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Fibonacci numbers which are concatenation of three Fibonacci or Lucas numbers

Fatih Erduvan
MEB,
İzmit Namık Kemal Anatolian High School,
Kocaeli/Türkiye
Email: erduvanmat@hotmail.com

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Abstract. In this note, we show that there is no Fibonacci number that can be expressed as a concatenation of three Fibonacci or Lucas numbers under a certain constraint. That is, we solve to the Diophantine equations

$$F_n = 10^{d+l} F_{m_1} + 10^l F_{m_2} + F_{m_3}$$

and

$$F_n = 10^{d+l} L_{m_1} + 10^l L_{m_2} + L_{m_3}$$

in non-negative integers (n, m_1, m_2, m_3) with $m_2 \leq m_1$, where d and l represent the number of digits of the (F_{m_2}, L_{m_2}) and (F_{m_3}, L_{m_3}), respectively. Moreover, we give these equations as a problem to the researchers without the constraint $m_2 \leq m_1$.

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1. Introduction

Let $(F_n)_{n\geq 2}$ and $(L_n)_{n\geq 2}$ be the Fibonacci and Lucas sequences given by the recurrence relations $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$ with the initial conditions, $F_0 = 0,\ F_1 = 1$, $L_0 = 2,\ L_1 = 1$, respectively. Binet formulas for these numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \text{ and } L_n = \alpha^n + \beta^n,$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$. It can be proved that

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \text{ for } n \geq 1, \tag{1. 1}$$

and

$$\alpha^{n-1} \leq L_n \leq 2\alpha^n \text{ for } n \geq 0 \tag{1. 2}$$

by induction. In mathematical terms, concatenation refers to the process of sequentially joining two or more objects (typically numbers or integer sequences). In this process, the objects are added in order, resulting in a new object. For example, the concatenation of the numbers a and b is represented as $\overline{ab}$. Diophantine equations involving concatenation are among the popular topics in recent times. In [3], the authors considered Diophantine equations with concatenations of members of binary recurrences. In this study, the authors also dealt with the Diophantine equation

$$F_k = \overline{F_m F_n} = 10^d F_m + F_n$$

in non-negative integers with $m > 0$, where d denotes the number of digits of F_n. In [1], Alan solved the Diophantine equations

$$F_n = 10^d L_m + L_k \text{ and } L_n = 10^d F_m + F_k$$

in non-negative integers. In [2], the investigators coped with the Diophantine equations

$$F_n = 10^d F_m + L_k \text{ and } F_n = 10^d L_m + F_k$$

in non-negative integers. Here d denotes the number of digits of the L_k and F_k. Moreover, similar papers on Padovan and Perrin numbers can be found in [4] and [8]. Triple concatenation is a more general form of binary concatenation. Thus, we consider extending this to the problem of finding Fibonacci numbers as the concatenation of three Fibonacci sequences. That is, we tackle the Diophantine equations

$$F_n = \overline{F_{m_1} F_{m_2} F_{m_3}} = 10^{d+l} F_{m_1} + 10^l F_{m_2} + F_{m_3} \tag{1. 3}$$

and

$$F_n = \overline{L_{m_1} L_{m_2} L_{m_3}} = 10^{d+l} L_{m_1} + 10^l L_{m_2} + L_{m_3} \tag{1. 4}$$

in non-negative integers (n, m_1, m_2, m_3) with $m_2 \leq m_1$, where d and l represent the number of digits of the (F_{m_2}, L_{m_2}) and (F_{m_3}, L_{m_3}), respectively. In triple concatenation, unlike binary concatenation, while the equations were desired to be reordered three times, due to some difficulties during the operations, a restriction was imposed, and the equations were reordered only twice.

The investigation of such Diophantine equations offers significant insights into the inherent structural properties of number theory. In particular, problems involving concatenations of well-established recursive sequences such as the Fibonacci and Lucas numbers present both rich theoretical challenges and notable computational interest. These types of problems are not only relevant within pure mathematics but also hold potential applications in various domains, including pattern recognition, the distribution of prime numbers, digital sequence analysis, and cryptographic systems. Moreover, research into the digit-based behavior of recursive sequences contributes to the development of novel techniques for uncovering intricate relationships among integers. Accordingly, the triple concatenation-based Diophantine equations examined in this study are expected to enrich the existing number theory literature and provide valuable perspectives for addressing broader mathematical inquiries.

2. Preliminaries

Baker's method is a technique commonly used in Diophantine equations and number theory, particularly for solving exponential Diophantine equations. The method is named after the mathematician A.D. Baker and is considered a powerful tool for finding solutions to certain types of equations in number theory. Now let's give some definitions and lemmas. The logarithmic height of an algebraic number η is defined as

$$h(\eta) = \frac{1}{d}\left(\log a_0 +_{i=1}^{d} \log\left(\max\left\{|\eta^{(i)}|, 1\right\}\right)\right), \tag{2. 5}$$

where a_0 is the leading coefficient of the minimal polynomial of η and d is the degree of η over $\mathbb{Q}$ and the $\left(\eta^{(i)}\right)_{1\leq i\leq d}$ are conjugates of η over $\mathbb{Q}$. Let η and γ be algebraic numbers. Then, the following properties hold (see [7]):

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2,$$
$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma),$$
$$h(\eta^m) = |m|h(\eta).$$

Now, we give the result of Matveev [6], which is one of our main tools.

Lemma 2.1. *Let $\gamma_1, \gamma_2, ..., \gamma_t$ be nonzero elements of a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, ..., b_t$ are rational integers and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

is nonzero. Then

$$\log|\Lambda| > -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1A_2...A_t,$$

where

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\} \text{ for all } i = 1, ..., t \text{ and } B \geq \max\{|b_1|, ..., |b_t|\}.$$

The following two lemmas are given in [5] and [10]. Let θ be a real number. Set $||\theta|| = \min\{|\theta - n| : n \in \mathbb{Z}\}$.

Lemma 2.2. *Let M be a positive integer and p/q be a convergent of the continued fraction of the irrational number γ, such that $q > 6M$. Let A, B, μ be some real numbers with $A > 0$ and $B > 1$. If $\epsilon := ||\mu q|| - M||\gamma q|| > 0$, then there exists no solution to the inequality*

$$0 < |n\gamma - r + \mu| < AB^{-w},$$

in positive integers n, r, and w with

$$n \leq M \text{ and } w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

Lemma 2.3. *Let $s, \Lambda \in \mathbb{R}$. If $0 < s < 1$ and $|\Lambda| < s$, then*

$$|\log(1 + \Lambda)| < \frac{-\log(1 - s)}{s} \cdot |\Lambda|$$

and

$$|\Lambda| < \frac{s}{1 - e^{-s}} \cdot \left|e^{\Lambda} - 1\right|.$$

Continued fractions play an important role in number theory and mathematical analysis. They are a powerful tool in defining irrational numbers and studying their relationships with rational numbers. They are also used in solving certain types of Diophantine equations and in numerical approximation problems. The following lemma which can be found in [9] will be used later.

Lemma 2.4. *Let x be a real number and $x = [a_0; a_1, a_2, a_3, ...]$ and $p, q \in \mathbb{Z}$. If*

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}$$

then $\frac{p}{q}$ is a convergent of the continued fraction of x. Furthermore, if M and n are non-negative integers such that $q_n > M$, then

$$\left| x - \frac{p}{q} \right| > \frac{1}{(b+2)q^2},$$

where $b := \max\{a_i : i = 0, 1, 2, ..., n\}$.

Now, we give the following two lemmas that determine the relationships among the variables in the equations (1. 3) and (1. 4).

Lemma 2.5. *Assume that the equation (1. 3) holds. Then, we have that the inequalities*

(a) $d < \frac{m_2+3}{4}$,
(b) $l < \frac{m_3+3}{4}$,
(c) $F_{m_2} < 10^d < 10F_{m_2}$,
(d) $F_{m_3} < 10^l < 10F_{m_3}$,
(e) $m_1 + m_2 + m_3 - 5 < n < m_1 + m_2 + m_3 + 9$,
(f) $n - m_3 \geq 6$.

Proof. (a) We write $d = \lfloor \log_{10} F_{m_2} \rfloor + 1$ since d is the number of digits of F_{m_2}. Then, we find

$$d = \lfloor \log_{10} F_{m_2} \rfloor + 1 \leq \log_{10} F_{m_2} + 1 \leq \log_{10} \alpha^{m_2-1} + 1 < \frac{m_2+3}{4}.$$

So, it follows that $d < \frac{m_2+3}{4}$. Here, we have used that $\log_{10} \alpha < 0.25$.

(b) It is proved similarly to proof of Lemma 2.5 (a).

(c) We note that $d = \lfloor \log_{10} F_{m_2} \rfloor + 1$ since d is the number of digits of F_{m_2}. Then, we obtain

$$F_{m_2} = 10^{\log_{10} F_{m_2}} < 10^d \leq 10^{\log_{10} F_{m_2} + 1} < 10F_{m_2}.$$

(d) It is proved similarly to proof of Lemma 2.5 (c).

(e) Using Lemma 2.5 (c),(d) and considering the inequality (1. 1), we can write

$$\alpha^{n-2} \leq F_n < 100F_{m_2}F_{m_3}F_{m_1} + 10F_{m_3}F_{m_2} + F_{m_3} < \alpha^{m_1+m_2+m_3+7}$$

and

$$\alpha^{n-1} \geq F_n > F_{m_3}F_{m_2}F_{m_1} + F_{m_3}F_{m_2} + F_{m_3} > F_{m_1}F_{m_2}F_{m_3} > \alpha^{m_1+m_2+m_3-6}.$$

Thus, we get $m_1 + m_2 + m_3 - 5 < n < m_1 + m_2 + m_3 + 9$.

(f) Taking into account the inequality

$$F_n > 10F_{m_3}F_{m_1} + F_{m_3}F_{m_2} + F_{m_3} \geq 12F_{m_3},$$

we obtain that $n - m_3 \geq 6$. $\square$

Lemma 2.6. *Assume that the equation (1. 4) holds. Then, we have that the inequalities*

(a) $d < \frac{m_2+6}{4}$,
(b) $l < \frac{m_3+6}{4}$,
(c) $L_{m_2} < 10^d < 10L_{m_2}$,
(d) $L_{m_3} < 10^l < 10L_{m_3}$,
(e) $m_1 + m_2 + m_3 - 2 < n < m_1 + m_2 + m_3 + 16$,
(f) $n - m_3 \geq 7$.

Proof. One can prove it similarly to the proof of the Lemma 2.5. $\square$

3. MAIN THEOREMS

Before stating the Theorem 3.1, let us note the following. If $m_1 = 0$ in the equation (1. 3) then we have $F_n = \overline{F_{m_2}F_{m_3}}$. This equation was also solved in [3]. Moreover, since the values of F_1 and F_2 are the same, we assume that $m_1 \geq 2$ in the equation (1. 3).

Theorem 3.1. *Let d and l be the number of digits of the F_{m_2} and F_{m_3}. The Diophantine equation*

$$F_n = \overline{F_{m_1}F_{m_2}F_{m_3}} = 10^{d+l}F_{m_1} + 10^l F_{m_2} + F_{m_3}$$

has no solution in non-negative integers (n, m_1, m_2, m_3) with $m_1 \geq 2$ and $m_2 \leq m_1$.

Proof. To initiate the proof of the theorem, we first derive a lower bound for n. F_n has at least three digits since $m_1 \geq 2$ and $m_2, m_3 \geq 0$. Thus, we can take $n \geq 12$ for $m_1 \geq 2$ and $m_2, m_3 \geq 0$. Using the equation (1. 3) and Binet formula for Fibonacci numbers, we get

$$\alpha^n - 10^{d+l}\alpha^{m_1} = \beta^n - 10^{d+l}\beta^{m_1} + \sqrt{5}10^l F_{m_2} + \sqrt{5}F_{m_3}.$$

Multiplying both sides of the above equality by $\frac{1}{10^{d+l}\alpha^{m_1}}$ and taking the absolute value, we obtain

$$\left|\frac{\alpha^{n-m_1}}{10^{d+l}} - 1\right| \leq \frac{|\beta|^n}{10^{d+l}\alpha^{m_1}} + \frac{1}{\alpha^{2m_1}} + \frac{\sqrt{5}F_{m_2}}{10^d\alpha^{m_1}} + \frac{\sqrt{5}F_{m_3}}{10^{d+l}\alpha^{m_1}} < \frac{2.85}{\alpha^{m_1}}. \tag{3. 6}$$

Here, we consider $n \geq 12, m_1 \geq 2$, $d, l \geq 1$, and Lemma 2.5(c),(d). Now, let us apply Lemma 2.1 with $(\gamma_1, b_1) := (\alpha, n - m_1)$ and $(\gamma_2, b_2) := (10, -(d + l))$. Note that $D = 2$. Let $\Lambda_1 := \frac{\alpha^{n-m_1}}{10^{d+l}} - 1$. Assume that $\Lambda_1 = 0$. Then $\alpha^{n-m_1} = 10^{d+l}$, which is not possible. Because α^{n-m_1} is not rational for $n - m_1 > 0$. Therefore, $\Lambda_1 \neq 0$. In addition, since $h(\gamma_1) = h(\alpha) = \frac{\log\alpha}{2}$ and $h(\gamma_2) = h(10) = \log 10$, we can say $A_1 := 0.49$ and $A_2 := 4.61$. From Lemma 2.5(a),(b),(e), we get

$$d + l < \frac{m_2 + m_3 + 6}{4} < \frac{n - m_1 + 11}{4} < n - m_1 + 3 \leq n + 1$$

for $m_1 \geq 2$. By recalling $B \geq \max\{|n - m_1|, |-(d + l)|\}$ and using the above inequality, we can choose $B := n + 1$. Thence, the inequality (3. 6) and Lemma 2.1, imply that

$$2.85 \cdot \alpha^{-m_1} > |\Lambda_1| > \exp\left(T \cdot (1 + \log(n + 1))\right),$$

where $T = -1.4 \cdot 30^5 \cdot 2^{4.5} \cdot 2^2 \cdot (1 + \log 2) \cdot 0.49 \cdot 4.61$. When the logarithm of the above inequality in base e is taken from both sides and multiplied by -1, this inequality covert to

$$m_1 \log\alpha < 1.18 \cdot 10^{10} \cdot (1 + \log(n + 1)) + \log(2.85). \tag{3. 7}$$

We rearrange the equation (1. 3) as

$$\alpha^n - \sqrt{5}10^{d+l}F_{m_1} - \sqrt{5}10^l F_{m_2} - \alpha^{m_3} = \beta^n - \beta^{m_3}$$

i.e.,

$$\alpha^n(1-\alpha^{m_3-n}) - \sqrt{5}10^l(10^d F_{m_1} + F_{m_2}) = \beta^n - \beta^{m_3}.$$

After performing the necessary calculations, we obtain

$$\begin{aligned}\left|1 - \frac{\sqrt{5}10^l(10^d F_{m_1} + F_{m_2})}{\alpha^n(1-\alpha^{m_3-n})}\right| &\leq \frac{1}{\alpha^n}\left|\frac{1}{1-\alpha^{m_3-n}}\right| \cdot \left(\frac{1}{\alpha^n} + \frac{1}{\alpha^{m_3}}\right) \\ &\leq \frac{1.07}{\alpha^n} \qquad (3.\ 8)\end{aligned}$$

for $n - m_3 \geq 6, m_3 \geq 0$, and $n \geq 12$. To applying Lemma 2.1, we take

$$(\gamma_1, b_1) := (\alpha, -n), (\gamma_2, b_2) := (10, l), (\gamma_3, b_3) := \left(\frac{\sqrt{5}(10^d F_{m_1} + F_{m_2})}{1-\alpha^{m_3-n}}, 1\right).$$

The number field containing γ_1, γ_2, and γ_3 is $K = \mathbb{Q}(\sqrt{5})$ and so $D = 2$. We show that

$$\Lambda_2 := 1 - \frac{\sqrt{5}10^l(10^d F_{m_1} + F_{m_2})}{\alpha^n(1-\alpha^{m_3-n})}$$

is nonzero. Suppose that $\Lambda_2 = 0$. Then

$$\alpha^n - \alpha^{m_3} = \sqrt{5}10^l(10^d F_{m_1} + F_{m_2}).$$

Conjugating in $\mathbb{Q}(\sqrt{5})$ gives us

$$\beta^n - \beta^{m_3} = -\sqrt{5}10^l(10^d F_{m_1} + F_{m_2}).$$

These imply that $L_n = L_{m_3}$, which leads to a contradiction. From Lemma 2.5(a),(e), we get

$$\begin{aligned}h(\gamma_3) &= h\left(\frac{\sqrt{5}(10^d F_{m_1} + F_{m_2})}{1-\alpha^{m_3-n}}\right) \\ &\leq h(\sqrt{5}) + d \cdot h(10) + h(F_{m_1}) + h(F_{m_2}) + (n-m_3)h(\alpha) + 2\log 2 \\ &< \frac{\log 80}{2} + \left(\frac{m_2+3}{4}\right)\log 10 + 2(m_1-1)\frac{\log\alpha}{2} + (n-m_3)\frac{\log\alpha}{2} \\ &< \frac{\log 80}{2} + 5\left(\frac{m_1+3}{4}\right)\log\alpha + (m_1-1)\log\alpha + (2m_1+9)\frac{\log\alpha}{2} \\ &< 5.68 + 3.25 m_1 \log\alpha,\end{aligned}$$

where we have used that $m_2 \leq m_1$. So, we can choose $A_1 := 0.49$, $A_2 := 4.61$, and $A_3 := 11.36 + 6.5m_1\log\alpha$. Lemma 2.5(b),(f) tell us

$$l < \frac{m_3+3}{4} \leq \frac{n-6+3}{4} < n$$

and so we can say $B := n$. In that case, using (3. 8) and Lemma 2.1, we obtain

$$1.07 \cdot \alpha^{-n} > |\Lambda_2| > \exp\left(R \cdot (1+\log n)(11.36 + 6.5m_1\log\alpha)\right)$$

i.e.,

$$n\log\alpha-\log 1.07<2.2\cdot 10^{12}\cdot(1+\log n)\left(11.36+6.5m_1\log\alpha\right), \tag{3. 9}$$

where $R=-1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2\cdot(1+\log 2)\cdot 0.49\cdot 4.61$. Using the inequalities (3. 7) and (3. 9), we get $n<1.42\cdot 10^{27}$. Now, we only need to lower the bound. Let

$$z_1:=(n-m_1)\log\alpha-(d+l)\log 10$$

and $\Lambda_1:=e^{z_1}-1$. For $m_1\geq 3$, we have

$$|\Lambda_1|=|e^{z_1}-1|<2.85\cdot\alpha^{-m_1}<0.7$$

from (3. 6). To applying Lemma 2.3, we can choose $s:=0.7$. So,

$$|(n-m_1)\log\alpha-(d+l)\log 10|<-\frac{\log 0.3}{0.7}\cdot\frac{2.85}{\alpha^{m_1}}<4.91\cdot\alpha^{-m_1}$$

i.e.,

$$0<\left|\frac{\log\alpha}{\log 10}-\frac{d+l}{(n-m_1)}\right|<\frac{2.14}{(n-m_1)\cdot\alpha^{m_1}}. \tag{3. 10}$$

Suppose that $m_1\geq 142$. Then, we can write

$$\frac{\alpha^{m_1}}{4.28}>1.1\cdot 10^{29}>n>n-m_1$$

and so we have

$$\left|\frac{\log\alpha}{\log 10}-\frac{d+l}{(n-m_1)}\right|<\frac{2.14}{(n-m_1)\cdot\alpha^{m_1}}<\frac{1}{2(n-m_1)^2}.$$

It follows from Lemma 2.4 that the rational number $\frac{d+l}{n-m_1}$ is a convergent to $\gamma=\frac{\log\alpha}{\log 10}$. Now let $[a_0;a_1,a_2,...]$ be the continued fraction expansion of γ and let $\dfrac{p_r}{q_r}$ be its r-th convergent and $\frac{d+l}{n-m_1}=\dfrac{p_t}{q_t}$ for some t. Then we have $q_{54}>2\cdot 10^{27}>n>n-m_1$. Thus, $t\in\{0,1,2,...,53\}$. Furthermore, $b=\max\{a_i|i=0,1,2,...,53\}=106$. By Lemma 2.4, we get

$$\left|\gamma-\frac{p_t}{q_t}\right|>\frac{1}{(b+2)(n-m_1)^2}=\frac{1}{108\cdot(n-m_1)^2}.$$

Thus, we obtain

$$\frac{2.14}{\alpha^{142}}\geq\frac{2.14}{\alpha^{m_1}}>\frac{1}{108\cdot(n-m_1)}>\frac{1}{216\times 10^{27}},$$

which is a contradiction. That is, $m_1\leq 141$. Using the inequalities (3. 9) and $m_1\leq 141$, we get $n<8.27\cdot 10^{16}$. When we reduce again the upper bound of n, we find that $m_1\leq 97$ and $n<5.7\cdot 10^{16}$. Put

$$z_2:=l\log 10-n\log\alpha+\log\left(\frac{\sqrt{5}(10^dF_{m_1}+F_{m_2})}{1-\alpha^{m_3-n}}\right).$$

From (3. 8), it is seen that

$$|\Lambda_2|=|1-e^{z_2}|<(1.07)\cdot\alpha^{-n}<0.004$$

for $n \geq 12$. According to Lemma 2.3, we can choose $s := 0.004$, and so we get

$$\left| l \log 10 - n \log \alpha + \log \left(\frac{\sqrt{5}(10^d F_{m_1} + F_{m_2})}{1 - \alpha^{m_3 - n}} \right) \right| < -\frac{\log(0.996)}{0.004} \cdot \frac{1.07}{\alpha^n} < \frac{1.08}{\alpha^n}$$

i.e.,

$$0 < \left| l \frac{\log 10}{\log \alpha} - n + \frac{\log \left(\frac{\sqrt{5}(10^d F_{m_1} + F_{m_2})}{1 - \alpha^{m_3 - n}} \right)}{\log \alpha} \right| < 2.25 \cdot \alpha^{-n}. \tag{3. 11}$$

Now, we can use Lemma 2.2 with

$$\gamma := \frac{\log 10}{\log \alpha}, \mu := \frac{\log \left(\frac{\sqrt{5}(10^d F_{m_1} + F_{m_2})}{1 - \alpha^{m_3 - n}} \right)}{\log \alpha}, A := 2.25,\ B := \alpha, w := n$$

and $l < n < M := 5.7 \cdot 10^{16}$. Let p/q be a convergent of the continued fraction of the γ. We find that $q_{45} > 6M$ for γ and compute

$$\epsilon := ||\mu q_{45}|| - M||\gamma q_{45}|| > 4.47 \cdot 10^{-8}$$

for $2 \leq m_2 \leq m_1 \leq 97, 6 \leq n - m_3 < m_1 + m_2 + 9$, and $1 \leq d < \frac{m_2+3}{4}$. A computer program tells us that the value of $\frac{\log(Aq_{45}/\epsilon)}{\log B}$ is less than 142.27. According to Lemma 2.2 we can say $n \leq 142$ and so $m_3 \leq 135$. Finally, we find that there is no Fibonacci number that is a concatenation of three Fibonacci numbers for $2 \leq m_2 \leq m_1 \leq 97, 0 \leq m_3 \leq 135$, and $12 \leq n \leq 142$. □

Theorem 3.2. *Let d and l be the number of digits of the L_{m_2} and L_{m_3}. The Diophantine equation*

$$F_n = \overline{L_{m_1} L_{m_2} L_{m_3}} = 10^{d+l} L_{m_1} + 10^l L_{m_2} + L_{m_3}$$

has no solution in non-negative integers (n, m_1, m_2, m_3) with $m_2 \leq m_1$.

Proof. Since $m_1, m_2, m_3 \geq 0$, F_n has at least three digits and so we can take $n \geq 12$. From the equation (1. 4), we can write

$$\alpha^n - \sqrt{5}10^{d+l}\alpha^{m_1} = \beta^n + \sqrt{5}10^{d+l}\beta^{m_1} + \sqrt{5}10^l L_{m_2} + \sqrt{5}L_{m_3}.$$

After appropriate rearrangement, we obtain

$$\left| \frac{\alpha^{n-m_1}}{\sqrt{5}10^{d+l}} - 1 \right| \leq \frac{|\beta|^n}{\sqrt{5}10^{d+l}\alpha^{m_1}} + \frac{1}{\alpha^{2m_1}} + \frac{L_{m_2}}{10^d \alpha^{m_1}} + \frac{L_{m_3}}{10^{d+l}\alpha^{m_1}} < \frac{2.11}{\alpha^{m_1}}. \tag{3. 12}$$

To apply Lemma 2.1 we choose

$$(\gamma_1, b_1) := (\alpha, n - m_1), (\gamma_2, b_2) := (10, -(d + l)), (\gamma_3, b_3) := \left(\sqrt{5}, -1\right).$$

Here, $D = 2$ and $\Lambda_1 := \frac{\alpha^{n-m_1}}{\sqrt{5}10^{d+l}} \neq 0$. Also, we can say $A_1 := 0.49$, $A_2 := 4.61$, and $A_3 := 1.61$, since $h(\gamma_1) = \frac{\log \alpha}{2}, h(\gamma_2) = \log 10$, and $h(\gamma_3) = \frac{\log 5}{2}$. We choose $B := n + 4$. Because Lemma 2.6(a),(b),(e) show that

$$d + l < \frac{m_2 + m_3 + 12}{4} < \frac{n - m_1 + 14}{4} < n - m_1 + 4 \leq n + 4$$

for $m_1 \geq 0$. Thus, considering (3. 12) and Lemma 2.1, we obtain

$$2.11 \cdot \alpha^{-m_1} > |\Lambda_1| > \exp\left(R \cdot (1+\log(n+4)) \cdot 1.61\right),$$

where $R = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1+\log 2) \cdot 0.49 \cdot 4.61$. By a straightforward calculation, it follows that

$$m_1 \log \alpha - \log 2.11 < 3.53 \cdot 10^{12} \cdot (1+\log(n+4). \tag{3. 13}$$

We rearrange the equation (1. 4) as

$$\alpha^n(1-\sqrt{5}\alpha^{m_3-n}) - \sqrt{5}10^l(10^d L_{m_1} + L_{m_2}) = \beta^n + \sqrt{5}\beta^{m_3}$$

and so

$$\begin{aligned} \left|1 - \frac{\sqrt{5}10^l(10^d L_{m_1} + L_{m_2})}{\alpha^n(1-\sqrt{5}\alpha^{m_3-n})}\right| &\leq \frac{1}{\alpha^n}\left|\frac{1}{1-\sqrt{5}\alpha^{m_3-n}}\right| \cdot \left(\frac{1}{\alpha^n} + \frac{\sqrt{5}}{\alpha^{m_3}}\right) \\ &\leq \frac{2.43}{\alpha^n} \end{aligned} \tag{3. 14}$$

for $n - m_3 \geq 7, m_3 \geq 0$ and $n \geq 12$. We take the notation of Lemma 2.1, as

$$(\gamma_1, b_1) := (\alpha, -n), (\gamma_2, b_2) := (10, l), (\gamma_3, b_3) := \left(\frac{\sqrt{5} \cdot (10^d L_{m_1} + L_{m_2})}{1-\sqrt{5}\alpha^{m_3-n}}, 1\right)$$

and

$$\Lambda_2 := 1 - \frac{\sqrt{5}10^l(10^d L_{m_1} + L_{m_2})}{\alpha^n(1-\sqrt{5}\alpha^{m_3-n})}.$$

It is easy to show that Λ_2 is nonzero. The numbers $\gamma_1, \gamma_2, \gamma_3 \in K = \mathbb{Q}(\sqrt{5})$ and so $D = 2$. Furthermore, we find that

$$l < \frac{m_3+6}{4} < \frac{n-1}{4} < n,$$

and

$$\begin{aligned} h(\gamma_3) &= h\left(\frac{\sqrt{5} \cdot (10^d L_{m_1} + L_{m_2})}{1-\sqrt{5}\alpha^{m_3-n}}\right) \\ &\leq 2h(\sqrt{5}) + d \cdot h(10) + h\left(L_{m_1}\right) + h\left(L_{m_2}\right) + (n-m_3)\, h\left(\alpha\right) + 4\log 2 \\ &< \log 80 + \left(\frac{m_2+6}{4}\right)\log 10 + 2m_1\frac{\log\alpha}{2} + (n-m_3)\frac{\log\alpha}{2} \\ &< \log 80 + 5\left(\frac{m_1+6}{4}\right)\log\alpha + m_1\log\alpha + (2m_1+16)\frac{\log\alpha}{2} \\ &< 11.85 + 3.25 m_1 \log\alpha, \end{aligned}$$

by Lemma 2.6(a),(b),(e),(f) with $m_2 \leq m_1$. So, we can choose $B := n, A_1 := 0.49, A_2 := 4.61$, and $A_3 := 23.7 + 6.5m_1\log\alpha$. By using (3. 14) and Lemma 2.1, we obtain

$$2.43 \cdot \alpha^{-n} > |\Lambda_2| > \exp\left(T \cdot (1+\log n)\left(23.7 + 6.5m_1\log\alpha\right)\right)$$

i.e.,

$$n\log\alpha - \log(2.43) < 2.2 \cdot 10^{12} \cdot (1+\log n)\left(23.7 + 6.5m_1\log\alpha\right). \tag{3. 15}$$

The inequalities (3. 13) and (3. 15), follow that $n < 5.06 \cdot 10^{29}$. Let

$$z_1 := (n - m_1)\log\alpha - (d + l)\log 10 - \log\sqrt{5}.$$

From (3. 12), we can write

$$|\Lambda_1| = |e^{z_1} - 1| < \frac{2.11}{\alpha^{m_1}} < 0.81$$

for $m_1 \geq 2$. Choosing $s := 0.81$, we get

$$\begin{aligned} |\,(n - m_1)\log\alpha - (d + l)\log 10 - \log\sqrt{5}| &< -\frac{\log(0.19)}{0.81} \cdot \frac{2.11}{\alpha^{m_1}} \\ &< \frac{4.33}{\alpha^{m_1}} \end{aligned}$$

i.e.,

$$0 < \left|(n - m_1)\frac{\log\alpha}{\log 10} - (d + l) - \frac{\log\sqrt{5}}{\log 10}\right| < 1.89 \cdot \alpha^{-m_1} \tag{3. 16}$$

by Lemma 2.3. Now, we can use Lemma 2.2. Put

$$\gamma := \frac{\log\alpha}{\log 10} \notin \mathbb{Q}, \mu := -\frac{\log\sqrt{5}}{\log 10}, A := 1.89, B := \alpha, w := m_1$$

and $n - m_1 < n < M := 5.06 \cdot 10^{29}$. Considering p/q be a convergent of the continued fraction of the γ, we compute that $q_{61} > 6M$ and $\epsilon > 0.02$. Hence, the inequality (3. 16) has a solution for

$$m_1 < \frac{\log(Aq_{61}/\epsilon)}{\log B} < 157.92.$$

So, $m_1 \leq 157$. If we consider $m_1 \leq 157$ and the inequality (3. 15) together, we obtain $n < 9.44 \cdot 10^{16}$. Taking $M := 9.44 \cdot 10^{16}$ and running once more the reduction cycle on the inequality (3. 16), we have that $m_1 \leq 93$ and $n < 5.7 \cdot 10^{16}$. Now, let

$$z_2 := l\log 10 - n\log\alpha + \log\left(\frac{\sqrt{5}(10^d L_{m_1} + L_{m_2})}{1 - \sqrt{5}\alpha^{m_3 - n}}\right).$$

From (3. 14), it is seen that

$$|\Lambda_2| = |1 - e^{z_2}| < (2.43) \cdot \alpha^{-n} < 0.008$$

for $n \geq 12$. In that case, using Lemma 2.3, we get

$$\begin{aligned} \left|l\log 10 - n\log\alpha + \log\left(\frac{\sqrt{5}(10^d L_{m_1} + L_{m_2})}{1 - \sqrt{5}\alpha^{m_3 - n}}\right)\right| &< -\frac{\log(0.992)}{0.008} \cdot \frac{2.43}{\alpha^n} \\ &< 2.44 \cdot \alpha^{-n}, \end{aligned}$$

i.e.,

$$0 < \left|l\frac{\log 10}{\log\alpha} - n + \frac{\log\left(\frac{\sqrt{5}(10^d L_{m_1} + L_{m_2})}{1 - \sqrt{5}\alpha^{m_3 - n}}\right)}{\log\alpha}\right| < 5.08 \cdot \alpha^{-n}. \tag{3. 17}$$

Put

$$\gamma := \frac{\log 10}{\log \alpha}, \mu := \frac{\log\left(\frac{\sqrt{5}(10^d L_{m_1} + L_{m_2})}{1-\sqrt{5}\alpha^{m_3-n}}\right)}{\log \alpha}, A := 5.08,\ B := \alpha, w := n,$$

and $l < n < M := 5.7 \cdot 10^{16}$. We find that $q_{47} > 6M$ and

$$\epsilon := ||\mu q_{47}|| - M||\gamma q_{47}|| > 6.12 \cdot 10^{12}$$

for $0 \leq m_2 \leq m_1 \leq 93, 7 \leq n - m_3 < m_1 + m_2 + 16$, and $1 \leq d < \frac{m_2+6}{4}$. Thus, according to Lemma 2.2 we see that

$$n \leq \frac{\log(Aq_{47}/\epsilon)}{\log B} < 142.58$$

Therefore, $n \leq 142$. A computer program gives us that there is no Fibonacci number that is concatenation of three Lucas numbers for $0 \leq m_2 \leq m_1 \leq 93, 0 \leq m_3 \leq 135$, and $12 \leq n \leq 142$. □

4. Conclusion and Conjecture

In this study, we have demonstrated that the Diophantine equations (1. 3) and (1. 4), which involve the concatenation of three Fibonacci and Lucas numbers respectively, do not admit any solutions under the constraint $m_2 \leq m_1$. This result builds on earlier works that focused on binary concatenations, extending the scope of investigation to ternary cases. The addition of this third term introduces new structural complexity, offering a more comprehensive perspective on how recurrence sequences behave under digit-based transformations.

Under the imposed constraint, exhaustive computational checks within the range $2 \leq m_1 \leq 100, 0 \leq m_2, m_3 \leq 100$ and $12 \leq n \leq 142$ yielded only a limited number of solutions. Specifically, for equation (1. 3), the only solution found is

$$(F_n, F_{m_1}, F_{m_2}, F_{m_3}) = (233, 2, 3, 3)$$

while for equation (1. 4), a small set of solutions was obtained

$$(F_n, L_{m_1}, L_{m_2}, L_{m_3}) \in \left\{ \begin{array}{c} (144, 1, 4, 4), (233, 2, 3, 3), \\ (377, 3, 7, 7), (4181, 4, 18, 1) \end{array} \right\}.$$

These sparse results highlight the rarity and structural rigidity of such concatenation-based representations within linear recurrence sequences.

However, when the condition $m_2 \leq m_1$ is removed, additional isolated solutions emerge. Despite this, a complete characterization of all such solutions remains elusive. The lack of a general theoretical framework for analyzing the unrestricted forms of equations (1. 3) and (1. 4) points to a deeper open problem in the interplay between recurrence sequences and digital concatenation. Developing such a framework could be an important direction for future research.

Furthermore, this line of inquiry can be naturally extended to other well-known recursive sequences, such as the Padovan, Perrin, and Tribonacci numbers. Investigating whether similar concatenation-based Diophantine equations exhibit solution patterns in those sequences could uncover new structural analogies and deepen our understanding of recurrence relations and digital representations. We leave the full resolution of these broader problems as an open question, inviting further exploration in both theoretical and computational number theory.

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