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Analytical Solution of Fractional Order Diffusion Equations Using Iterative Laplace Transform Method

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Abstract. In this present article, by using the Iterative Laplace Transform Method (ILTM), the diffusion equation of fractional order is solved. The ILTM, which works as a combination of two methods, the iterative method and the other is the Laplace transform method, is applied to several diffusion equations to obtain analytical solutions. The proposed method gives the closed-form of series solutions in terms of the Mittag-Leffler function, which is a queen of functions in fractional calculus. The main aim of this work is to present a simple but reliable algorithm for the solution of diffusion equations of the multi-dimensional type, which clearly describes the materials of density dynamics in the diffusion process. The results obtained by using the ILTM approach indicate that this approach is attractive computationally and implemented easily. Due to its straightforward approach and comfortable way of solving problems, the ILTM can be utilized to solve nonlinear fractional problems in various applied and engineering sciences.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Iterative Laplace Transform, Mittag-Leffler Function, Exact solutions, Diffusion Equation, Multi-dimensional.

1. Introduction

The partial differential equation (PDE), which describes the density dynamic in material that undergoes diffusion, is known as the diffusion equation. The processes exhibition of the behaviour of diffusion-like are also described by using the diffusion equation; for instance, in population genetics, the diffusion equation is

$$\frac{\partial\psi(r,t)}{\partial t} = \Delta.(A(\psi(r,t),r)\Delta\psi(r,t)),$$

where $\psi(r,t)$ represents density of diffusion materials, $r=(x,y,z)$ is position and t is time. $A(\psi(r,t),r)$ represents the diffusion coefficient for the density A at position r. In literature, several approaches have been implemented to solve these types of equations; the approaches include the numerical methods [19],

variational iteration method (VIM) [1], homotopy perturbation method (HPM) [3]. In the investigation of complex systems, the idea of entropy and fractional calculus (FC) is attractive and most prevalent. In this era, FC is making progress in most branches of science, especially in biology and engineering sciences.
The diffusion equation is not a new idea. Its signs can be found in the past as in 1885. Adolf Fick gave Fick's law of diffusion; finally, after 1885, Fick's second law became known as the famous diffusion equation. The process which we are calling diffusion is a process of mesh movement of molecules or atoms of greater chemical potential or highest concentration to an area of the region of smaller chemical potential or lowest concentration. The mathematicians and scientists gave a generalization of the wave and diffusion equations in the physical processes such as diffusion wave hybrid, slow and classical diffusion, and classical wave equation in [7]. Diffusion equation has several applications, such as cosmology, biochemistry, acoustics, electromagnetism, filtration, phase transition, electrochemistry, dynamics in biological groupings, and microbiology [8]. In the chemical potential of diffusing, diffusion can be found by a gradient where the gradient is a change or a variation in something, particularly in a number or value, such as pressure, concentration, and temperature, with a change in several variables. The temperature gradient is a variation in temperature terminated with a distance, a concentration gradient is a variation in concentration terminated with a distance, and a pressure gradient is a variation in pressure terminated with a distance. In the industries, scientists and researchers have been struggling to get high-effectiveness outputs [13]. In several engineering systems, the entropy generation has different causes. It can be seen in the thermal systems that the initial starting of entropy generation is heat transfer, mass transfer, dissipation of viscous, heat coupling, conduction of electronics and chemicals reacting [4, 5]. In the applications mentioned above, the researchers and mathematicians utilized several methods to solve a bunch of differential equations of ordinary as well as arbitrary (fractional) order, especially the fractional partial differential equations (FPDEs). The FPDEs play an important role in modeling real-life phenomena more reliably and accurately than an integer-order partial differential equation (PDE). In applied and engineering sciences, the nonlinear FPDEs play a vital role in describing different phenomena. The nonlinear FPDEs are the best way to be utilized in several fields, such as chemistry, physics, material science, thermodynamics, chemical kinetics, and medicines [21, 23, 11]. To solve the FPDEs, several methods can be seen in the literature such as VIM [14], modified Adomian decomposition method (MADM) [2], Optimal homotopy asymptotic method and homotopy perturbation method (OHAM, HPM) [10, 16].
In 2006, the iterative method (IM) was introduced to solve the functional equations [21]. The IM can be used to solve differential equations (DEs) of both integer and fractional orders. In this research article, we used IM and Laplace transform method (LTM) as a combination of ILTM to approximate the solutions of FPDEs. In this present work, we will apply ILTM to solve the diffusion equations of the following forms.
(1) The 1D diffusion equation of fractional order has the form:

$$\frac{\partial^k \psi}{\partial t^k} = \frac{\partial^2 \psi}{\partial \xi^2} - \frac{\partial \psi}{\partial \xi} + \psi \frac{\partial^2 \psi}{\partial \xi^2} - \psi^2 + \psi, 0 < k \leq 1, t \geq 0$$

with initial condition

$$\psi(\xi, 0) = \mu(\xi)$$

(2) The 2D diffusion equation of fractional order has the form:

$$\frac{\partial^k \psi}{\partial t^k} = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial v^2}, 0 < k \leq 1, t \geq 0$$

with initial condition

$$\psi(\xi, v, 0) = \mu(\xi, v)$$

(3) The 3D diffusion equation of fractional order has the form:

$$\frac{\partial^k \psi}{\partial t^k} = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial v^2} + \frac{\partial^2 \psi}{\partial u^2}, 0 < k \leq 1, t \geq 0$$

with initial condition

$$\psi(\xi, v, u, 0) = \mu(\xi, v, u)$$

2. Preliminaries

In this section, we will present some useful definitions which will help through this article.

Definition 2.1. *Caputo type derivative of fractional order is given below:*[15]

$$D_\theta^k \psi(\xi, \theta) = \frac{1}{\Gamma(p-k)} \int_0^\theta (\theta - u)^{p-k-1} \psi^{(p)}(\xi, u) du, p-1 < k \leq p, p \in \{1, 2, 3, \ldots\} \tag{2. 1}$$

Definition 2.2. *Riemann-Liouville(RL) type integral of fractional order is given below:*[15]

$$I_\theta^k \psi(\xi, \theta) = \frac{1}{\Gamma(k)} \int_0^\theta (\theta - u)^{k-1} \psi(\xi, u) du, u > 0, p-1 < k \leq p, p \in \{1, 2, 3, \ldots\} \tag{2. 2}$$

where I_θ^k is the fractional integral operator.

Definition 2.3. *Laplace Transform(LT):*[17, 22, 6, 20]
The LT of $\psi(\xi), \xi > 0$ is

$$LT[\psi(\xi)] = \overline{\psi}(s) = \int_0^\infty e^{-su} \psi(u) du \tag{2. 3}$$

Definition 2.4. *LT of $D_\theta^k \psi(\xi, \theta)$* [18]

$$LT[D_\theta^k \psi(\xi, \theta)] = LT[\psi(\xi, \theta)] - \sum_{\alpha=0}^{p-1} \psi^\alpha(\xi, 0) s^{k-\alpha-1}, p-1 < k \leq p, p \in \{1, 2, 3, \ldots\} \tag{2. 4}$$

where $\psi^\alpha(\xi, 0)$ is α-th order derivative at $\theta = 0$ of $\psi(\xi, \theta)$.

Definition 2.5. *Mittag-Leffler Function* [9]

$$E_a(w) = \sum_{p=0}^\infty \frac{w^p}{\Gamma(ap+1)}, a \in \boldsymbol{C}, Re(a) > 0 \tag{2. 5}$$

The generalized form of eq. 2. 5 in two parameters is given below:

$$E_{a,b}(w) = \sum_{p=0}^\infty \frac{w^p}{\Gamma(ap+b)}, a, b \in \boldsymbol{C}, Re(a) > 0, Re(b) > 0 \tag{2. 6}$$

3. Fundamental Concept of ILTM

The fundamental concept of this method is illustrated in this section [18]. Let us consider a general form of nonlinear, non-homogeneous fractional diffusion equation along with the initial condition:

$$D_\xi^k \psi(\xi,\theta) + A_1(\psi(\xi,\theta)) + A_2(\psi(\xi,\theta)) = B(\xi,\theta), p-1 < k \leq p, p \in \{1,2,3,\ldots\} \qquad (3.\ 7)$$

$$\psi^\alpha(\xi,0) = h_\alpha(\xi), \alpha = 0,1,2,\ldots,p-1$$

where $D_\xi^k \psi(\xi,\theta)$ represents Caputo type fractional order derivative of $\psi(\xi,\theta)$, A_1 represents a differential operator or linear type, A_2 represents a differential operator of nonlinear type and $B(\xi,\theta)$ represents a function of ξ and θ alone.

Apply LT on eq. 3. 7 , we obtain

$$LT[D_\xi^k \psi(\xi,\theta)] + LT[A_1(\psi(\xi,\theta)) + A_2(\psi(\xi,\theta))] = LT[B(\xi,\theta)] \qquad (3.\ 8)$$

using def.2.4 in eq. 3. 8 , we obtain

$$LT[\psi(\xi,\theta)] = \frac{1}{s^k}\sum_{\alpha=0}^{p-1}\psi^\alpha(\xi,0)s^{k-\alpha-1} + \frac{1}{s^k}LT[B(\xi,\theta)] - \frac{1}{s^k}LT[A_1(\psi(\xi,\theta)) + A_2(\psi(\xi,\theta))]$$

By applying the inverse of LT, we obtain

$$\psi(\xi,\theta) = LT^{-1}[\frac{1}{s^k}(\sum_{\alpha=0}^{p-1}\psi^\alpha(\xi,0)s^{k-\alpha-1} + LT[B(\xi,\theta)])] - LT^{-1}[\frac{1}{s^k}LT[A_1(\psi(\xi,\theta)) + A_2(\psi(\xi,\theta))]] \qquad (3.\ 9)$$

Now utilizing IM,

$$\psi(\xi,\theta) = \sum_{j=0}^{\infty}\psi_j(\xi,\theta) \qquad (3.\ 10)$$

Using the fact that A_1 is linear and A_2 is the nonlinear differential operator.

$$A_1(\sum_{j=0}^{\infty}\psi_j(\xi,\theta)) = \sum_{j=0}^{\infty}A_1(\psi_j(\xi,\theta)) \qquad (3.\ 11)$$

and a decomposition of A_2

$$A_2(\sum_{j=0}^{\infty}\psi_j(\xi,\theta)) = A_2(\psi_0(\xi,\theta)) + \sum_{j=0}^{\infty}\{A_2(\sum_{\alpha=0}^{j}\psi_\alpha(\xi,\theta)) - A_2(\sum_{\alpha=0}^{j-1}\psi_\alpha(\xi,\theta))\} \qquad (3.\ 12)$$

By putting eqs. 3. 10 , 3. 11 and 3. 12 in eq. 3. 9 , we obtain

$$\sum_{j=0}^{\infty}\psi_j(\xi,\theta) = LT^{-1}[\frac{1}{s^k}(\sum_{\alpha=0}^{p-1}\psi^\alpha(\xi,0)s^{k-\alpha-1} + LT[B(\xi,\theta)])] - LT^{-1}[\frac{1}{s^k}LT[\sum_{j=0}^{\infty}A_1(\psi_j(\xi,\theta)) +$$

$$A_2(\psi_0(\xi,\theta)) + \sum_{j=0}^{\infty}\{A_2(\sum_{\alpha=0}^{j}\psi_\alpha(\xi,\theta)) - A_2(\sum_{\alpha=0}^{j-1}\psi_\alpha(\xi,\theta))\}]] \qquad (3.\ 13)$$

The recurrence relation is defined as,

$$\psi_0(\xi,\theta) = LT^{-1}[\frac{1}{s^k}(\sum_{\alpha=0}^{p-1}\psi^\alpha(\xi,0)s^{k-\alpha-1} + LT[B(\xi,\theta)])],$$

$$\psi_0(\xi,\theta) = -LT^{-1}[\frac{1}{s^k}LT[A_1(\psi_0(\xi,\theta)) + A_2(\psi_0(\xi,\theta))]],$$
$$\vdots$$
$$\psi_{p+1}(\xi,\theta) = -LT^{-1}[\frac{1}{s^k}LT[A_1(\psi_p(\xi,\theta)) - \{A_2(\sum_{\alpha=0}^{p}\psi_\alpha(\xi,\theta)) - A_2(\sum_{\alpha=0}^{p-1}\psi_\alpha(\xi,\theta))\}]], p \geq 1$$

The approximate solution will be

$$\psi(\xi,\theta) \approxeq \psi_0(\xi,\theta) + \psi_1(\xi,\theta) + \psi_2(\xi,\theta) + \psi_3(\xi,\theta) + \psi_4(\xi,\theta) + \ldots + \psi_p(\xi,\theta), p = 1,2,3,\ldots \quad (3.\ 14)$$

4. Applications

In this section, we will utilize ILTM on fractional diffusion equations.

Example 4.1. *The 1D diffusion equation of fractional order:*[12]

$$\frac{\partial^k\psi}{\partial t^k} = \frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial\psi}{\partial\xi} + \psi\frac{\partial^2\psi}{\partial\xi^2} - \psi^2 + \psi, 0 < k \leq 1, t \geq 0 \quad (4.\ 15)$$

with initial condition

$$\psi(\xi,0) = e^{\xi}$$

Apply LT to eq. 4. 15 , we obtain

$$LT[\psi(\xi,t)] = \frac{e^{\xi}}{s} + \frac{1}{s^k}[LT(\frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial\psi}{\partial\xi} + \psi\frac{\partial^2\psi}{\partial\xi^2} - \psi^2 + \psi)] \quad (4.\ 16)$$

Apply inverse of LT to eq. 4. 16 , we obtain

$$\psi(\xi,t) = e^{\xi} + LT^{-1}[\frac{1}{s^k}[LT(\frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial\psi}{\partial\xi} + \psi\frac{\partial^2\psi}{\partial\xi^2} - \psi^2 + \psi)]]$$

Utilizing IM,

$$\psi_0(\xi,t) = e^{\xi},$$
$$\psi_1(\xi,t) = LT^{-1}[\frac{1}{s^k}[LT(\frac{\partial^2\psi_0}{\partial\xi^2} - \frac{\partial\psi_0}{\partial\xi} + \psi_0\frac{\partial^2\psi_0}{\partial\xi^2} - \psi_0^2 + \psi_0)]]$$
$$= e^{\xi}\frac{t^k}{\Gamma(k+1)},$$
$$\psi_2(\xi,t) = LT^{-1}[\frac{1}{s^k}[LT(\frac{\partial^2\psi_1}{\partial\xi^2} - \frac{\partial\psi_1}{\partial\xi} + \psi_1\frac{\partial^2\psi_1}{\partial\xi^2} - \psi_1^2 + \psi_1)]]$$
$$= e^{\xi}\frac{t^{2k}}{\Gamma(2k+1)},$$
$$\psi_3(\xi,t) = LT^{-1}[\frac{1}{s^k}[LT(\frac{\partial^2\psi_2}{\partial\xi^2} - \frac{\partial\psi_2}{\partial\xi} + \psi_2\frac{\partial^2\psi_2}{\partial\xi^2} - \psi_2^2 + \psi_2)]]$$
$$= e^{\xi}\frac{t^{3k}}{\Gamma(3k+1)},$$
$$\psi_4(\xi,t) = LT^{-1}[\frac{1}{s^k}[LT(\frac{\partial^2\psi_3}{\partial\xi^2} - \frac{\partial\psi_3}{\partial\xi} + \psi_3\frac{\partial^2\psi_3}{\partial\xi^2} - \psi_3^2 + \psi_3)]]$$
$$= e^{\xi}\frac{t^{4k}}{\Gamma(4k+1)},$$

The approximated solution will be

$$\psi(\xi,t)=\psi_0(\xi,t)+\psi_1(\xi,t)+\psi_2(\xi,t)+\psi_3(\xi,t)+\psi_4(\xi,t)+\dots$$
$$=e^{\xi}+e^{\xi}\frac{t^k}{\Gamma(k+1)}+e^{\xi}\frac{t^{2k}}{\Gamma(2k+1)}+e^{\xi}\frac{t^{3k}}{\Gamma(3k+1)}+e^{\xi}\frac{t^{4k}}{\Gamma(4k+1)}+\dots$$
$$=e^{\xi}[1+\frac{t^k}{\Gamma(k+1)}+\frac{t^{2k}}{\Gamma(2k+1)}+\frac{t^{3k}}{\Gamma(3k+1)}+\frac{t^{4k}}{\Gamma(4k+1)}+\dots]$$
$$=e^{\xi}\sum_{j=0}^{\infty}\frac{(t^k)^j}{\Gamma(jk+1)}$$
$$=e^{\xi}E_k(t^k) \tag{4. 17}$$

When $k=1$ in eq. 4. 17 , then solution becomes

$$\psi(\xi,t)=e^{\xi}\sum_{j=0}^{\infty}\frac{t^j}{j!}$$

and then the exact solution will be

$$\psi(\xi,t)=e^{(\xi+t)}$$

Example 4.2. *The 2D diffusion equation of fractional order:*[12]

$$\frac{\partial^k\psi}{\partial t^k}=\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial v^2},0<k\leq 1,t\geq 0 \tag{4. 18}$$

with initial condition

$$\psi(\xi,v,0)=(1-v)e^{\xi}$$

Apply LT on eq. 4. 18 , we obtain

$$LT[\psi(\xi,v,t)]=\frac{1}{s}(1-v)e^{\xi}+\frac{1}{s^k}[LT[\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial v^2}]] \tag{4. 19}$$

Apply inverse of LT on eq. 4. 19 , we obtain

$$\psi(\xi,v,t)=(1-v)e^{\xi}+LT^{-1}[\frac{1}{s^k}[LT[\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial v^2}]]] \tag{4. 20}$$

Utilizing IM,

$$\psi_0(\xi,v,t)=(1-v)e^{\xi},$$
$$\psi_1(\xi,v,t)=LT^{-1}[\frac{1}{s^k}[LT[\frac{\partial^2\psi_0}{\partial\xi^2}+\frac{\partial^2\psi_0}{\partial v^2}]]]$$
$$=(1-v)e^{\xi}\frac{t^k}{\Gamma(k+1)},$$
$$\psi_2(\xi,v,t)=LT^{-1}[\frac{1}{s^k}[LT[\frac{\partial^2\psi_1}{\partial\xi^2}+\frac{\partial^2\psi_1}{\partial v^2}]]]$$
$$=(1-v)e^{\xi}\frac{t^{2k}}{\Gamma(2k+1)},$$
$$\psi_3(\xi,v,t)=LT^{-1}[\frac{1}{s^k}[LT[\frac{\partial^2\psi_2}{\partial\xi^2}+\frac{\partial^2\psi_2}{\partial v^2}]]]$$

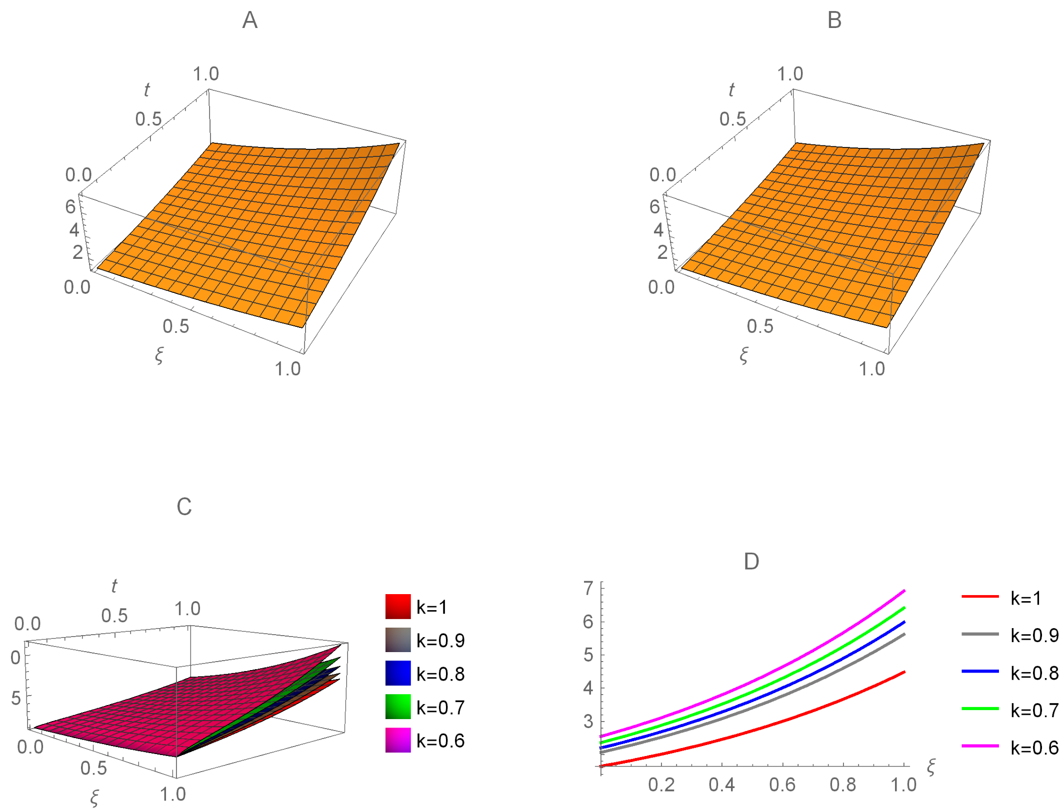

FIGURE 1. Example 1—A: Exact Solution when $k = 1$, B: Approximated solution using ILTM when $k = 1$, C: Approximated solutions for several values of fractional order $k \in \{1, 0.9, 0.8, 0.7, 0.6\}$,D: For $t = 0.5$ the 2D plot of ILTM solution for several values of fractional order $k \in \{1, 0.9, 0.8, 0.7, 0.6\}$.The convergence of solutions in fractional and integer senses is seen.

$$= (1 - v)e^{\xi} \frac{t^{3k}}{\Gamma(3k + 1)},$$

$$\psi_4(\xi, v, t) = LT^{-1}[\frac{1}{s^k}[LT[\frac{\partial^2 \psi_3}{\partial \xi^2} + \frac{\partial^2 \psi_3}{\partial v^2}]]]$$

$$= (1 - v)e^{\xi} \frac{t^{4k}}{\Gamma(4k + 1)},$$

The approximated solution will be

$$\psi(\xi, v, t) = \psi_0(\xi, v, t) + \psi_1(\xi, v, t) + \psi_2(\xi, v, t) + \psi_3(\xi, v, t) + \psi_4(\xi, v, t) + \dots$$

$$= (1-v)e^{\xi} + (1-v)e^{\xi}\frac{t^k}{\Gamma(k+1)} + (1-v)e^{\xi}\frac{t^{2k}}{\Gamma(2k+1)} + (1-v)e^{\xi}\frac{t^{3k}}{\Gamma(3k+1)} + (1-v)e^{\xi}\frac{t^{4k}}{\Gamma(4k+1)} + \ldots$$
$$= (1-v)e^{\xi}[1 + \frac{t^k}{\Gamma(k+1)} + \frac{t^{2k}}{\Gamma(2k+1)} + \frac{t^{3k}}{\Gamma(3k+1)} + \frac{t^{4k}}{\Gamma(4k+1)} + \ldots]$$
$$= (1-v)e^{\xi}\sum_{j=0}^{\infty}\frac{(t^k)^j}{\Gamma(jk+1)}$$
$$= (1-v)e^{\xi}E_k(t^k) \quad (4.\ 21)$$

When $k = 1$ in eq. 4. 21 , then solution becomes

$$\psi(\xi, v, t) = (1-v)e^{\xi}\sum_{j=0}^{\infty}\frac{t^j}{j!}$$

and then the exact solution will be

$$\psi(\xi, v, t) = (1-v)e^{\xi+t}$$

Example 4.3. *The 3D diffusion equation of fractional order:*[12]

$$\frac{\partial^k\psi}{\partial t^k} = \frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial v^2} + \frac{\partial^2\psi}{\partial u^2}, 0 < k \le 1, t \ge 0 \quad (4.\ 22)$$

with initial condition

$$\psi(\xi, v, u, 0) = \sin\xi\sin v\sin u$$

Apply LT on eq. 4. 22 , we obtain

$$LT[\psi(\xi, v, u, t)] = \frac{1}{s}\sin\xi\sin v\sin u + \frac{1}{s^k}[LT[\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial v^2} + \frac{\partial^2\psi}{\partial u^2}]] \quad (4.\ 23)$$

Apply inverse of LT on eq. 4. 23 , we obtain

$$\psi(\xi, v, u, t) = \sin\xi\sin v\sin u + LT^{-1}[\frac{1}{s^k}[LT[\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial v^2} + \frac{\partial^2\psi}{\partial u^2}]]]$$

Utilizing IM,

$$\psi_0(\xi, v, u, t) = \sin\xi\sin v\sin u,$$
$$\psi_1(\xi, v, u, t) = LT^{-1}[\frac{1}{s^k}[LT[\frac{\partial^2\psi_0}{\partial\xi^2} + \frac{\partial^2\psi_0}{\partial v^2} + \frac{\partial^2\psi_0}{\partial u^2}]]]$$
$$= (-3)\sin\xi\sin v\sin u\frac{t^k}{\Gamma(k+1)},$$
$$\psi_2(\xi, v, u, t) = LT^{-1}[\frac{1}{s^k}[LT[\frac{\partial^2\psi_1}{\partial\xi^2} + \frac{\partial^2\psi_1}{\partial v^2} + \frac{\partial^2\psi_1}{\partial u^2}]]]$$
$$= (-3)^2\sin\xi\sin v\sin u\frac{t^{2k}}{\Gamma(2k+1)},$$
$$\psi_3(\xi, v, u, t) = LT^{-1}[\frac{1}{s^k}[LT[\frac{\partial^2\psi_2}{\partial\xi^2} + \frac{\partial^2\psi_2}{\partial v^2} + \frac{\partial^2\psi_2}{\partial u^2}]]]$$
$$= (-3)^3\sin\xi\sin v\sin u\frac{t^{3k}}{\Gamma(3k+1)},$$

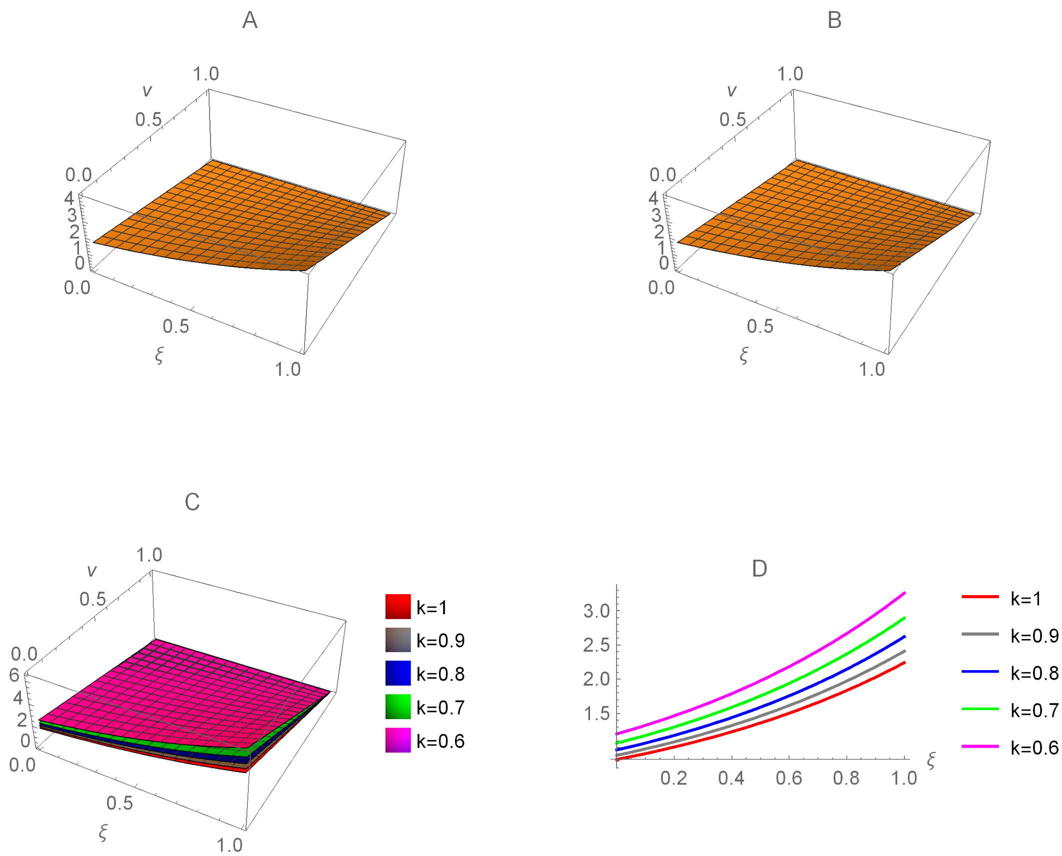

FIGURE 2. Example 2—A: Exact Solution when $k = 1$ and $t = 0.5$, B: Approximated solution using ILTM when $k = 1$ and $t = 0.5$, C: Approximated solutions for several values of fractional order $k \in \{1, 0.9, 0.8, 0.7, 0.6\}$ and $t = 0.5$,D: The 2D plot of ILTM solution for several values of fractional order $k \in \{1, 0.9, 0.8, 0.7, 0.6\}$ and $t = v = 0.5$.The convergence of solutions in fractional and integer senses is seen.

$$\psi_4(\xi, v, u, t) = LT^{-1}[\frac{1}{s^k}[LT[\frac{\partial^2\psi_3}{\partial\xi^2} + \frac{\partial^2\psi_3}{\partial v^2} + \frac{\partial^2\psi_3}{\partial u^2}]]]$$
$$= (-3)^4 \sin\xi \sin v \sin u \frac{t^{4k}}{\Gamma(4k+1)},$$

The approximated solution will be

$$\psi(\xi, v, u, t) = \psi_0(\xi, v, u, t) + \psi_1(\xi, v, u, t) + \psi_2(\xi, v, u, t) + \psi_3(\xi, v, u, t) + \psi_4(\xi, v, u, t) + \dots$$
$$= \sin\xi \sin v \sin u + (-3) \sin\xi \sin v \sin u \frac{t^k}{\Gamma(k+1)} + (-3)^2 \sin\xi \sin v \sin u \frac{t^{2k}}{\Gamma(2k+1)} +$$

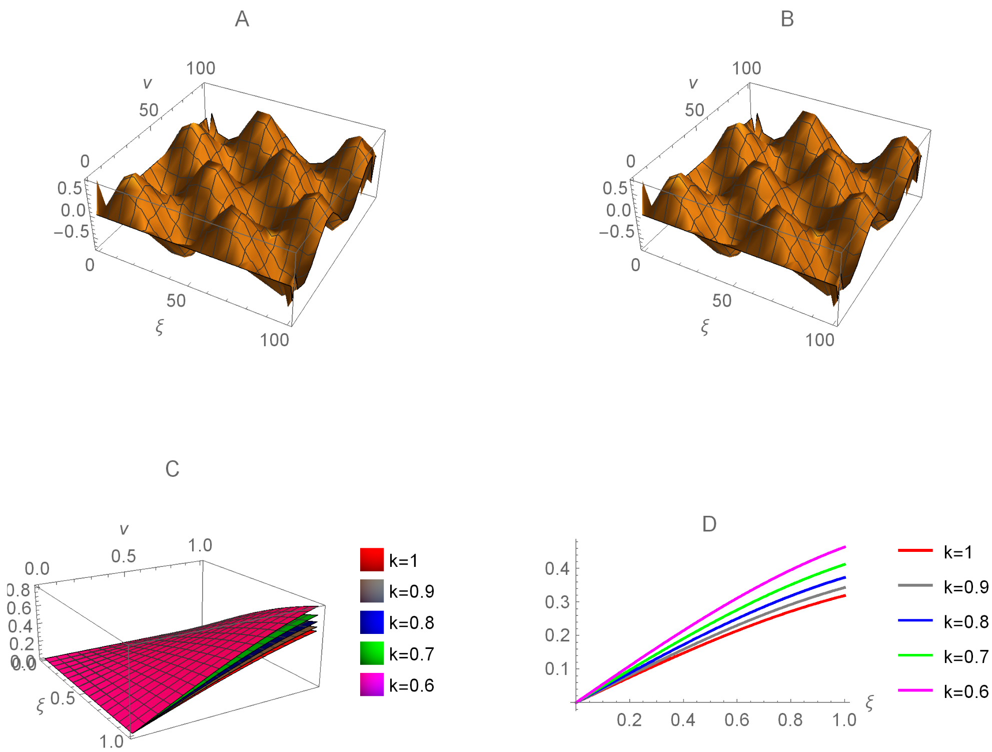

FIGURE 3. Example 3—A: Exact Solution when $k = 1$ and $u, t = 0.5$, B: Approximated solution using ILTM when $k = 1$ and $u, t = 0.5$, C: Approximated solutions for several values of fractional order $k \in \{1, 0.9, 0.8, 0.7, 0.6\}$ and $u = t = 0.5$,D: The 2D plot of ILTM solution for several values of fractional order $k \in \{1, 0.9, 0.8, 0.7, 0.6\}$ and $u = t = v = 0.5$.The convergence of solutions in fractional and integer senses is seen.

$$(-3)^3 \sin\xi \sin v \sin u \frac{t^{3k}}{\Gamma(3k+1)} + (-3)^4 \sin\xi \sin v \sin u \frac{t^{4k}}{\Gamma(4k+1)} + \dots$$

$$= \sin\xi \sin v \sin u[1 + (-3)\frac{t^{k}}{\Gamma(k+1)} + (-3)^2 \frac{t^{2k}}{\Gamma(2k+1)} +$$

$$(-3)^3 \frac{t^{3k}}{\Gamma(3k+1)} + (-3)^4 \frac{t^{4k}}{\Gamma(4k+1)} + \dots]$$

$$= \sin\xi \sin v \sin u[1 + \frac{-3t^k}{\Gamma(k+1)} + \frac{(-3t^k)^2}{\Gamma(2k+1)} +$$

$$\frac{(-3t^k)^3}{\Gamma(3k+1)} + \frac{(-3t^k)^4}{\Gamma(4k+1)} + \dots]$$

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