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# The First General Zagreb Index of the Zero Divisor Graph for the Ring $\mathbb{Z}_{pq^k}$

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**Abstract.** This study investigates the application of graph theory in analyzing the zero divisor graph of a commutative ring, with a specific focus on its connection to the topological index. For an undirected graph $\Gamma$ with consists of a non-empty set of vertices, $V$, and a set of edges, $E$, the first general Zagreb index is defined as a graph invariant that measures the sum of the degree of each vertex to the power of $\alpha \neq 0$. Meanwhile, the zero divisor graph $\Gamma$ of the commutative ring, $R$ is the (undirected) graph with vertices the zero-divisors of $R$, and distinct vertices $a$ and $b$ are adjacent if and only if $ab = 0$. In this paper, the general formulas of the first general Zagreb index of the zero divisor graph for the ring of integers modulo $pq^k$ are computed for the cases $\delta = 1$, 2, and 3. This research focuses on the ring defined as the integers modulo $pq^k$, where $k$ is a positive integer, $p$ and $q$ are primes $p < q$. Two examples are given to demonstrate the main findings.

**AMS (MOS) Subject Classification Codes: 05C07, 05C09, 13D09, 13B35**
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## 1. Introduction

In general, topological molecular descriptors, also known as topological index, are numerical values that encode certain topological features of molecular graphs. Topological indices play an important role in studying quantitative structure–activity relationships (QSAR) and quantitative structure–property relationships (QSPR) for predicting different physico–chemical properties of chemical compounds [20]. Many studies on topological indices of graphs have been conducted in recent years [1, 4, 5, 11–13, 16, 24, 25, 27, 28].

This paper focuses on the first general Zagreb index also known as the general zeroth-order Randić index, a degree-based topological index commonly used by chemists. The first general Zagreb index was formulated by Li and Zheng in 2005 [18]. Matejić et al. [20] found various upper and lower bounds of the general zeroth-order Randić index. Liu et al. in [19] computed the first general Zagreb index of the D-sum graphs in terms of their factor graphs. Jamil et al. [14] investigated some sharp upper and lower bounds on the zeroth-order general Randić index in terms of connectivity, minimum degree, and independent number. By Elumalai and Mansour in [6], the expected value of the zeroth-order Randić index has been quantified for all bargraphs with n cells. Mondal et al. in [21] obtained some degree-based, degree-distance-based, and distance-based topological indices of the zero divisor graphs $\Gamma(\mathbb{Z}_{r^n})$ and $\Gamma(\mathbb{Z}_r \times \mathbb{Z}_s \times \mathbb{Z}_t)$ through algebraic polynomials. Anjum et al. [3] presented the generalized $\rho$ dependent polynomials for the calculations of eccentricity, distance, total distance, and degree-based topological indices of the identity graph of $\mathbb{Z}_\rho$.

This study focuses on the zero divisor graph of commutative rings. An undirected graph $\Gamma = (V, E)$ consists of a nonempty set $V$ of vertices and a collection $E$ of edges. A nonzero element of a ring, $R$ is said to be a zero divisor if the product of that nonzero element with another nonzero element of the ring is equal to zero [7]. Assuming that $Z\left(R\right)^*$ represents the set of all zero divisors in a ring $R$, the zero divisor graph of $R$, denoted as $\Gamma\left(R\right)$, is a graph with vertices representing the nonzero zero divisors of $R$ has been proposed and studied by Anderson and Livingston in [2]. Two distinct vertices, $u$ and $v$ are adjacent to one another if $u \cdot v = v \cdot u = 0$.

The concept of a fuzzy zero divisor graph in a commutative ring has been introduced by Kuppan and Sankar in [17]. Rayer and Jeyaraj [26] presented that the boundaries are related to topological indices for the zero divisor graph. some graph parameters of the zero divisor graph $\Gamma(R)$ of a finite commutative ring $R$ for $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \times \mathbb{Z}_{2p}$ have been investigated by Movahedi and Akhbari [23]. Morales et al. [22] computed the offensive alliance (global, independent, and independent global) numbers of $\Gamma(\mathbb{Z}_n)$ for some cases of $n$.

This paper presents the theoretical findings of an endeavor to compute the first general Zagreb index of the zero divisor graph for the ring of integers modulo $pq^k$, denoted as $M_1^\delta\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right)$ where $k$ is a positive integer, $p$ and $q$ are prime numbers with $p < q$. The first general Zagreb index is found for the cases $\delta = 1, 2$ and 3.

## 2. Preliminaries

This section presents a concise overview of the fundamental concepts and definitions of ring theory, graph theory, and topological index.

**Definition 2.1.** [18] First General Zagreb Index
Let $\Gamma$ be a connected graph and $deg\left(u\right)$ be the vertex degree u in the graph. Then,

$$M_1^\delta\left(\Gamma\right) = \sum_{u \in V} \left(\deg\left(u\right)\right)^\delta$$

where $\delta$ is an arbitrary real number.

**Definition 2.2.** [10] First Zagreb Index
Let $\Gamma$ be a connected graph. Then, the first Zagreb index,

$$M_1(\Gamma) = \sum_{u\in V(\Gamma)} \deg(u)^2.$$

**Definition 2.3.** [8] Forgotten Topological Index
Let $\Gamma$ be a connected graph. Then, the forgotten topological index (also called F-index) written as

$$FT(\Gamma) = \sum_{u\in V(\Gamma)} \deg(u)^3.$$

Note that $M_1^2(\Gamma) = M_1(\Gamma)$ when $\delta = 2$ and $M_1^3(\Gamma) = FT(\Gamma)$ when $\delta = 3$.

**Proposition 2.4.** [9] For any graph $\Gamma$, $\sum_{u\in V(\Gamma)} \deg(u) = 2|E(\Gamma)|$.

**Proposition 2.5.** [15] Let $\mathbb{Z}_n$ be the ring of integers $\mathbb{Z}_n$, where n can be expressed as $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, such that $p_1, p_2, \ldots, p_m$ are distinct prime numbers, and $k_1, k_2, \ldots, k_m \in \mathbb{N}$. Then, the ring $\mathbb{Z}_n$ has at least a zero divisor, and the number of its zero divisors is

$$n - n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_m}\right) - 1.$$

## 3. Results and Discussion

From now on, assume that $k$ is a positive integer, and $p$ and $q$ are prime numbers, with $p < q$. In this section, some results related to $M_1^\delta\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right)$ are included. First, the set of all zero divisors denoted as $Z\left(\mathbb{Z}_{pq^k}\right)^*$ and the number of zero divisors in the ring of integers modulo $pq^k$, denoted as $\left|Z\left(\mathbb{Z}_{pq^k}\right)^*\right|$, are listed in the following propositions.

**Proposition 3.1.** $Z\left(\mathbb{Z}_{pq^k}\right)^* = A_1 \cup A_2 = \{p, 2p, 3p, \ldots, p\left(q^k-1\right)\} \cup \{q, 2q, 3q, \ldots, q\left(pq^{k-1}-1\right)\}$.

*Proof.* Suppose $u$ is a zero divisor of the ring $\mathbb{Z}_{pq^k}$. $Z\left(\mathbb{Z}_{pq^k}\right)^*$ is given in the following:

a) Let $u \in \mathbb{Z}_{pq^k}$ with $\gcd(a,p) > 1$ and $A_1$ is $Z\left(\mathbb{Z}_{pq^k}\right)^*$ for $u$. Then, $A_1 = \{p, 2p, 3p, \ldots, p\left(q^k-1\right)\}$.
b) Let $u \in \mathbb{Z}_{pq^k}$ with $\gcd(a,q) > 1$ and $A_2$ is $Z\left(\mathbb{Z}_{pq^k}\right)^*$ for $u$. Then, $A_2 = \{q, 2q, 3q, \ldots, q\left(pq^{k-1}-1\right)\}$.

Thus, $Z\left(\mathbb{Z}_{pq^k}\right)^* = A_1 \cup A_2 = \{p, 2p, 3p, \ldots, p\left(q^k-1\right)\} \cup \{q, 2q, 3q, \ldots, q\left(pq^{k-1}-1\right)\}$. □

The degree of a vertex $u$ in $\Gamma\left(\mathbb{Z}_{pq^k}\right)$ is divided into three conditions, as presented in Propositions 3.2, 3.3, and 3.4.

**Proposition 3.2.** Let $u \in Z\left(\mathbb{Z}_{pq^k}\right)^*$ with $\gcd\left(u, pq^k\right) = p$. Then $\deg(u) = p - 1$.

*Proof.* Given that $\deg(u) = q - 1$ with $\gcd\left(u, pq^k\right) = p$. Let $u \in Z\left(\mathbb{Z}_{pq^k}\right)^*$ with $\gcd\left(u, pq^k\right) = p$, and let $v \in Z\left(\mathbb{Z}_{pq^k}\right)^*$ with $\gcd\left(v, pq^k\right) = q^j$ here $u$ and $v$ are adjacent if and only if $j = k$. Since $\gcd\left(v, pq^k\right) = q^j$ and $j = k$, so $v \in q^k\mathbb{Z}_{pq^k}$ and $\left|q^k\mathbb{Z}_{pq^k}\right| = \frac{pq^k}{q^k} - 1$. Thus, since $0 \notin Z\left(\mathbb{Z}_{p^k q}\right)^*$, so $\deg(u) = p - 1$. □

**Proposition 3.3.** Let $u \in Z\left(\mathbb{Z}_{pq^k}\right)^*$ with $\gcd\big(u, pq^k\big) = q^i$ for $i = 1, 2, \ldots, k$. Then $\deg(u) = q^i - 1$.

*Proof.* Given that $\deg(u) = q^i - 1$ with $\gcd\big(u, pq^k\big) = q^i$ for $i = 1, 2, \ldots, k$. Let $u \in Z\left(\mathbb{Z}_{pq^k}\right)^*$ with $\gcd\big(u, pq^k\big) = q^i$, and let $v \in Z\left(\mathbb{Z}_{pq^k}\right)^*$ with $\gcd\big(v, pq^k\big) = pq^j$, $\gcd\big(v, pq^k\big) = q^j$ or $\gcd\big(v, pq^k\big) = p$ where $u$and $v$ are adjacent if and only if $i + j \geq k$. Since $\gcd\big(v, pq^k\big) = pq^j$, $\gcd\big(v, pq^k\big) = q^j$ or $\gcd\big(v, pq^k\big) = p$ where $j \geq k - i$, so $v \in pq^{k-i}\mathbb{Z}_{pq^k}$ and $\left|pq^{k-i}\mathbb{Z}_{pq^k}\right| = \frac{pq^k}{pq^{k-i}} - 1$. Thus, $\deg(u) = q^i - 1$. ☐

**Proposition 3.4.** Let $u \in Z\left(\mathbb{Z}_{pq^k}\right)^*$ with $\gcd\big(u, pq^k\big) = pq^i$. Then

$$\deg(u) = \begin{cases} pq^i - 1, & \text{for } i \leq \left\lfloor \frac{k-1}{2} \right\rfloor, \\ pq^i - 2, & \text{for } i > \left\lfloor \frac{k-1}{2} \right\rfloor. \end{cases}$$

*Proof.* First, assume $i \leq \left\lfloor \frac{k-1}{2} \right\rfloor$. Let $u \in Z\left(\mathbb{Z}_{pq^k}\right)^*$ with $\gcd\big(u, pq^k\big) = pq^i$ and let $v \in Z\left(\mathbb{Z}_{pq^k}\right)^*$ with $\gcd\big(v, pq^k\big) = pq^j$ and $\gcd\big(v, pq^k\big) = q^j$ where $u$ and $v$ are adjacent if and only if $i + j \geq k$. Since $\gcd\big(v, pq^k\big) = pq^j$, $\gcd\big(v, pq^k\big) = q^j$ and $j \geq k - i$, so $v \in q^{k-i}\mathbb{Z}_{pq^k}$ and $\left|q^{k-i}\mathbb{Z}_{pq^k}\right| = \frac{pq^k}{q^{k-i}} - 1$. Thus, $\deg(u) = pq^i - 1$.
Next, assume $i > \left\lfloor \frac{k-1}{2} \right\rfloor$. Let $u \in Z\left(\mathbb{Z}_{pq^k}\right)^*$ with $\gcd\big(u, pq^k\big) = pq^i$ and let $b \in Z\left(\mathbb{Z}_{pq^k}\right)^*$ with $\gcd\big(v, pq^k\big) = pq^j$ and $\gcd\big(v, pq^k\big) = q^j$ where $u$ and $v$ are adjacent if and only if $i + j \geq k$. Since $\gcd\big(v, pq^k\big) = pq^j$, $\gcd\big(v, pq^k\big) = q^j$ or $\gcd\big(v, pq^k\big) = p$ where $j \geq k - i$, so $v \in q^{k-i}\mathbb{Z}_{pq^k}$ and $\left|q^{k-i}\mathbb{Z}_{pq^k}\right| = \frac{pq^k}{q^{k-i}} - 2$. Thus, $\deg(u) = pq^i - 2$. ☐

Since $\left|\mathbb{Z}_{pq^k}\right| = |\mathbb{Z}_p| \cdot \left|\mathbb{Z}_{q^k}\right|$, then the number of vertices in $\Gamma\left(\mathbb{Z}_{pq^k}\right)$ for a given degree is divided into three cases as given in Propositions 3.5, 3.6, and 3.7.

**Proposition 3.5.** Let $u \in V'$, thus $u \in Z\left(\mathbb{Z}_{pq^k}\right)^*$:$\gcd(u, p) = p$ then $|V'_i| = \big(q^k - q^{k-i}\big)$.

*Proof.* Given that $V'_i = \{u \in Z\left(\mathbb{Z}_{pq^k}\right)^*$:$\gcd(u, p) = p\}$ then $|V'_i| = \big(q^k - q^{k-i}\big)$ and $V'_j = \{v \in \mathbb{Z}_{q^k}$:$\gcd(v, p) = p^0 = 1\}$ then $|V'_j| = p^0 = 1$. So $|V'| = |V'_i| \cdot |V'_j| = \big(q^k - q^{k-i}\big)$. ☐

**Proposition 3.6.** Let $u \in V'$, thus $u \in Z\left(\mathbb{Z}_{pq^k}\right)^*$:$\gcd\big(u, q^k\big) = q^i$ then

a) $|V'| = \big(q^{k-i} - q^{k-(i+1)}\big)(p-1)$ for $1 \leq i \leq k-1$,
b) $|V'| = \big(q^{k-i}\big)(p-1)$ for $i = k$.

*Proof.* Given that $V'_i = \{u \in Z\left(\mathbb{Z}_{pq^k}\right)^*$:$\gcd\big(u, q^k\big) = q^i\}$ then

a) $|V'_i| = \big(q^{k-i} - q^{k-(i+1)}\big)$ for $1 \leq i \leq k-1$ and $V'_j = \{v \in \mathbb{Z}_p$:$\gcd(v, p) = p\}$ then $|V'_j| = p^1 - p^0 = p - 1$. So $|V'| = |V'_i| \cdot |V'_j| = \big(q^{k-i} - q^{k-(i+1)}\big)(p-1)$.
b) $|V'_i| = q^{k-i}$ for $i = k$ and $V'_j = \{v \in \mathbb{Z}_p$:$\gcd(v, p) = p\}$ then $|V'_j| = p^1 - p^0 = p - 1$. So $|V'| = |V'_i| \cdot |V'_j| = \big(q^{k-i}\big)(p-1)$.

☐

**Proposition 3.7.** Let $u \in V'$, thus $u \in Z\left(\mathbb{Z}_{pq^k}\right)^*$:$\gcd\big(u, p^k q\big) = p^i q$ then $|V'_i| = \big(q^{k-i} - q^{k-(i+1)}\big)$ for $1 \leq i \leq k-1$.

*Proof.* Given that $V'_i = \{u \in Z\left(\mathbb{Z}_{pq^k}\right)^* : \gcd\left(a, pq^k\right) = pq^i\}$ then $|V'_i| = \left(q^{k-i} - q^{k-(i+1)}\right)$ for $1 \leq i \leq k-1$ and $V'_j = \{v \in \mathbb{Z}_p : \gcd(v, p) = p^0 = 1\}$ then $|V'_j| = p^0 = 1$. So $|V'| = |V'_i| \cdot |V'_j| = \left(q^{k-i} - q^{k-(i+1)}\right)$. □

**Theorem 3.8.** The number of edges for $\Gamma\left(\mathbb{Z}_{pq^k}\right)$ is

$$\left|E\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right)\right| = \frac{1}{2}\left[\left(q^k - q^{k-1}\right)\left((p-1)\left(k - \frac{q^{k-1}-1}{q^{k-1}(q-1)}\right) + p(k-1) - \frac{q^{k-1}-1}{q^{k-1}(q-1)} - \frac{q^{k-1} - q^{\left\lfloor \frac{k-1}{2} \right\rfloor}}{q^{\left\lfloor \frac{3(k-1)}{2} \right\rfloor}(q-1)} + (p-1)\left(q^k - 1\right)\right],$$

*Proof.* Using Proposition 2.4, we have

$$\left|E\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right)\right| = \frac{1}{2}\sum_{u \in V\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right)} \deg(u).$$

Using it from Proposition 3.2 to Proposition 3.7, we then have

$$\begin{aligned}\left|E\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right)\right| = \frac{1}{2}\Bigg[&(p-1)\sum_{i=1}^{1}\left(q^k - q^{k-i}\right) + (p-1)\sum_{i=1}^{k-1}\left(q^{k-i} - q^{k-(i+1)}\right)\left(q^i - 1\right)\\ &+ (p-1)\sum_{i=1+k-1}^{k} q^{k-i}\left(q^i - 1\right) + \sum_{i=1}^{\left\lfloor \frac{k-1}{2} \right\rfloor}\left(q^{k-i} - q^{k-(i+1)}\right)\left(pq^i - 1\right)\\ &+ \sum_{i=1+\left\lfloor \frac{k-1}{2} \right\rfloor}^{k-1}\left(q^{k-i} - q^{k-(i+1)}\right)\left(pq^i - 2\right).\end{aligned}$$

Using summation rules and geometric sequences,

$$\begin{aligned}\left|E\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right)\right| = \frac{1}{2}\Bigg[&(p-1)\left(q^k - q^{k-1}\right) + (p-1)\sum_{i=1}^{k-1}\left(q^k - q^{k-i} - q^{k-1} + q^{k-(i+1)}\right)\\ &+ (p-1)\left(q^k - 1\right) + \sum_{i=1}^{\left\lfloor \frac{k-1}{2} \right\rfloor}\left(q^{k-i} - q^{k-(i+1)}\right)\left(pq^i - 1\right)\\ &- \sum_{i=1+\left\lfloor \frac{k-1}{2} \right\rfloor}^{k-1}\left(q^{k-i} - q^{k-(i+1)}\right)\\ = \frac{1}{2}\Bigg[&(p-1)\left(q^k - q^{k-1}\right) + (p-1)\left(q^k - q^{k-1}\right)\left(\sum_{i=1}^{k-1} 1 - \sum_{i=1}^{k-1} q^{-i}\right)\\ &+ \left(q^k - q^{k-1}\right)\left(\sum_{i=1}^{k-1} p - \sum_{i=1}^{k-1} q^{-i} - \sum_{i=1+\left\lfloor \frac{k-1}{2} \right\rfloor}^{k-1} q^{-i}\right) + (p-1)\left(q^k - 1\right)\Bigg].\end{aligned}$$

Therefore, we obtain a simplified general formula is simplified that yields

$$\left|E\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right)\right| = \frac{1}{2}\left[\left(q^k - q^{k-1}\right)\left((p-1)\left(k - \frac{q^{k-1}-1}{q^{k-1}(q-1)}\right) + p(k-1) - \frac{q^{k-1}-1}{q^{k-1}(q-1)} - \frac{q^{k-1} - q^{\left\lfloor \frac{k-1}{2} \right\rfloor}}{q^{\left\lfloor \frac{3(k-1)}{2} \right\rfloor}(q-1)} + (p-1)\left(q^k - 1\right)\right].$$

□

Next, the main results in this research for the cases $\delta = 1, 2$ and $3$ are presented in the following theorems.

**Theorem 3.9.**

$$M_1^1\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right) = \left(q^k - q^{k-1}\right)\left[(p-1)\left(k - \frac{q^{k-1}-1}{q^{k-1}(q-1)}\right) + p(k-1) - \frac{q^{k-1}-1}{q^{k-1}(q-1)} - \frac{q^{k-1} - p^{\left\lfloor \frac{k-1}{2} \right\rfloor}}{q^{\left\lfloor \frac{3(k-1)}{2} \right\rfloor}(q-1)}\right] + (p-1)\left(q^k - 1\right).$$

*Proof.* Using Definition 2.1 when $\delta = 1$ and approaches similar to Theorem 3.8, we have

$$\begin{aligned} M_1^1\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right) &= \sum_{a \in V\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right)} deg(a)^1 \\ &= (p-1)\sum_{i=1}^{1}\left(q^k - q^{k-i}\right)^1 + (p-1)\sum_{i=1}^{k-1}\left(q^{k-i} - q^{k-(i+1)}\right)\left(q^i - 1\right)^1 \\ &\quad + (p-1)\sum_{i=1+k-1}^{k} q^{k-i}\left(q^i - 1\right)^1 + \sum_{i=1}^{\left\lfloor \frac{k-1}{2} \right\rfloor}\left(q^{k-i} - q^{k-(i+1)}\right)\left(pq^i - 1\right)^1 \\ &\quad + \sum_{i=1+\left\lfloor \frac{k-1}{2} \right\rfloor}^{k-1}\left(q^{k-i} - q^{k-(i+1)}\right)\left(pq^i - 2\right)^1 \\ &= \left(q^k - q^{k-1}\right)\left[(p-1)\left(k - \frac{q^{k-1}-1}{q^{k-1}(q-1)}\right) + p(k-1) - \frac{q^{k-1}-1}{q^{k-1}(q-1)} - \frac{q^{k-1} - p^{\left\lfloor \frac{k-1}{2} \right\rfloor}}{q^{\left\lfloor \frac{3(k-1)}{2} \right\rfloor}(q-1)}\right] + (p-1)\left(q^k - 1\right). \end{aligned}$$

□

We discovered that $M_1^2\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right) = M_1\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right)$ in the following.

**Theorem 3.10.**

$$\begin{aligned}M_1^2\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right) &= M_1\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right)\\ &= \left(q^k - q^{k-1}\right)\Bigg[(p-1)^2 + (p-1)\left(\frac{q\left(q^{k-1}-1\right)}{q-1} + \frac{q^{k-1}-1}{q^{k-1}(q-1)}\right.\\ &\left. - 2(k-1)\right) + \frac{p^2 q\left(q^{k-1}-1\right)}{q-1} + \frac{q^{k-1}-1}{q^{k-1}(q-1)} + 3\left(\frac{q^{k-1}-q^{\left\lfloor\frac{k-1}{2}\right\rfloor}}{q^{\left\lfloor\frac{3(k-1)}{2}\right\rfloor}(q-1)}\right)\\ &- 2p\left\lceil\frac{k-1}{2}\right\rceil - 2p(k-1)\Bigg] + (p-1)\left(q^k-1\right)^2.\end{aligned}$$

*Proof.* By applying the similar method as in Theorem 3.9 but for $\delta = 2$, we obtain

$$\begin{aligned}M_1^2\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right) &= M_1\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right)\\ &= \sum_{a\in V\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right)} deg(a)^2\\ &= (p-1)\sum_{i=1}^{1}\left(q^k - q^{k-i}\right)^2 + (p-1)\sum_{i=1}^{k-1}\left(q^{k-i} - q^{k-(i+1)}\right)\left(q^i-1\right)^2\\ &+ (p-1)\sum_{i=1+k-1}^{k} q^{k-i}\left(q^i-1\right)^2 + \sum_{i=1}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\left(q^{k-i} - q^{k-(i+1)}\right)\left(pq^i-1\right)^2\\ &+ \sum_{i=1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{k-1}\left(q^{k-i} - q^{k-(i+1)}\right)\left(pq^i-2\right)^2\\ &= \left(q^k - q^{k-1}\right)\Bigg[(p-1)^2 + (p-1)\left(\frac{q\left(q^{k-1}-1\right)}{q-1} + \frac{q^{k-1}-1}{q^{k-1}(q-1)}\right.\\ &\left. - 2(k-1)\right) + \frac{p^2 q\left(q^{k-1}-1\right)}{q-1} + \frac{q^{k-1}-1}{q^{k-1}(q-1)} + 3\left(\frac{q^{k-1}-q^{\left\lfloor\frac{k-1}{2}\right\rfloor}}{q^{\left\lfloor\frac{3(k-1)}{2}\right\rfloor}(q-1)}\right)\\ &- 2p\left\lceil\frac{k-1}{2}\right\rceil - 2p(k-1)\Bigg] + (p-1)\left(q^k-1\right)^2.\end{aligned}$$

□

The following theorem provides the first general Zagreb index when $\delta = 3$ and the F-index of $\Gamma\left(\mathbb{Z}_{2^k q}\right)$ are determined.

**Theorem 3.11.**

$$\begin{aligned}M_1^3\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right) &= FT\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right)\\ &= \left(q^k - q^{k-1}\right)\Bigg[(p-1)^3 + (p-1)\left(\frac{q^2\left(q^{2(k-1)}-1\right)}{q^2-1} + 3(k-1)\right.\\ &\quad \left. - \frac{3q\left(q^{k-1}-1\right)}{q-1} - \frac{q^{k-1}-1}{q^{k-1}(q-1)}\right) + \frac{p^3q^2\left(q^{2(k-1)}-1\right)}{q^2-1} + 3p(k-1)\\ &\quad - \frac{3p^2q\left(q^{k-1}-1\right)}{q-1} - \frac{q^{k-1}-1}{q^{k-1}(q-1)} - \frac{3p^2q^{\left\lfloor\frac{k+1}{2}\right\rfloor}\left(q^{\left\lceil\frac{k-1}{2}\right\rceil}-1\right)}{q-1}\\ &\quad + 9p\left\lceil\frac{k-1}{2}\right\rceil - \frac{7\left(q^{k-1}-q^{\left\lfloor\frac{k-1}{2}\right\rfloor}\right)}{q^{\left\lfloor\frac{3k-1}{2}\right\rfloor}(q-1)}\Bigg] + (p-1)\left(q^k-1\right)^3.\end{aligned}$$

*Proof.* Similarly, we then have

$$\begin{aligned}M_1^3\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right) &= FT\left(\Gamma\left(\mathbb{Z}_{pq^k}\right)\right)\\ &= \sum_{i=1}^{1}\left(q^k - q^{k-i}\right)(p-1)^3 + (p-1)\sum_{i=1}^{k-1}\left(q^{k-i} - q^{k-(i+1)}\right)\left(q^i-1\right)^3\\ &\quad + (p-1)\sum_{i=1+k-1}^{k} q^{k-i}\left(q^i-1\right)^3 + \sum_{i=1}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\left(q^{k-i} - q^{k-(i+1)}\right)\left(pq^i-1\right)^3\\ &\quad + \sum_{i=1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{k-1}\left(q^{k-i} - q^{k-(i+1)}\right)\left(pq^i-2\right)^3\\ &= \left(q^k - q^{k-1}\right)\Bigg[(p-1)^3 + (p-1)\left(\frac{q^2\left(q^{2(k-1)}-1\right)}{q^2-1} + 3(k-1)\right.\\ &\quad \left. - \frac{3q\left(q^{k-1}-1\right)}{q-1} - \frac{q^{k-1}-1}{q^{k-1}(q-1)}\right) + \frac{p^3q^2\left(q^{2(k-1)}-1\right)}{q^2-1} + 3p(k-1)\\ &\quad - \frac{3p^2q\left(q^{k-1}-1\right)}{q-1} - \frac{q^{k-1}-1}{q^{k-1}(q-1)} - \frac{3p^2q^{\left\lfloor\frac{k+1}{2}\right\rfloor}\left(q^{\left\lceil\frac{k-1}{2}\right\rceil}-1\right)}{q-1}\\ &\quad + 9p\left\lceil\frac{k-1}{2}\right\rceil - \frac{7\left(q^{k-1}-q^{\left\lfloor\frac{k-1}{2}\right\rfloor}\right)}{q^{\left\lfloor\frac{3k-1}{2}\right\rfloor}(q-1)}\Bigg] + (p-1)\left(q^k-1\right)^3.\end{aligned}$$

□

Two examples of the first general Zagreb index $\Gamma\left(\mathbb{Z}_{pq^k}\right)$ are provided for some values of $k$ when $\delta = 2$ and $\delta = 3$:

a) $\delta = 2$, $p$ is even while $q$ is odd,
b) $\delta = 3$, $p$ and $q$ are both odd.

**Example 3.12.** The graph $\Gamma\left(\mathbb{Z}_{54}\right)$ when $p = 2, k = 3$ and $q = 3$, is shown in Figure 1.

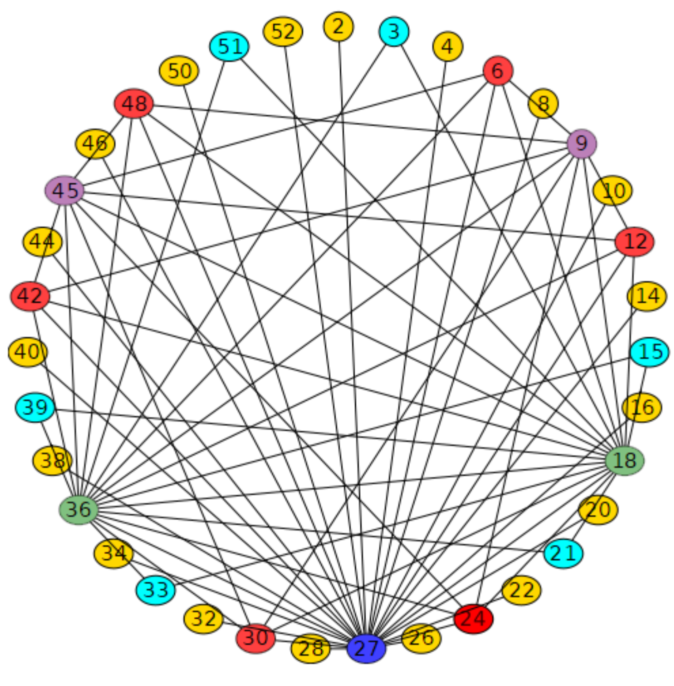


FIGURE 1. The graph $\Gamma\left(\mathbb{Z}_{54}\right)$

By using Proposition 3.1, $Z\left(\mathbb{Z}_{54}\right)^*$ is determined as follows:

$$\begin{aligned} Z\left(\mathbb{Z}_{54}\right)^* &= A_1 \cup A_2 \\ &= \{2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32,34,36,38,40,42,44, \\ &\quad 46,48,50,52\} \cup \{3,6,9,12,15,18,21,24,27,30,33,36,39,42,45,48,51\} \\ &= \{2,3,4,6,8,9,10,12,14,15,16,18,20,21,22,24,26,27,28,30,32,33,34, \\ &\quad 36,38,39,40,42,44,45,46,48,50,51,52\}. \end{aligned}$$

The graph $\Gamma(\mathbb{Z}_{54})$ has 67 edges, as shown in Figure 1. However, using Theorem 3.8, we have

$$\begin{aligned}|E(\Gamma(\mathbb{Z}_{54}))| &= \frac{1}{2}\left[\left(3^3 - 3^{3-1}\right)\left((2-1)\left(3 - \frac{3^{3-1}-1}{3^{3-1}(3-1)}\right) + 2(3-1) - \frac{3^{3-1}-1}{3^{3-1}(3-1)}\right.\right. \\ &\left.\left. - \frac{3^{3-1} - 3^{\lfloor \frac{3-1}{2} \rfloor}}{3^{\lfloor \frac{3(3-1)}{2} \rfloor}(3-1)}\right) + (2-1)\left(3^3-1\right)\right] \\ &= 67.\end{aligned}$$

Referring to Figure 1, yellow represents vertices of degree one, cyan represents vertices of degree two, red represents degree five, purple represents degree eight, green represents degree 16, and blue represents degree 26, which has a single vertex. By Definition 2.1, the first general Zagreb index when $\delta = 2$ is calculated in the following:

$$\begin{aligned}M_1^2(\Gamma(\mathbb{Z}_{54})) &= \sum_{u \in V(\Gamma(\mathbb{Z}_{54}))} \deg(u)^2 \\ &= \deg(2)^2 + \deg(3)^2 + ... + \deg(52)^2 \\ &= 1508.\end{aligned}$$

Using Theorem 3.10 is a straightforward approach to determine the value of $M_1^2(\Gamma(\mathbb{Z}_{54}))$ under the condition that $\delta = 2$:

$$\begin{aligned}M_1^2(\Gamma(\mathbb{Z}_{54})) &= \left(3^3 - 3^{3-1}\right)\left[(2-1)^2 + \frac{2^2(3)\left(3^{3-1}-1\right)}{3-1} + \frac{3^{3-1}-1}{3^{3-1}(3-1)}\right. \\ &+ (2-1)\left(\frac{3\left(3^{3-1}-1\right)}{3-1} - 2(3-1) + \frac{3^{3-1}-1}{3^{3-1}(3-1)}\right) - 2(2)\left\lceil \frac{3-1}{2} \right\rceil \\ &\left. - 2(2)(3-1) + 3\left(\frac{3^{3-1} - 3^{\lfloor \frac{3-1}{2} \rfloor}}{3^{\lfloor \frac{3(3-1)}{2} \rfloor}(3-1)}\right)\right] + (2-1)\left(3^3-1\right)^2 \\ &= 1508.\end{aligned}$$

**Example 3.13.** The graph $\Gamma(\mathbb{Z}_{75})$ when $p = 3, k = 2$ and $q = 5$, is illustrated in Figure 2. According to Proposition 3.1, $Z(\mathbb{Z}_{75})^*$ is listed as follows:

$$\begin{aligned}Z(\mathbb{Z}_{75})^* &= A_1 \cup A_2 \\ &= \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57, 60, 63, 66, \\ &\quad 69, 72\} \cup \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70\} \\ &= \{3, 5, 6, 9, 10, 12, 15, 18, 20, 21, 24, 25, 27, 30, 33, 35, 36, 39, 40, , 42, 45, 48, \\ &\quad 50, 51, 54, 55, 57, 60, 63, 65, 66, 69, 70, 72\}.\end{aligned}$$

Using Proposition 3.1, the vertices of $\Gamma(\mathbb{Z}_{75})$ are the set of 34 integers. Figure 2 illustrates that the $\Gamma(\mathbb{Z}_{75})$ has a total of 86 edges. However, by Theorem 3.8, we obtain

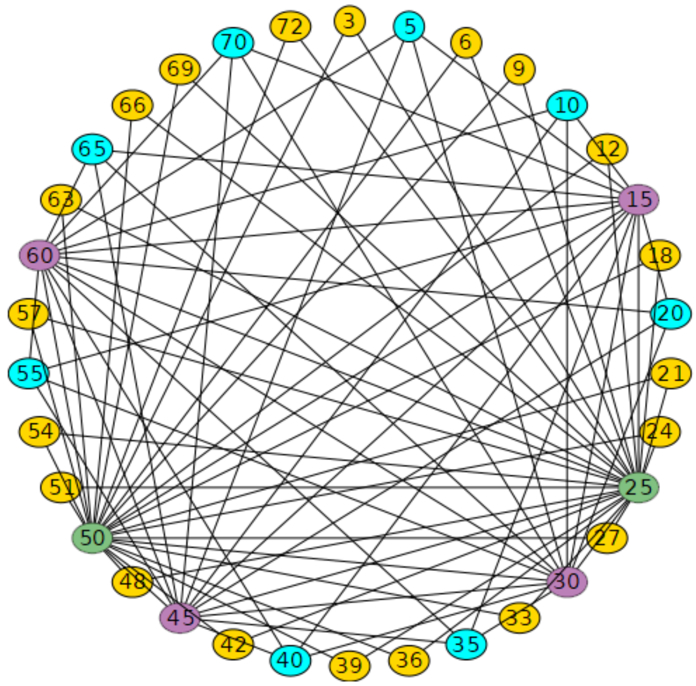


FIGURE 2. The graph $\Gamma(\mathbb{Z}_{75})$

$$\begin{aligned}|E(\Gamma(\mathbb{Z}_{75}))| = &\ \frac{1}{2}\Bigg[\left(5^2 - 5^{2-1}\right)\Bigg((3-1)\left(2 - \frac{5^{2-1}-1}{5^{2-1}(5-1)}\right) + 3(2-1) - \frac{5^{2-1}-1}{5^{2-1}(5-1)} \\ &- \frac{5^{2-1} - 5^{\lfloor \frac{2-1}{2} \rfloor}}{5^{\lfloor \frac{3(2-1)}{2} \rfloor}(5-1)}\Bigg) + (3-1)\left(5^2 - 1\right)\Bigg] \\ = &\ 86.\end{aligned}$$

Note that in Figure 2, the vertices are colored as follows: yellow represents vertices of degree two, cyan represents vertices of degree four, purple represents degree 13, and green represents degree 24. The computation of $M_1^3(\Gamma(\mathbb{Z}_{75}))$ when $\delta = 3$ can be calculated by Definition 2.1, and involves the following procedure:

$$\begin{aligned}M_1^3(\Gamma(\mathbb{Z}_{75})) &= \sum_{u \in V(\Gamma(\mathbb{Z}_{75}))} \deg(u)^3 \\ &= \deg(3)^3 + \deg(5)^3 + ... + \deg(72)^3 \\ &= 37108.\end{aligned}$$

The computation of $M_1^3\left(\Gamma\left(\mathbb{Z}_{75}\right)\right)$ by applying Theorem 3.11 as demonstrated in the following:

$$\begin{aligned}
M_1^3\left(\Gamma\left(\mathbb{Z}_{75}\right)\right) = \left(5^2-5^{2-1}\right)\Bigg[&(3-1)^3+(3-1)\left(\frac{5^2\left(5^{2(2-1)}-1\right)}{5^2-1}+3\left(2-1\right)\right. \\
&\left.-\frac{3(5)\left(5^{2-1}-1\right)}{5-1}-\frac{5^{2-1}-1}{5^{2-1}\left(5-1\right)}\right)+\frac{3^3 5^2\left(5^{2(2-1)}-1\right)}{5^2-1}+3(3)\left(2-1\right) \\
&-\frac{3(3^2)(5)\left(5^{2-1}-1\right)}{5-1}-\frac{5^{2-1}-1}{5^{2-1}\left(5-1\right)}-\frac{3(3)^2 5^{\left\lfloor\frac{2+1}{2}\right\rfloor}\left(5^{\left\lceil\frac{2-1}{2}\right\rceil}-1\right)}{5-1} \\
&+9(3)\left\lceil\frac{2-1}{2}\right\rceil-\frac{7\left(5^{2-1}-5^{\left\lfloor\frac{2-1}{2}\right\rfloor}\right)}{5^{\left\lfloor\frac{3(2-1)}{2}\right\rfloor}\left(5-1\right)}\Bigg]+\left(3-1\right)\left(5^2-1\right)^3 \\
=\ & 37108.
\end{aligned}$$

## 4. Conclusion

The set of all zero divisors, the number of zero divisors, the degree of a vertex, and the number of vertices and edges are determined in this paper. Furthermore, we computed the general formulas of the first general Zagreb index $\Gamma\left(\mathbb{Z}_{pq^k}\right)$ for the cases $\delta = 1, 2$ and $3$. In addition, we provided two examples of the first general Zagreb index for $\Gamma\left(\mathbb{Z}_{54}\right)$ and $\Gamma\left(\mathbb{Z}_{75}\right)$ to illustrate the findings.

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## References

[1] A. Abdussakir, L. A. Puspitasari, W. H. Irawan, and E. Alisah. *Eccentric connectivity index of identity graph of the cyclic group and finite commutative ring with unity*. J. Phys. Conf. Ser., **1375**, no. 1 (2019) p. 012067.
[2] D. F. Anderson and P. S. Livingston. *The zero-divisor graph of a commutative ring*. J. Algebra, **217**, no. 2 (1999): 434–447.
[3] R. Anjum, M. U. Mirza, and N. Niaz. *Generalized $\rho$-dependent polynomials of topological indices of the identity graph for the ring $\mathbb{Z}_\rho$*. Punjab Univ. J. Math., **55**, no. 9-10 (2023): 397–421.
[4] T. Asir and V. Rabikka. *The Wiener index of the zero-divisor graph of $\mathbb{Z}_n$*. Discrete Appl. Math. (1979), **319**, (2022): 461–471.
[5] X. Chen. *General sum-connectivity index of a graph and its line graph*. Appl. Math. Comput., **443**, (2023) p. 127779.
[6] S. Elumalai and T. Mansour. *On the general zeroth-order Randić index of bargraphs*. Discrete Math. Lett., **2**, (2019): 6–9.
[7] J. B. Fraleigh. *A first course in abstract algebra*. Pearson Education India, 2003.
[8] B. Furtula and I. Gutman. *A forgotten topological index*. J. Math. Chem., **53**, no. 4 (2015): 1184–1190.

[9] D. S. Gunderson. *Handbook of mathematical induction*. Chapman and Hall, 2014.

[10] I. Gutman and K. C. Das. *The first Zagreb index 30 years after*, MATCH Commun. Math. Comput. Chem, **50**, no. 1 (2004): 83–92.

[11] A. Gürsoy, A. Ülker, and N. K. Gürsoy. *Sombor index of zero-divisor graphs of commutative rings*. Analele ştiinţifice ale Universităţii" Ovidius" Constanţa, Seria Matematică, **30**, no. 2 (2022): 231–257.

[12] A. Gürsoy, N. K. Gürsoy, and A. Ülker. *Computing forgotten topological index of zero-divisor graphs of commutative rings*. Turk. J. Math., **46**, no. 5 (2022): 1845–1863.

[13] R. Ismail, F. Ali, R. Qasim, M. Naeem, W. K. Mashwani, and S. Khan. *Several Zagreb indices of power graphs of finite non-abelian groups*. Heliyon, (2023).

[14] M. K. Jamil, A. Javed, E. Bonyah, and I. Zaman. *Some upper bounds on the first general Zagreb index*. Journal of Mathematics, **2022**, (2022): 1–4.

[15] R. Juliana. *Karakteristik graf pembagi nol pada gelanggang bilangan bulat modulo*, Fraktal: Jurnal Matematika dan Pendidikan Matematika, **3**, no. 2 (2022): 1–8.

[16] A. N. A. Koam, A. Ahmad, A. Haider, and M. A. Ansari. *Computation of eccentric topological indices of zero-divisor graphs based on their edges*. AIMS Mathematics, **7**, no. 7 (2022): 11509–11518.

[17] A. Kuppan and J. R. Sankar. *Fuzzy zero divisor graph in a commutative ring*. TWMS J. of Apl. & Eng. Math., **11**, (2021): 42–50.

[18] X. Li and J. Zheng. *A unified approach to the extremal trees for different indices*. MATCH Commun. Math. Comput. Chem., **54**, no. 1 (2005): 195–208.

[19] J.-B. Liu, S. Javed, M. Javaid, and K. Shabbir. *Computing first general Zagreb index of operations on graphs*. IEEE access, **7**, (2019): 47494–47502.

[20] M. Matejic, Ş. Altındağ, E. Milovanovic, and I. Milovanovic. *On the bounds of zeroth-order general Randić index*. Filomat, **36**, no. 19 (2022): 6443–6456.

[21] S. Mondal, M. Imran, N. De, and A. Pal. *Topological indices of total graph and zero divisor graph of commutative ring: a polynomial approach*. Complexity, **2023**, (2023): 1–16.

[22] J. Á. J. Morales, J. R. Valencia, R. J. Morales, and G. R. Hernández. *On the offensive alliance number for the zero divisor graph of* $\mathbb{Z}_n$. Mathematical Biosciences and Engineering, **20**, no. 7 (2023): 12118–12129.

[23] F. Movahedi and M. H. Akhbari. *Some graph parameters of the zero-divisor graphs of finite commutative rings*. J. Math. Ext., **17**, (2023).

[24] Z. S. Mufti, A. Wajid, T. U. Islam, and N. Ali. *A consequential computation of degree-based topological indices of grasmere geometric graph*. Punjab Univ. J. Math., **52**, no. 8 (2020).

[25] H. S. Ramane, I. Gutman, K. Bhajantri, and D. V. Kitturmath. *Sombor index of some graph transformations*. Communications in Combinatorics and Optimization, **8**, no. 1 (2023): 193–205.

[26] C. J. Rayer and R. S. Jeyaraj. *Applications on topological indices of zero-divisor graph associated with commutative rings*. Symmetry (Basel), **15**, no. 2 (2023) p. 335.

[27] N. H. Sarmin, N. I. Alimon, and A. Erfanian. *Topological indices of the non-commuting graph for generalized quaternion group*. Bulletin of the Malaysian Mathematical Sciences Society, **43**, no. 5 (2020): 3361–3367.

[28] I. Zaman and F. M. Bhatti. *On the Structural Properties and Some Topological Indices of Young-Fibonacci Graphs*. Punjab Univ. J. Math., **54**, no. 12, (2023): 723–737.