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Hybrid Fractional Problem with Periodic Boundary Conditions

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Abstract. The primary focus of the current study is to construct the analytic solution to hybrid fractional problem including periodic boundary conditions. Utilization of the separation of variables method (SVM) provides the solution in a Fourier series form in terms of corresponding eigenfunctions which are the solutions of corresponding fractional Sturm-Liouville problem in the sense of hybrid fractional derivative. The significant motivation of this study is that fractional diffusion problem with periodic boundary conditions in the constant proportional Caputo hybrid derivative (CPCHD), a combination of Riemann-Liouville integral and Caputo derivative, is considered through SVM. Special cases of CPCHD are taken into account and obtained results are compared to analyze the effect of chosen proportions. Moreover, the established solutions are given in terms of bivariate Mittag-Leffler function emerging in diverse applications. As a result, the novelty of this research is that fractional diffusion problems with periodic boundary conditions in the sense of CPCHD is considered and their solutions are obtained by means of bivariate Mittag-Leffler function. Examples are provided to present accuracy and efficiency of the proposed method as well as influence of the proportions in CPCHD for hybrid fractional problem.

AMS (MOS) Subject Classification Codes: 65M70; 26A33
Key Words: Periodic boundary conditions, Bivariate Mittag-Leffler function, Hybrid Fractional Derivative, Spectral method.

1. Introduction

Fractional differential equations (FDEs) are mostly preferred to model various scientific processes which leads to growing interest of many scientist in the diverse fields of sciences. Moreover, FDEs with diverse fractional derivatives get the attention of a number of researches since the correct choices of fractional derivative also play a significant role in the reflection of the processes by FDEs. Hence, the definition of different fractional derivatives such as Riemann-Liouville, Caputo and Atangana-Baleanu fractional derivatives, etc. have been given [20, 8, 22, 3, 9, 4, 16, 14, 28, 12]. The main reason of utilizing FDEs is that the fractional mathematical models reflect the behavior of the process under consideration much more better than the other mathematical models. However, establishing the analytical solution of FDEs is harder which leads to construct approximate or truncated solutions of them. The selection of the suitable fractional derivative should be based on the real data of the process. There are various methods to define new fractional derivatives which can be categorized as local or non-local fractional derivatives based on their properties. Utilization of the Caputo derivative and proportional derivative together leads to a new defined fractional derivative, called the constant proportional Caputo hybrid derivative (CPCHD) given as

$$\begin{aligned} {}_{0}^{CPC}D_{\xi}^{\alpha} f(\xi) &= \frac{1}{\Gamma(1-\alpha)} \int_{0}^{\xi} \left(K_1(\alpha) f(\tau) + K_0(\alpha) f'(\tau)\right)(\xi-\tau)^{-\alpha} d\tau, \\ &= K_1(\alpha)\, {}_{0}^{RL}I_{\xi}^{1-\alpha} f(\xi) + K_0(\alpha)\, {}_{0}^{C}D_{\xi}^{\alpha} f(\xi), \end{aligned} \tag{1. 1}$$

where $0 \leq \xi$ and the limit of functions K_0 and K_1 must have certain conditions [7]. The convergence condition of the integral in Eq. (1. 1) is that f must be differentiable and f and f' must be locally L^1 on the positive reals [7]. The notations ${}_{0}^{C}D_{\xi}^{\alpha}$ and ${}_{0}^{RL}I_{\xi}^{\alpha}$ denote Caputo derivative and Riemann-Liouville integral, respectively. From the definition of hybrid derivative, the physical interpretation of it can be concluded that the physical system modeled by hybrid derivative is influenced by memory of its instantaneous rate of change and current situation in some rate.

Fractional diffusion problems have been utilized broadly in the modeling of diverse processes in physics, engineering, and biology. Fractional diffusion equations play an essential role in the investigation for the behavior of charge carriers in materials [6]. Spectral method have been used to obtain the numerical or analytical solutions of diverse fractional differential problems [1, 21, 33]. Moreover, various methods based on operational matrix have been employed to accomplish numerical solutions of fractional differential problems. The integral operational matrices based on orthogonal Chelyshkov polynomials are used to establish numerical solutions to fractional-order two-point boundary value problems [10]. Furthermore, the generalized integral operational matrix derived from orthogonal Laguerre polynomials and operational matrix are employed to acquire numerical solutions to fractional-order differential equations with initial conditions [32]. Numerical solutions to generalized modified Caputo fractional differential equations are established by the generalized derivative and integral operational matrices [30]. Paraskevopoulos's algorithm with operational matrices of Vieta-Lucas polynomials provide numerical solutions to the multi-order linear and nonlinear Caputo fractional-order differential equations [31].

In this current work, the focus is on the establishment of the fractional diffusion problem (FDP) in the sense of CPCHD by employing SVM:

$${}_{0}^{CPC}D_{\xi}^{\alpha} \omega(\zeta,\xi) = \gamma^2 \omega_{\zeta\zeta}(\zeta,\xi), \tag{1. 2}$$

$$\begin{cases} \omega\left(-l,\xi\right)=\omega\left(l,\xi\right), \\ \omega_{\zeta}\left(-l,\xi\right)=\omega_{\zeta}\left(l,\xi\right), \end{cases} \tag{1. 3}$$

$$\omega\left(\zeta,0\right)=f(\zeta), \tag{1. 4}$$

where $0<\alpha<1, -l\leq\zeta\leq l, 0\leq\xi\leq T_0, \gamma\in\mathbb{R}$. The following forms of CPCHD are used in this study:

$$ {}_{0}^{CPC}D_{\alpha}^{\;1} f\left(\xi\right)=\left(1-\alpha\right)\ {}_{0}^{RL}I_{\xi}^{1-\alpha} f\left(\xi\right)+\alpha{}_{0}^{C}D_{\xi}^{\alpha} f\left(\xi\right), \tag{1. 5}$$

and

$$ {}_{0}^{CPC}D_{\alpha}^{\;2} f\left(\xi\right)=\left(1-\alpha^{2}\right)\ {}_{0}^{RL}I_{\xi}^{1-\alpha} f\left(\xi\right)+\alpha^{2}{}_{0}^{C}D_{\xi}^{\alpha} f\left(\xi\right). \tag{1. 6}$$

The main contribution of this research is to analyze the effect of proportions in CPCHD for hybrid fractional problems with periodic boundary conditions by establishing the solution in terms of bivariate Mittag-Leffler function via SVM. To this end, two special case of CPCHD are considered and the obtained outcomes are compared. The suitable choices of proportions in CPCHD are made based on the real data from the system under consideration which makes CPCHD more versatile and trustworthy compare to other fractional derivatives. Recently, numerous research on some viral diseases are mathematically modeled by fractional differential equations in CPCHD sense [17], [18], [29], [16].

Numerous systems such as electrostatic systems in periodic boundary conditions, in physics, engineering and other scientific fields are modeled by differential equations with periodic boundary conditions such as the time-harmonic Schrödinger equation [15], [5], [13].

From a physical point of view, the exhibition of the intrinsic nature of the physical process can be made by fractional mathematical models better than other mathematical models. Hence, the agreement between the solution and the process is excellent which leads to more accurate predictions and experimental measurement. The modeling of non-local processes by FDP lets us to investigate the processes under consideration better than other models. Moreover, the processes with memory can be investigated by modeling them with FDP. On the other hand, using suitable fractional derivative also plays a vital role in the investigation of the scientific processes.

The mathematical modeling of diffusion processes with various matters such as temperature, liquid and gas in a phase contains the diffusion coefficient γ^2 depending on the fractional order α which implies that α must be determined accurately [11]. The diffusion of diverse matters in gas dynamics and fluid mechanics have been studied by many scientist [27, 2, 24, 25, 34, 4]. From this point of view, the investigation of the diffusion processes arise in many applications which implies the importance of the subject. Diffusion processes are classified as sub-diffusion and super-diffusion processes. The diffusion of the matter in sub-diffusion processes for which $0<\alpha<1$ is slower than the one in super-diffusion processes.

The current work presents the investigation of the sub-diffusion processes by establishing the solution of time fractional diffusion problem including periodic boundary conditions.

$$ {}_{0}^{CPC}D_{\xi}^{\alpha}\omega\left(\zeta,\xi\right)=\gamma^{2}\omega_{\zeta\zeta}\left(\zeta,\xi\right), \tag{1. 7} $$

$$ \begin{cases}\omega\left(-l,\xi\right)=\omega\left(l,\xi\right),\\ \omega_{\zeta}\left(-l,\xi\right)=\omega_{\zeta}\left(l,\xi\right),\end{cases} \tag{1. 8} $$

$$ \omega\left(\zeta,0\right)=f(\zeta), \tag{1. 9} $$

where $0<\alpha<1,-l\leq\zeta\leq l,0\leq\xi\leq T_{0},\gamma\in\mathbb{R}$.

The novelty of this research is that the above problem in the sense of CPCHD is studied and the obtained solution is expressed in terms of bivariate Mittag-Leffler function.

The rest of the paper is designed in the following form: Preliminaries are presented in section 2. SVM for time fractional differential problems is introduced in section 3. Elucidatory examples are presented for different cases in section 4. Results and discussion are provided in section 5. In the final section 6, the summary and conclusions of the study is provided.

2. Preliminaries

This section is devoted to basic definitions and concepts utilized in this study [26],[23].

Definition 2.1. *Let $v\left(\zeta,\theta\right)$ be a real valued function. Its Riemann-Liouville time fractional integral of order $\alpha>0$ is denoted by $I_{\psi}^{\alpha}v\left(\zeta,\theta\right)$ and is defined as:*

$$ I_{\theta}^{\alpha}v\left(\zeta,\theta\right)=\frac{1}{\Gamma(\alpha)}\int_{0}^{\theta}\frac{v\left(\zeta,s\right)}{\left(\theta-s\right)^{1-\alpha}}ds. $$

Definition 2.2. *The Caputo time fractional derivative of $v\left(\zeta,\theta\right)$ is defined as:*

$$ D_{\theta}^{\alpha}v\left(\zeta,\theta\right)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_{0}^{\theta}\frac{\frac{\partial^{n}}{\partial s^{n}}v\left(\zeta,s\right)}{\left(\theta-s\right)^{1+\alpha-m}}ds, n-1<\alpha<n,\\ \dfrac{\partial^{n}}{\partial\theta^{n}}v\left(\zeta,\theta\right),\alpha=n.\end{cases} $$

Definition 2.3. *The Mittag-Leffler function of two parameters $E_{\alpha,\beta}\left(\zeta\right)$ is defined as*

$$ E_{\alpha,\beta}\left(\zeta\right)=\sum_{j=0}^{\infty}\frac{\zeta^{j}}{\Gamma(\alpha j+\beta)}, $$

where $Re\left(\alpha\right)>0,\zeta,\beta\in\mathbb{C}$.

Definition 2.4. *The bivariate Mittag-Leffler function $E_{\alpha,\,\beta,\,\kappa}^{(\gamma)}\left(\zeta,y\right)$ is defined as* [19]

$$ E_{\alpha,\beta,\,\kappa}^{(\gamma)}\left(x,y\right)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}}{\Gamma\left(\alpha+r\right)\Gamma\left(\beta+\kappa s\right)}\,\frac{x^{r}}{r!}\,\frac{y^{\kappa s}}{s!},\alpha,\beta,\,\gamma\in\mathbb{C},\,Re(\alpha),\,Re(\beta),\,Re(\kappa)>0. \tag{2. 10} $$

This function is an entire function of x and y for $Re(\alpha)>0$ $Re(\beta)>0$ since the series in (2. 10) is absolutely convergent as well as locally uniformly convergent.

3. THE SEPARATION OF VARIABLES METHOD FOR TIME FRACTIONAL DIFFUSION PROBLEM WITH PERIODIC BOUNDARY CONDITIONS

This section demonstrate the implementation of SVM for the solutions to the problem (1. 7)-(1. 9).

$$\omega\left(\zeta,\xi;\alpha\right)=X\left(\zeta\right)T\left(\xi;\alpha\right), \tag{3. 11}$$

where $-l\leq\zeta\leq l, 0\leq\xi\leq T_0$.
Employing (3. 11) in (1. 7) provides the following:

$$\frac{{}_{0}^{CPC}D_{\xi}^{\alpha}\left(T\left(\xi;\alpha\right)\right)}{T\left(\xi;\alpha\right)}=\gamma^{2}\frac{X''\left(\zeta\right)}{X\left(\zeta\right)}=-\lambda^{2}. \tag{3. 12}$$

Utilization of the boundary conditions (1. 8) and the Eq. (3. 12) yields:

$$X''\left(\zeta\right)+\lambda^{2}X\left(\zeta\right)=0, \tag{3. 13}$$

$$\begin{cases}X\left(-l\right)=X\left(l\right),\\ X'\left(-l\right)=X'\left(l\right),\end{cases} \tag{3. 14}$$

having the following solution

$$X\left(\zeta\right)=e^{r\zeta}, \tag{3. 15}$$

which yields the characteristic equation below:

$$r^{2}+\lambda^{2}=0. \tag{3. 16}$$

Case 1. If $\lambda=0$, there are two coincident roots $r_1=r_2$ which generates the solution in the following form:

$$X\left(\zeta\right)=k_{1}\zeta+k_{2}, \tag{3. 17}$$

$$X'\left(\zeta\right)=k_{1}. \tag{3. 18}$$

The boundary condition $\omega\left(-l,\xi\right)=\omega\left(l,\xi\right)$ leads to

$$X\left(-l\right)=-k_{1}l+k_{2}=k_{1}l+k_{2}=X(l)\Rightarrow k_{1}=0, \tag{3. 19}$$

implying the following

$$X\left(\zeta\right)=k_{2}. \tag{3. 20}$$

Moreover, the last boundary condition yields

$$X'\left(-l\right)=0=X'(l). \tag{3. 21}$$

Hence, the solution is established as

$$X_{0}\left(\zeta\right)=k_{2}. \tag{3. 22}$$

Case 2. If $\lambda > 0$, there are two distinct real roots r_1, r_2 which generates the following solution:

$$X(\zeta) = c_1 e^{r_1 \zeta} + c_1 e^{r_2 \zeta}. \tag{3. 23}$$

Utilization of the boundary condition $\omega(-l, \xi) = \omega(l, \xi)$ provides

$$X(-l) = c_1 e^{r_1(-l)} + c_2 e^{r_2(-l)} = c_1 e^{r_1 l} + c_2 e^{r_2 l} = X(l). \tag{3. 24}$$

$$c_1 \left(e^{r_1(-l)} - c_1 e^{r_1 l} \right) + c_2 \left(e^{r_2(-l)} - e^{r_2 l} \right) = 0. \tag{3. 25}$$

Linearly independence of $\left(e^{r_1(-l)} - c_1 e^{r_1 l}\right)$ and $\left(e^{r_2(-l)} - e^{r_2 l}\right)$ implies that $c_1 = 0 = c_2$ which yields $X(\zeta; \beta) = 0$, implying no solution for $\lambda > 0$.
Case 3. If $\lambda < 0$, there are two complex roots which generates the solution in the following form:

$$X(\zeta) = c_1 \cos(\lambda\zeta) + c_2 \sin(\lambda\zeta). \tag{3. 26}$$

The boundary condition $\omega(-l, \xi) = \omega(l, \xi)$ leads to

$$X(-l) = c_1 \cos(\lambda l) - c_2 \sin(\lambda l) = c_1 \cos(\lambda l) + c_2 \sin(\lambda l) = X(l), \tag{3. 27}$$

implying that

$$2c_2 \sin(\lambda l) = 0 \Rightarrow c_2 = 0. \tag{3. 28}$$

Hence, we have the solution below

$$X(\zeta) = c_1 \cos(\lambda\zeta), \tag{3. 29}$$

$$X'(\zeta) = -c_1 \lambda \sin(\lambda\zeta). \tag{3. 30}$$

In a similar way, second boundary condition yields

$$X'(-l) = c_1 \lambda \sin(\lambda l) = -c_1 \lambda \sin(\lambda l) = X'(l) \Rightarrow 2c_1 \lambda \sin(\lambda l) = 0, \tag{3. 31}$$

implying that

$$\sin(\lambda l) = 0, \tag{3. 32}$$

which produces the following outcomes:

$$\lambda_n = \frac{w_n}{l}, \lambda_1 < \lambda_2 < \lambda_3 < \ldots, \tag{3. 33}$$

where the equation $sin(w_n) = 0$ have the solutions $w_n = n\pi$.
Hence, the obtained solution is written as:

$$X_n(\zeta) = c_1 \cos\left(w_n \left(\frac{\zeta}{l} \right) \right), n = 1, 2, 3, \ldots \tag{3. 34}$$

Following ordinary differential equation is obtained by using the eigenvalue λ_n in the second equation (3. 12):

$$\frac{{}_{0}^{CPC}D_{\xi}^{\alpha}\left(T\left(\xi;\alpha\right)\right)}{T\left(\xi;\alpha\right)}=-\gamma^{2}\lambda_{n}^{2}, \tag{3. 35}$$

yielding the following outcome [7]

$$T_{n}\left(\xi;\alpha\right)=E_{\alpha,1,\ 1}^{1}\left(\frac{-\gamma^{2}\lambda_{n}^{2}}{K_{0}\left(\alpha\right)}\xi^{\alpha},\frac{-K_{1}\left(\alpha\right)}{K_{0}\left(\alpha\right)}\xi\right),n=0,1,2,3,\ldots \tag{3. 36}$$

Each solution $u_n\left(\zeta,t;\alpha\right)$ corresponding to the eigenvalue λ_n is established as

$$u_{n}\left(\zeta,\xi;\alpha\right)=X_{n}\left(\zeta\right)T_{n}\left(\xi;\alpha\right)=E_{\alpha,1,\ 1}^{1}\left(\frac{-\gamma^{2}\lambda_{n}^{2}}{K_{0}\left(\alpha\right)}\xi^{\alpha},\frac{-K_{1}\left(\alpha\right)}{K_{0}\left(\alpha\right)}\xi\right)\cos\left(w_{n}\left(\frac{\zeta}{l}\right)\right),n=0,1,2,3,\ldots \tag{3. 37}$$

implying the general solution below:

$$\omega\left(\zeta,\xi;\alpha\right)=A_{0}+\sum_{n=1}^{\infty}A_{n}\cos\left(w_{n}\left(\frac{\zeta}{l}\right)\right)E_{\alpha,1,\ 1}^{1}\left(\frac{-\gamma^{2}\lambda_{n}^{2}}{K_{0}\left(\alpha\right)}\xi^{\alpha},\frac{-K_{1}\left(\alpha\right)}{K_{0}\left(\alpha\right)}\xi\right), \tag{3. 38}$$

which satisfies FDE with periodic boundary conditions.
Utilization of initial condition leads to determination of unknown coefficients in (3. 38):

$$\omega\left(\zeta,0\right)=f\left(\zeta\right)=A_{0}+\sum_{n=1}^{\infty}A_{n}\cos\left(w_{n}\left(\frac{\zeta}{l}\right)\right). \tag{3. 39}$$

The employment of the inner product in $L^2[-l,l]$ leads to the determination of the unknown coefficients A_n for $n=0,1,2,3,\ldots$:

$$A_{0}=\frac{1}{2l}\int_{-l}^{l}f(\zeta)d\zeta, \tag{3. 40}$$

$$A_{n}=\frac{1}{l}\int_{-l}^{l}f(\zeta)\cos\left(w_{n}\left(\frac{\zeta}{l}\right)\right). \tag{3. 41}$$

4. Elucidatory Examples

This section presents some examples of time FDP:

$$\omega_{\xi}(\zeta,\xi)=\omega_{\zeta\zeta}(\zeta,\xi),$$

$$\begin{cases}\omega\left(-1,\xi\right)=\omega\left(1,\xi\right),\\ \omega_{\zeta}\left(-1,\xi\right)=\omega_{\zeta}\left(1,\xi\right),\end{cases}$$

$$\omega\left(\zeta,0\right)=\cos\left(\pi\zeta\right), \tag{4. 42}$$

which are satisfied by the following outcome:

$$\omega\left(\zeta,\xi\right)=\cos\left(\pi\zeta\right)e^{-\pi^2\xi}, \tag{4. 43}$$

where $-1\leq\zeta\leq1, 0\leq\xi\leq T_0$.

The corresponding time FDP is given as

$${}_0^{CPC}D_\xi^\alpha\omega\left(\zeta,\xi\right)=\omega_{\zeta\zeta}\left(\zeta,\xi\right), \tag{4. 44}$$

$$\begin{cases}\omega\left(-1,\xi\right)=\omega\left(1,\xi\right),\\ \omega_\zeta\left(-1,\xi\right)=\omega_\zeta\left(1,\xi\right),\end{cases} \tag{4. 45}$$

$$\omega\left(\zeta,0\right)=\cos\left(\pi\zeta\right), \tag{4. 46}$$

where $0<\alpha<1, -1\leq\zeta\leq1,\ 0\leq\xi\leq T_0$.
Utilization of SVM provides the equations below:

$$\frac{{}_0^{CPC}D_\xi^\alpha\left(T\left(\xi;\alpha\right)\right)}{T\left(\xi;\alpha\right)}=\frac{X''\left(\zeta\right)}{X\left(\zeta\right)}=-\lambda^2. \tag{4. 47}$$

By employing the boundary conditions (4. 45) and Eq. (4. 47) together leads to following:

$$X''\left(\zeta\right)+\lambda^2X\left(\zeta\right)=0, \tag{4. 48}$$

$$\begin{cases}X\left(-l\right)=X\left(l\right),\\ X'\left(-l\right)=X'\left(l\right).\end{cases} \tag{4. 49}$$

The eigenvalues of the problem (4. 48)-(4. 49) are acquired as

$$X_n\left(\zeta\right)=\cos\left(n\pi\zeta\right), n=1,2,3,\ldots \tag{4. 50}$$

Employing the eigenvalues λ_n in the Eq. (4. 47) yields

$$\frac{{}_0^{CPC}D_\xi^\alpha\left(T\left(\xi;\alpha\right)\right)}{T\left(\xi;\alpha\right)}=-\lambda^2, \tag{4. 51}$$

which is satisfied by

$$T_n\left(\xi;\alpha\right)=E^1_{\alpha,1,\ 1}\left(\frac{-n^2\pi^2}{K_0\left(\alpha\right)}\xi^\alpha,\frac{-K_1\left(\alpha\right)}{K_0\left(\alpha\right)}\xi\right), n=0,1,2,3,\ldots \tag{4. 52}$$

The solution corresponding to λ_n is represented as

$$\omega_n\left(\zeta,\xi;\alpha\right)=E^1_{\alpha,1,\ 1}\left(\frac{-n^2\pi^2}{K_0\left(\alpha\right)}\xi^\alpha,\frac{-K_1\left(\alpha\right)}{K_0\left(\alpha\right)}\xi\right)\cos\left(n\pi\zeta\right), n=0,1,2,3,\ldots \tag{4. 53}$$

By means of Superposition Principle, the following is obtained:

$$\omega\left(\zeta,\xi;\alpha\right)=A_0+\sum_{n=1}^{\infty}A_n\cos\left(n\pi\zeta\right)\ E^1_{\alpha,1,\ 1}\left(\frac{-n^2\pi^2}{K_0\left(\alpha\right)}\xi^\alpha,\frac{-K_1\left(\alpha\right)}{K_0\left(\alpha\right)}\xi\right), \tag{4. 54}$$

where A_n for $n = 0, 1, 2, 3, \ldots$ are obtained by means of L^2 inner product and initial condition

$$\omega(\zeta, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi\zeta), \tag{4. 55}$$

as follows:

$$A_0 = \frac{1}{2}\int_{-1}^{1} \cos(\pi\zeta) d\zeta = \left(\frac{1}{2\pi}\sin(\pi\zeta)\right)_{\zeta=-1}^{\zeta=1} = 0. \tag{4. 56}$$

$$A_n = \int_{-1}^{1} \cos(\pi\zeta)\cos(n\pi\zeta) d\zeta, \tag{4. 57}$$

which implies that $A_n = 0$ for $n \neq 1$ and

$$A_1 = \int_{-1}^{1} \cos^2(\pi\zeta) d\zeta = \int_{-1}^{1} \left(\frac{1}{2} + \frac{\cos(2\pi\zeta)}{2}\right) d\zeta = \left(\frac{\zeta}{2} + \frac{\sin(2\pi\zeta)}{4\pi}\right)\Bigg|_{\zeta=-1}^{\zeta=1} = 1. \tag{4. 58}$$

Therefore, the following outcome is obtained

$$\omega(\zeta, \xi; \alpha) = \cos(\pi\zeta) E^{1}_{\alpha,1,\,1}\left(\frac{-\pi^2}{K_0(\alpha)}\xi^{\alpha}, \frac{-K_1(\alpha)}{K_0(\alpha)}\xi\right). \tag{4. 59}$$

Replacing α by 1 in (4. 59) leads to the solution to (4. 42) indicates the accuracy of the method. Especially, the problem (4. 44)-(4. 46) have the solution for the following cases of K_0 and K_1:

Case 1: $K_0(\alpha) = \alpha, K_1(\alpha) = 1 - \alpha$,

$$\omega(\zeta, \xi; \alpha) = \cos(\pi\zeta) E^{1}_{\alpha,1,\,1}\left(\frac{-\pi^2}{\alpha}\xi^{\alpha}, \frac{\alpha - 1}{\alpha}\xi\right). \tag{4. 60}$$

Case 2: $K_0(\alpha) = \alpha^2, K_1(\alpha) = 1 - \alpha^2$,

$$\omega(\zeta, \xi; \alpha) = \cos(\pi\zeta) E^{1}_{\alpha,1,\,1}\left(\frac{-\pi^2}{\alpha^2}\xi^{\alpha}, \frac{\alpha^2 - 1}{\alpha^2}\xi\right). \tag{4. 61}$$

5. Results and Discussion

The 2D solutions of the problem (4. 42) for various values of α for Case 1 and Case 2 are presented in Fig. 1-4. It is clear from Fig.1-4 that the truncated solution of the time fractional diffusion problem with periodic boundary conditions for the case where $K_0(\alpha) = \alpha, K_1(\alpha) = 1-\alpha$ is closer to the exact solution for all values of α. Furthermore, it can be observed that as α tends to 1, the truncated solutions for two cases get closer to each other as well as the exact solution. Moreover, the diffusion rate decreases for all cases when α tends to 0 or time variable ξ increases. The elucidatory examples reveal that the obtained solutions in series form always converges rapidly against the analytical solution

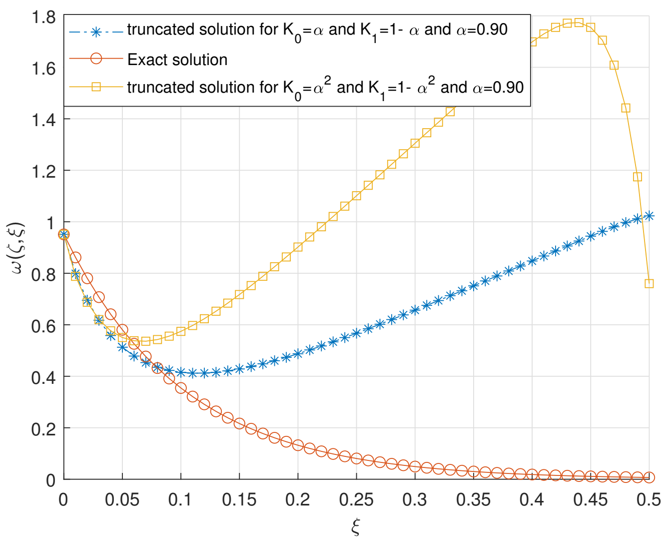

FIGURE 1. The 2D solution graphics at $\alpha = 0.9$ for Example for various functions $K_0\left(\alpha\right)$ and $K_1\left(\alpha\right)$ at $\zeta = 0.1$.

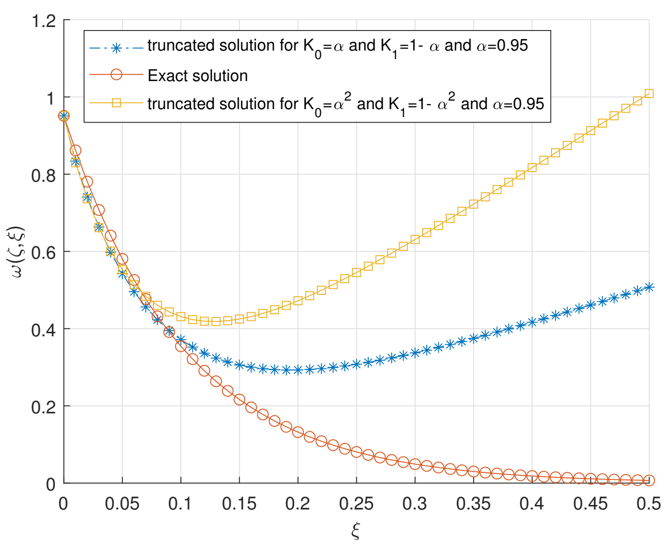

FIGURE 2. The 2D solution graphics at $\alpha = 0.95$ for Example for various functions $K_0\left(\alpha\right)$ and $K_1\left(\alpha\right)$ at $\zeta = 0.1$.

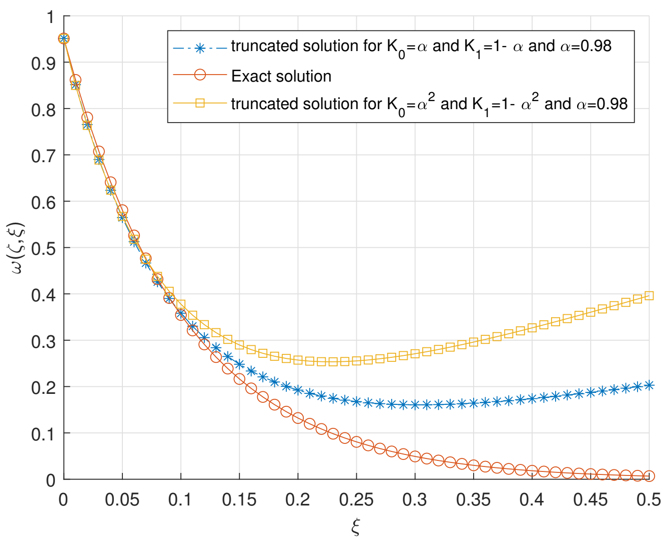

FIGURE 3. The 2D solution graphics at $\alpha = 0.98$ for Example for various functions $K_0(\alpha)$ and $K_1(\alpha)$ at $\zeta = 0.1$.

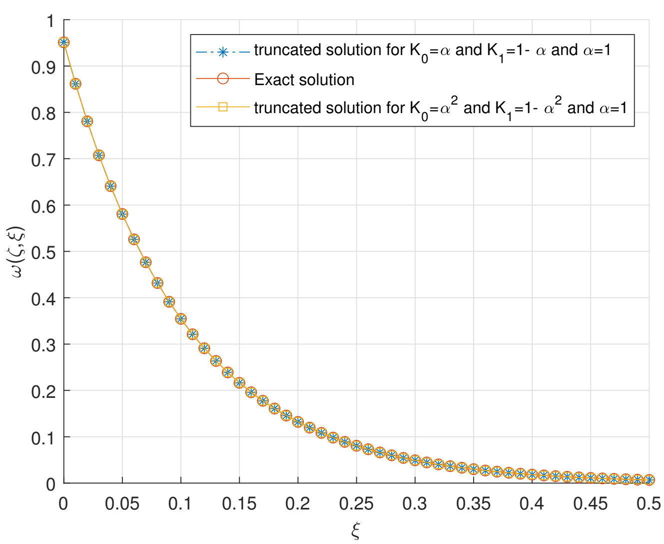

FIGURE 4. The 2D solution graphics at $\alpha = 1$ for Example for various functions $K_0(\alpha)$ and $K_1(\alpha)$ at $\zeta = 0.1$.

6. Conclusion

The solution of FDEs in hybrid derivative is the focus of this research. By means of SVM, the solutions are obtained in series form including bivariate Mittag-Leffler function. The accuracy of the method is investigated by substituting $\alpha = 1$ which leads to the solution to corresponding diffusion problem. The novelty of this research is to discuss the effect of proportions in CPCHD which is used for the modeling of scientific problems with periodic boundary conditions through analyzing the solutions obtained by SVM. The presented examples illustrate that SVM is an effective method in the determination of the solution to the time fractional diffusion problems.

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