f(t_2)|\le\frac{1}{2}|t_1-t_2|,\forall t_1,t_2\in[1,2].\}$ *so that* $P=\frac{1}{2}$.

$$|\kappa| = \left|\left(\Gamma(\frac{5}{6}) - \left[2\Big(\frac{(2.3)^{\frac{1}{3}} - 2^{\frac{1}{3}}}{\frac{1}{3}}\Big)^{-1/6} + 2.5\Big(\frac{(2.4)^{\frac{1}{3}} - 2^{\frac{1}{3}}}{\frac{1}{3}}\Big)^{-1/6}\right]\right)^{-1}\right|$$
$$\approx 0.21228832 < 1.$$

Also,

$$\Omega_1 = \frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{7}{6})}\Bigg[0.21228832\bigg(2\Big(\frac{(2.3)^{\frac{1}{3}} - 2^{\frac{1}{3}}}{\frac{1}{3}}\Big)^{1/6} + 2.5\Big(\frac{(2.4)^{\frac{1}{3}} - 2^{\frac{1}{3}}}{\frac{1}{3}}\Big)^{1/6}\bigg)$$
$$+ \Big(\frac{(3)^{\frac{1}{3}} - 2^{\frac{1}{3}}}{\frac{1}{3}}\Big)^{1/3}\Bigg]$$
$$\approx 1.89126819.$$

Then, $L(P+1)\Omega_1 = (\frac{1}{7})(\frac{3}{2})(1.89126819) \approx 0.40527190 < 1$. *Then, by the Theorem 3.2, the problem (4. 24) possesses a solution in* $C_{\frac{1}{6},\frac{1}{3}}[2,3]$.

5. Conclusion

In this research paper we have proved existence and uniqueness of the solutions of generalized iterative fractional differential equations via terminal condition with the help of fixed point theorems. The generality of generalized Katugampola fractional derivative in the discussed problem enhances applicability of the problem.

- If $\alpha \to 1$ in equation (1. 1), our problem will reduce to Hilfer fractional iterative differential equation with nonlocal condition and hence their existence and uniqueness results can be on similar lines with the help of results obtained in this article.
- If $\sigma = 0,\ \alpha \to 1$ in equation (1. 1), the problem reduces to iterative fractional differential equation having Riemann-Liouville fractional derivative operator.
- $\alpha \to 0$ in the equation (1. 1), the problem reduces to Hilfer-Hadamard iterative fractional derivative for which existence and uniqueness results can also proved with the help of fixed point techniques employed in the article.

6. Conflict of interest:

There is no conflict between the authors.

7. Funding:

This research is not funded by any government or private organization.

8. Acknowledgments

The Authors are thankful to referee for their comments and suggestions.

REFERENCES

[1] R. P. Agarwal, M. Meehan, and D. O'Regan. *Fixed Point Theory and Applications*, Cambridge University Press, United Kingdom, (2001).

[2] M. A. Anwar, S. U. Rehman, F. Ahmad, and M. I. Qadir. *A Numerical Iterative Scheme for Solving Nonlinear Boundary Value Problems of Fractional Order* $0 < \alpha < 1$, Punjab Univ. J. Math., **51**, no.1 (2019): 115-126.

[3] D. Baleanu, and B. Shiri. *Nonlinear higher order fractional terminal value problems*, AIMS Mathematics, **7**, no.5 (2022): 7489-7506.

[4] D. Baleanu, and B. Shiri. *Generalized fractional differential equations for past dynamic*, AIMS Math, no.7 (2022): 14394-14418.

[5] S. P. Bhairat, and D. B. Dhaigude. *Existence of solutions of generalized differential equation with nonlocal condition*, Mathematica Bohemica, **144**, no.2 (2019): 203-220.

[6] S. P. Bhairat, and D. B. Dhaigude. *Existence and stability of fractional differential equations involving generalized Katugampola derivative*, Studia Universitatis Babes-Bolyai Mathematica, **65**, no.1(2020).

[7] S. Bouriah, M. Benchohra, and S. G. Ozyurt. *Terminal value problem for neutral fractional functional differential equations with Hilfer-Katugampola fractional derivative*, Filomat, **37**, no.21 (2023): 7131-7147.

[8] S. I. Butt, B. Bayraktar, and J. E. N. Valdes. *Some new inequalities of Hermite Hadamard type via Katugampola fractional integral*, Punjab Univ. J. Math., **55**, no.7-8 (2023): 269-289.

[9] L. Byszewski. *Theorems about the existence and uniqueness of solutions of semilinear evolution nonlocal Cauchy problem*, Journal of Mathematical Analysis and Applications, **162**, (1991): 497-505.

[10] L. Byszewski, and H. Alca. *Existence of solutions of a semilinear functional-differential evolution nonlocal problem*, Nonlinear Analysis., **34**, (1998): 65-72.

[11] L. Byszewski, and V. Lakshmikantham. *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space*. Applied Analysis., **40**, (1990): 11-19.

[12] W. Chen, X. Xu, and S. Zhu. *Analytically pricing double barrier options based on a time-fractional Black-Scholes equation*, Computers & Mathematics with Applications, **69**, no.12 (2015): 1407-1419.

[13] K. Diethelm, and N. Ford. *A note on the well-posedness of terminal value problems for fractional differential equations*, Journal of integral equations and applications, **30**, no.3 (2018).

[14] K. Diethelm, and F. Uhlig. *A new approach to shooting methods for terminal value problems of fractional differential equations*, Journal of Scientific Computing, **97**, no.38 (2023).

[15] E. Eder. *The functional differential equation* $x' = x(x(t))$, Journal of Differential Equations, **54**, (1984):390-400.

[16] T. G. Hallam. *A comparison principle for terminal value problems in ordinary differential equations*, Transactions of the American Mathematical Society **169**, (1972): 49-57.

[17] R. W. Ibrahim, A. Kilicman, and F. H. Damag. *Existence and uniqueness for a class of iterative fractional differential equations*, Advanced Differential Equation, **1**, (2015): 1-13. doi:10.1186/s13662-015-0421-y.

[18] R. W. Ibrahim. *Existence of Iterative Cauchy Fractional Differential Equations*, Journal of Mathematics, (2013).

[19] H. Irmak, and T. Yildiz. *Comprehensive notes on various effects of some operators of fractional-order derivatives to certain functions in the complex domains and some of related implications*, Punjab Univ. J. Math., **54**, no. 5 (2022): 285-296.

[20] U.N.Katugampola. *New approach to a generalized fractional integral*, Applied Mathematics Computation, **218**, (2011): 860-865.

[21] U.N.Katugampola. *A new approach to generalized fractional derivatives*, Bulletin of Mathematical Analysis and Application, **6**, (2014): 1-15.

[22] U.N.Katugampola. *Existence and uniqueness results for a class of generalized fractional differenital equations*, Available at https://arxiv.org/abs/1411.5229, (2016).

[23] V. Kharat, and T. Jagtap. *Existence of iterative fractional differential equation with non local condition*, The Journal of Indian Mathematical Society, **83**, (2016): 97-106.

[24] C. Li, M. Li, and H. Zhou. *Terminal value problems for a generalized fractional ordinary differential equation*, Math Meth Appl Sci., (2021): 1-17.

[25] A.A.Kilbas, H.M. Srivastava, and J. J.Trujillo. *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies 204. Elsevier, Amsterdam, (2006).

[26] D. Mohammed, and M. S. Imakhlaf. *Existence and uniqueness of solutions of a terminal value problem for fractional-order differential equations*, Journal of Mathematical Extension, **15**, (2022): 1-23.
[27] A. A. Nanwate, and S. P. Bhairat. *On nonlocal terminal value problem in generalized fractional sense*, **11**, (2022): 62-74.
[28] D. S. Oliveira, and E. Capelas de Oliveira. *Hilfer-Katugampola fractional derivative*, Available at https://arxiv.org/abs/1705.07733, (2017).
[29] I. Podlubny. Fractional Differential Equations, *An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Mathematics in Science and Engineering 198. Academic Press, San Diego, (1999).
[30] I. A. Rus. *Ulam stability of ordinary differential equations*, Studia Universitatis Babes-Bolyai Mathematica, LIV, **4**, (2009): 125-133.
[31] A. Shaikh, V. Jadhav, B. Sontakke, and K. Nisar. *Analysis of Multi Term Fractional Differential Equations using Variational Iteration Method*, Punjab Univ. J. Math., **54**, no.1(2022):15-31. 10.52280/pujm.2022.540102.
[32] B. Shiri, G. C. Wu, and D. Baleanu. *Collocation methods for terminal value problems of tempered fractional differential equations*, Applied Numerical Mathematics, **156**, (2020): 385-395.
[33] B. Shiri, G. C. Wu, and D. Baleanu. *Terminal value problems for the nonlinear systems of fractional differential equations*, Appl. Numer. Math, **170**, (2021): 162-178.
[34] H. Sussain Shah, and M. Rehman. *A note on terminal value problems for fractional differential equations on infinite interval*, Applied Mathematics. Letters, **52**, (2016): 118-125.
[35] W. Shreve. *Terminal value problems for second order nonlinear differential equations*, SIAM Journal Applied Mathematics, **18**, no.4 (1970): 783-791.
[36] J. Wang, L. Lv, and Y. Zhou. *Ulam stability and data dependence for fractional differential equations with Caputo derivative*, Electronic Journal of Qualitative Theory of Differential. Equation., **63**, (2011): 1-10.
[37] J. Wang, and Y. Zhang. *Nonlocal initial value problems for differential equations with Hilfer fractional derivative*, Applied Mathematics and Computation., **266**, (2015): 850-859.
[38] G. C. Wu, B. Shiri, and Q. Fan, et al. *Terminal Value Problems of Non-homogeneous Fractional Linear Systems with General Memory Kernels*, J Nonlinear Math Phys, **30**, (2023):303-314. https://doi.org/10.1007/s44198-022-00085-2.
[39] G. Yang, B. Shiri, H. Kong, and G. C. Wu. *Intermediate value problems for fractional differential equations*, Comput. Appl. Math., **40**, (2021):195.
[40] H Zhang, F Liu, I Turner, and Q Yang. *Numerical solution of the time fractional Black Scholes model governing European options*, Computers & Mathematics with Applications, **71**, (2016): 1772-1783.