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Existence and Ulam stability of first order nonsingular impulsive and delay integro differential system using a concept of the delayed matrix exponential

Bakhtawar Pervaiz[1], Akbar Zada[2], Ioan-Lucian Popa[3] and Afef Kallekh[4]
[1,2]Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan.
[3]Department of Computing, Mathematics and Electronics, "1 Decembrie 1918" University of Alba Iulia, Alba Iulia, 510 0 09, Romania.
[3]Faculty of Mathematics and Computer Science, Transilvania University of Brasov, Iuliu Maniu Street 50, 500091 Brasov, Romania.
[4]Department of mathematics, Faculty of Science, King Khalid University, Abha 61413, Saudi Arabia.
Corresponding author: Akbar Zada (Email: akbarzada@uop.edu.pk)
E-mail: [1]bakhtawarpervaiz77@gmail.com; [2]zadababo@yahoo.com [3]lucian.popa@uab.ro; [4]aklac@kku.edu.sa

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Abstract. In this article, we discuss the existence and uniqueness of the solution along with β–Hyers–Ulam–Rassias stability of nonsingular impulsive delay integro differential system via the fixed point method. The aforementioned investigations are carried out on compact intervals and then the results are extended to unbounded intervals. We utilize Banach fixed point theorem and Grönwall's type inequality as a main tool to achieve our desired objectives. At the end, we provide an example along with graphical representation to verify the applicability of the reported results.

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1. Introduction

In 1940, at Wisconsin University, Ulam for the first time put forwarded a question concerning the relation of approximate solution between homomorphisms [18]. Hyers positively replied to the question over the Banach spaces [7]. This famous theory was extended by Aoki [2] as well as Rassias [15]. In these articles the authors studied the norm of differences and Cauchy differences, $g(s+t) - g(s) - g(t)$, respectively. Answers to Ulam's question, its attractions and inductions for various situations, gave birth to an important area of research known as Hyers-Ulam stability. For detailed information about this type of stability, we recommend [4, 12] and reference therein. Delay differential systems play a crucial role to characterize evolution problems in physiological systems, control theory and automatic engines etc. [17]. Boichuk *et al.,* examined Fredholms boundary–value problems for differential systems with a single delay [5]. Khusainov and Shuklin established an idea of matrix exponential with delay and utilized the concept to obtain the solutions of linear problems with delay under the restriction of permutable matrices [8]. Diblik and Khusainov utilized the notion of [6] to establish a discrete matrix exponential map with delay. Motivated from the work done in [6, 8] different researchers studied first order and fractional order delay differential problems, for more details we recommend [10, 11, 14]. There are many implications of differential systems. However, depending on the situations, a small change in the real world can endure differential systems with abrupt deviation, e.g. heart beats, blood flows, changes in population, control theory, etc. These behaviors can not be easily scrutinized with simple differential systems. To deal with such circumstances impulsive differential systems play a crucial role, for more information see [1,3,19]. You et al. [20] extended the notion described in [13] to study exponential stability of nonsingular delayed systems of the form:

$$\begin{cases} E\mathcal{Y}'(\omega) = A\mathcal{Y}(\omega) + B\mathcal{Y}(\omega-\varrho),\ \omega \geq 0,\ \varrho \geq 0, \\ \mathcal{Y}(\omega) = g(\omega), -\varrho \leq \omega \leq 0, \end{cases}$$

$$\begin{cases} E\mathcal{Y}'(\omega) = A\mathcal{Y}(\omega) + B\mathcal{Y}(\omega-\varrho) + f(\omega,\mathcal{Y}(\omega)), \omega \geq 0, \varrho \geq 0, \\ \mathcal{Y}(\omega) = f(\omega), -\varrho \leq \omega \leq 0, \end{cases}$$

$$\begin{cases} E\mathcal{Y}'(\omega) = A\mathcal{Y}(\omega) + B\mathcal{Y}(\omega-\varrho) + g(\omega,\mathcal{Y}(\omega-\varrho)), \omega \geq 0, \varrho \geq 0, \\ \mathcal{Y}(\omega) = f(\omega), -\varrho \leq \omega \leq 0, \end{cases}$$

where E, A and B, are permutable square matrices with E is nonsingular. Furthermore, f and g are suitable functions satisfying some assertions. In addition, the authors studied the relative controllability of the system:

$$\begin{cases} E\mathcal{Y}'(\omega) = A\mathcal{Y}(\omega) + B\mathcal{Y}(\omega-\varrho) + y(\omega,\mathcal{Y}(\omega)) + CU(\omega), \omega \in J, \varrho \geq 0, \\ \mathcal{Y}(\omega) = f(\omega), -\varrho \leq \omega \leq 0. \end{cases}$$

Nowadays, Ulam's type stability is one of the central topics of research because of its fruitful applications in various fields of interest. Motivated by the work done in [20];

• We introduce some new class of first order nonsingular delay integro differential system with instantaneous impulsive effects.

• We study the existence, uniqueness of solution and β-Hyers Ulam Rassias stability (HURS).

• We utilize Gronwall lemma and fixed point techniques as basic tools to establish our main results.

• Investigations are carried out on compact intervals and then the results are generalized to unbounded intervals.

• For the applicability of our obtained theoretical results we provide simulated numerical examples at the end of the paper.

The outlines of the remaining parts of the article are as follows; In the second part of the article we establish the statement of the problem. The basics notations, lemmas, remarks and definitions are provided in the third part. The existence of solution and Ulam's type stabilities are carried out in the fourth and fifth parts of the paper respectively. In the final part we present an example for the validity of our obtained results.

2. Problem Formulation

In this section, we introduce the statement of the problem.
We examine the existence of solution and β–HURS stability for the following mentioned nonsingular impulsive integro delay differential system:

$$\begin{cases} \mathcal{B}\mathcal{E}'(\omega) = N\mathcal{E}(\omega) + A\mathcal{E}(\omega-\varrho) + \Phi\bigg(\omega, \varepsilon, \int_{\varepsilon_0}^{\varepsilon} x(\omega,\varepsilon,\mathcal{E}(\varepsilon))d\varepsilon, \int_{\varepsilon_0}^{\varepsilon} z(\omega,\varepsilon,\mathcal{E}(\varepsilon-\varrho))d\varepsilon\bigg), \\ \omega,\ \varepsilon \geq 0,\ \varrho \geq 0,\ k \in \mathbb{N}_1^m := \overline{1,m}, \\ \mathcal{E}(\omega) = \varphi(\omega), -\varrho \leq \omega \leq 0, \\ \Delta\mathcal{E}(\omega_k) = \mathcal{E}(\omega_k^+) - \mathcal{E}(\omega_k) = \Upsilon_k(\mathcal{E}(\omega_k)),\ k \in \mathbb{N}_1^m := \overline{1,m}, \end{cases} \tag{2. 1}$$

where $\mathcal{B}$, N and A represents constant permutable square matrices of finite dimension n, also A is nonsingular matrix and the operator $\varphi \in C^1([-\varrho,0],\Re^n)$. In addition, $\Phi \in \mathcal{C}([0,+\infty] \times \Re^n; \Re^n)$. Furthermore, $\mathcal{E}(\omega_k^+) = \lim_{\omega\to\omega_k^+} \mathcal{E}(\omega)$ represents the right-handed limit and $\mathcal{E}(\omega_k) = \lim_{\omega\to\omega_k^-} \mathcal{E}(\omega)$ is the left-handed limit of $\mathcal{E}(\omega)$ at $\omega = \omega_k$ with $\mathcal{E}(\omega_k^-) = \mathcal{E}(\omega_k)$ and $x, z \in \mathcal{C}([0,+\infty] \times [0,+\infty] \times \Re^n; \Re^n)$. Furthermore, $\Phi \in \mathcal{C}([0,+\infty] \times [0,+\infty] \times \Re^n \times \Re^n; \Re^n)$ and $\Upsilon_k \in \mathcal{C}([0,+\infty], \Re^n)$ are suitable functions.

3. Basics

Here we discuss essential basic definitions and concepts to establish our results. For each interval $\mathcal{J} = [0,\tau]$ the subset of $\Re$ and $\mathcal{Z} \subseteq \Re^k$, let $\mathcal{C}(\mathcal{J},\mathcal{Z})$, be the Banach space of all continuous mappings from $\mathcal{J} \to \mathcal{Z}$ endowed with norm $||\mathcal{E}||_{\mathcal{C}} = \sup_{\omega\in\mathcal{J}}\{||\mathcal{E}(\omega)||$, for all $\mathcal{E} \in \mathcal{C}(\mathcal{J},\mathcal{Z})\}$, also we denote $\mathcal{C}'(\mathcal{J},\mathcal{Z}) = \{\mathcal{E} \in \mathcal{C}(\mathcal{J},\mathcal{Z}) : \mathcal{E}' \in \mathcal{C}(\mathcal{J},\mathcal{Z})\}$. In addition, we represents the Banach space of piecewise continuous mappings by $\mathcal{PC}(\mathcal{J},\mathcal{Z}) := \{\mathcal{E} : \mathcal{J} \to \mathcal{Z}, \mathcal{E} \in \mathcal{C}((\omega_k,\omega_{k+1}),\mathcal{Z}), k \in \mathbb{N}_0^m := 0,1,\ldots,m\}$. Furthermore, there exits $\mathcal{E}(\omega_k^+),\ \mathcal{E}(\omega_k^-)$ such that $\mathcal{E}(\omega_k^+) = \mathcal{E}(\omega_k^-)$, for $k \in \mathbb{N}_0^m$, endowed with norm $\|\mathcal{E}\|_{\mathcal{PC}} = \sup\{\|\mathcal{E}(\omega)\|,\ for\ all\ \omega \in \mathcal{J}\}$.

Definition 3.1. *Let $\mathcal{U}$ be the vector space over field K, then $||.||_\beta : \mathcal{U} \to [0,\infty)$ is known as β-norm provided the following assertions are satisfied:*
$(\mathbf{a})$ $||\mathcal{E}||_\beta = 0$ *if and only if* $\mathcal{E} = 0$,
$(\mathbf{b})$ $||\eta\mathcal{E}||_\beta = |\eta|^\beta ||\mathcal{E}||_\beta$ *for every* $\eta \in K$ *and* $\mathcal{E} \in \mathcal{U}$,
$(\mathbf{c})$ $||\mathcal{E} + z||_\beta \leq ||\mathcal{E}||_\beta + ||z||_\beta$.
If the above assertions $(\mathbf{a})$, $(\mathbf{b})$ and $(\mathbf{c})$ are fulfilled, then ($\mathcal{U}$, $||.||_\beta$) is known as β–normed space.
Our space will be $P\beta$-Banach space endowed with norm $||\mathcal{E}||_{P\beta} = \sup\{||\mathcal{E}(\omega)||^\beta\}$, where $\omega \in \mathcal{J} = [0,\tau]$ and $0 < \beta < 1$. Now, we introduce $P\beta$-Banach space for this we assume the space $\mathcal{PC}(\mathcal{J},\mathcal{Z})$ and set the interval $\omega \in \mathcal{J}' = [0,\tau],\ \omega \neq \omega_k,\ k \in \mathbb{N}_1^m$.

Definition 3.2. *$\mathcal{PC}(\mathcal{J},\mathcal{Z}) := \{\mathcal{E} : \mathcal{E} \in \mathcal{C}((\omega_k, \omega_{k+1}), \mathcal{Z})\}$, there exist $\mathcal{E}(\omega_k^-)$ and $\mathcal{E}(\omega_k^+)$ such that $\mathcal{E}(\omega_k^-) = \mathcal{E}(\omega_k^+)$ for each $k \in M_0 = \{0\} \cup M$, provided $M = \{1, 2, \dots, m\}$ endowed with norm*

$$\|\mathcal{E}\|_{P\beta} = \sup\{\|\mathcal{E}(\omega)\|^{\beta}\},$$

where $\omega \in \mathcal{J}$ and $\beta \in (0,1)$. So, $(\mathcal{PC}(\mathcal{J},\mathcal{Z}), ||\cdot\|_{P\beta})$ is called $P\beta$-Banach space.

Let $\mathcal{O}$ and I be the zero and identity matrices, respectively.

Lemma 3.3. *([9], Lemma 12) Let $\mathcal{K}$ be a constant square matrix of order n. If $\|\mathcal{K}\| \leq \delta e^{\delta\vartheta}$, for $\delta \in \mathfrak{R}^+$, then*

$$\|e_{\vartheta}^{\mathcal{K}(\varsigma-\vartheta)}\| \leq e^{\delta\varsigma},\ \varsigma \in \mathfrak{R},$$

where

$$e_{\vartheta}^{\mathcal{K}\varsigma} = \begin{cases} \mathcal{O},\ \varsigma < -\vartheta, \\ I,\ -\vartheta \leq \varsigma < 0, \\ I + \mathcal{K}\varsigma + \mathcal{K}^2\frac{(\varsigma-\vartheta)^2}{2} + \cdots + \mathcal{K}^k\frac{(\varsigma-(k-1)\vartheta)^k}{k!},\ (k-1)\vartheta \leq \varsigma < k\vartheta,\ k \in \mathbb{N}_1^m, \end{cases}$$

which is called the delayed matrix exponential (see Definition 0.3 of [8]).

Remark 3.4. *([9]) The nonsingular impulsive delay integro differential system,*

$$\begin{cases} \mathcal{B}\mathcal{E}'(\omega) = N\mathcal{E}(\omega) + A\mathcal{E}(\omega-\varrho) + \Phi\left(\omega, \varepsilon, \int_{\omega_0}^{\omega} x(\omega,\varepsilon,\mathcal{E}(\varepsilon))d\varepsilon, \int_{\omega_0}^{\omega} z(\omega,\varepsilon,\mathcal{E}(\varepsilon-\varrho))d\varepsilon\right), \\ \omega,\ \varepsilon \geq 0,\ \varrho \geq 0,\ k \in \mathbb{N}_1^m := \overline{1,m}, \\ \mathcal{E}(\omega) = \varphi(\omega), -\varrho \leq \omega \leq 0, \\ \Delta\mathcal{E}(\omega_k) = \mathcal{E}(\omega_k^+) - \mathcal{E}(\omega_k) = \Upsilon_k(\mathcal{E}(\omega_k)),\ k \in \mathbb{N}_1^m, \end{cases}$$

has the solution,

$$\mathcal{E}(\omega) = \begin{cases} Z(\omega+\varrho)\phi(-\varrho) + \mathcal{B}^{-1}\int\limits_{-\varrho}^{0} Z(\omega-\varepsilon)[\mathcal{B}\dot{\phi}(\varepsilon) - N\phi(\varepsilon)]d\varsigma \\ +\mathcal{B}^{-1}\int\limits_{0}^{\omega} Z(\omega-\varepsilon)\Phi\left(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\mathcal{E}(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\mathcal{E}(u-\varrho))du\right)d\varepsilon \\ +\sum_{0\leq k\leq m} Z(\omega-\omega_k)\Upsilon_k(\mathcal{E}(\omega_k)), \end{cases}$$

where $Z(\omega) = e^{N\mathcal{B}^{-1}\omega}e_{\varrho}^{A_1\mathcal{B}^{-1}(\omega-\varrho)}$ $A_1 = e^{-NA^{-1}\varrho}B$ and $N\mathcal{B} = \mathcal{B}N$, $NA = AN$, $A\mathcal{B} = \mathcal{B}A$ are used. From above clearly $\dot{\phi}$ exists (i.e $\phi \in C^1([-\varrho, 0], \mathfrak{R}^n)$), we give the following definition.

Definition 3.5. *Trivial solution of the problem* (2. 1) *is known as exponentially stable, if there exists positive constants $\sigma_1, \sigma_2, \delta$ depending on the permutable matrices $\mathcal{B}$, M, N and $||\phi||_1$, where $||\phi||_1 := \max_{[-\varrho,0]}||\phi|| + \max_{[-\varrho,0]}||\phi||$, provided*

$$||\mathcal{E}(\omega)|| \leq \sigma_1 e^{-\sigma_2\omega},\ \omega \geq 0. \tag{3. 2}$$

Assume that $0 < \epsilon$, $0 \leq \psi$, and φ from $\mathcal{C}(\mathcal{J}, \mathcal{Z})$, one can set

$$\begin{cases} ||\mathcal{B}\mathcal{E}'(\omega) - N\mathcal{E}(\omega) - A\mathcal{E}(\omega - \varrho) \\ -\Phi\bigg(\omega, \varepsilon, \int_{\omega_0}^{\omega} x(\omega, \varepsilon, \mathcal{E}(\varepsilon))d\varepsilon, \int_{\omega_0}^{\omega} z(\omega, \varepsilon, \mathcal{E}(\varepsilon - \varrho))d\varepsilon\bigg)|| \leq \epsilon\varphi(\varepsilon),\ \omega, \varepsilon \geq 0, \varrho \geq 0, \\ ||\mathcal{E}(\omega) - \phi(\omega)|| \leq \epsilon\psi, -\varrho \leq \omega \leq 0, \\ \|\mathcal{E}(\omega_k^+) - \mathcal{E}(\omega_k) - \Upsilon_k(\mathcal{E}(\omega_k))\| \leq \epsilon\psi,\ k = \mathbb{N}_0^m. \end{cases} \tag{3. 3}$$

Using the above inequality (3. 3), one can define β–HURS for system (2. 1).

Definition 3.6. *Problem (2. 1) is β–HUR stable with respect to (w. r. t) $(\varphi^\beta, \psi^\beta)$ if there exists $\mathcal{K}_{f,\mathcal{M},\varphi,\psi,\beta} > 0$ provided for each $\epsilon > 0$ and for any solution $\mathcal{E} \in \mathcal{PC}(\mathcal{J}', \mathcal{Z}) \cap \mathcal{C}(\mathcal{J}', \mathcal{Z})$ of (3. 3) there exists Θ the solution of the problem (2. 1) in $\mathcal{PC}(\mathcal{J}', \mathcal{Z})$, satisfying*

$$\|\Theta(\omega) - \mathcal{E}(\omega)\|^\beta \leq \mathcal{K}_{\Phi,\mathcal{S},\mathcal{M},\varphi,\psi,\beta}\epsilon^\beta\Big(\varphi^\beta(\omega) + \psi^\beta(\omega)\Big),\ \omega \in \mathcal{J}.$$

Remark 3.7. *The direct consequence of problem (3. 3) is that a map $\Theta \in \mathcal{PC}(\mathcal{J}', \mathcal{Z}) \cap \mathcal{C}(\mathcal{J}', \mathcal{Z})$ is called the solution of the problem (3. 3) if and only if one can define mapping $f \in \mathcal{C}(\mathcal{J}', \mathcal{Z})$ with $\varphi \geq 0,\ \psi \geq 0$ and a sequence $\hbar_k,\ k \in \mathbb{N}_1^k$ satisfying the following inequality*

$$\begin{cases} \|f(\omega)\| \leq \epsilon\varphi(\omega),\ \|f_k\| \leq c\psi\ \ \omega \in \mathcal{I}' \text{ and } k \in M, \\ \mathcal{B}\Theta'(\omega) = N\Theta(\omega) + A\Theta(\omega - \varrho) \\ +\Phi\bigg(\omega, \varepsilon, \int_{\omega_0}^{\omega} x(\omega, \varepsilon, \mathrm{Z}(\varepsilon))d\varepsilon, \int_{\omega_0}^{\omega} z(\omega, \varepsilon, \mathrm{Z}(\varepsilon - \varrho))d\varepsilon\bigg) + f(\omega),\ \ \omega, \varepsilon \in \mathcal{I}', \\ \Theta(0) = \Theta_0 + f(\omega), \\ \Theta(\omega) = \phi(\omega), \\ \Delta\mathrm{Z}(\omega_k) = \Upsilon_k(\mathrm{Z}(\omega_k)),\ k \in \mathbb{N}_1^m. \end{cases}$$

Assume that

$$\mathcal{M} = \sup_{0 \leq \varepsilon \leq \omega \leq \tau} \|Z(\omega - \varepsilon)\|, \tag{3. 4}$$

where τ denotes the length of $[0, \tau]$.

Via Remark 3.7, the solution of the problem

$$\begin{cases} \mathcal{B}\mathcal{E}'(\omega) = N\mathcal{E}(\omega) + A\mathcal{E}(\omega - \varrho) \\ +\Phi\bigg(\omega, \varepsilon, \int_{\omega_0}^{\omega} x(\omega, \varepsilon, \mathcal{E}(\varepsilon))d\varepsilon, \int_{\omega_0}^{\omega} z(\omega, \varepsilon, \mathcal{E}(\varepsilon - \varrho))d\varepsilon\bigg) + f(\varepsilon),\ \varepsilon \geq 0, \varrho \geq 0, \\ \mathcal{E}(\varepsilon) = \phi(\varepsilon), -\varrho \leq \varepsilon \leq 0, \\ \Delta\mathcal{E}(\omega_k) = \Upsilon_k(\mathcal{E}(\omega_k)),\ k \in \mathbb{N}_1^m, \end{cases}$$

is given by

$$\begin{aligned}\mathcal{E}(\omega) &= Z(\omega+\varrho)\phi(-\varrho)+\mathcal{B}^{-1}\int\limits_{-\varrho}^{0} Z(\omega-\varepsilon)[\mathcal{B}\dot{\phi}(\varepsilon)-N\phi(\varepsilon)]d\varepsilon \\ &+\mathcal{B}^{-1}\int\limits_{0}^{\omega} Z(\omega-\varepsilon)\Bigg(\Phi\Bigg(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\mathcal{E}(u))du,\int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\mathcal{E}(u-\varrho))du\Bigg)+f(\varepsilon)\Bigg)d\varepsilon, \\ &+\sum_{0\le k\le m} Z(\omega-\omega_k)\Upsilon_k(\mathcal{E}(\omega_k)),\end{aligned}$$

where $Z(\omega)=e^{N\mathcal{B}^{-1}\omega}e_{\varrho}^{A_1\mathcal{B}^{-1}(\omega-\varrho)}$ $N_1=e^{-N\mathcal{B}^{-1}\varrho}B$ and $M\mathcal{B}=\mathcal{B}N$, $NA=AN$, $A\mathcal{B}=\mathcal{B}A$.
For the inequality (3. 3), we have

$$\begin{aligned}&\|\mathcal{E}(\omega)-Z(\omega+\varrho)\phi(-\varrho)-\mathcal{B}^{-1}\int\limits_{-\varrho}^{0} Z(\omega-\varepsilon)[\mathcal{B}\dot{\phi}(\varepsilon)-A\phi(\varepsilon)]d\varepsilon \\ &-\mathcal{B}^{-1}\int\limits_{0}^{\omega} Z(\omega-\varepsilon)\Bigg(\Phi\Bigg(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\mathcal{E}(u))du,\int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\mathcal{E}(u-\varrho))du\Bigg)\Bigg)d\varepsilon\| \\ &-\sum_{0\le k\le m}\|Z(\omega-\omega_k)\Upsilon_k(\mathcal{E}(\omega_k))\| \\ =\ &\|\mathcal{B}^{-1}\int_0^{\omega} Z(\omega-\varepsilon)f(\varepsilon)d\varepsilon\|+\sum_{0\le k\le m}\|Z(\omega-\omega_k)\hbar_k\| \\ \le\ &\|\mathcal{B}^{-1}\|\int_0^{\omega}\|Z(\omega-\varepsilon)\|\|f(\varepsilon)\|d\varepsilon+\sum_{0\le k\le m}\|Z(\omega-\omega_k)\|\|\hbar_k\| \\ \le\ &\mathcal{S}\int\limits_{0}^{\omega}\mathcal{M}\|f(\varepsilon)\|d\varepsilon+\sum_{0\le k\le m}\epsilon\mathcal{M}\psi \\ \le\ &\epsilon\mathcal{M}\Bigg(m\psi+\mathcal{S}\int\limits_{0}^{\omega}\varphi(\varepsilon)d\varepsilon\Bigg).\end{aligned}$$

Now, for our main result we state the well known Gronwall lemma.

Lemma 3.8. *([16]) For every $\omega\ge 0$ provided*

$$\mathfrak{u}(\omega) \le q(\omega)+\int\limits_{0}^{\omega} p(\varepsilon)\mathfrak{u}(\varepsilon)d\varepsilon+\sum_{0<\omega_k<\omega}\gamma_k\mathfrak{u}(\omega_k^-), \qquad (3.\ 5)$$

where q is nondecreasing and $\mathfrak{u}$, q, p belong to $\mathcal{PC}(\Re^+, \Re^+)$, and γ is positive constant, then for $\omega \in \Re^+$, one can see:

$$\mathfrak{u}(\omega) \quad \leq \quad q(\omega)\Big(1+\gamma_k\Big)^k \exp\Big(\int_0^{\omega} p(\varepsilon)d\varepsilon\Big), \ where\ k \in \mathbb{N}_1^m. \tag{3. 6}$$

Remark 3.9. *If we replace γ_k by $\gamma_k(\omega)$, then*

$$\mathfrak{u}(\omega) \quad \leq \quad q(\omega) \prod_{0<\omega_k<\omega} \Big(1+\gamma_k(\omega)\Big) \exp\Big(\int_0^{\omega} p(\varepsilon)d\varepsilon\Big), \ where\ k \in M. \tag{3. 7}$$

Definition 3.10. *The map f from $\mathcal{X}$ to $\mathcal{X}$, is said to be contraction if for each $\varrho, z \in \mathcal{X}$, there exists $k \in [0,1)$ such that*

$$d(f(\varrho), f(z)) \leq kd(\varrho, z),$$

where $(\mathcal{X}, d)$ represents the metric space.

Definition 3.11. *The mapping $f : \mathcal{X} \to \mathcal{X}$, in the complete metric space $(\mathcal{X}, d)$ has a unique fixed point if it is a contraction.*

4. Existence of Solution

To examine existence of solution for the considered problem, we establish the following conditions:
$[D_1]$: The linear problem, $\mathcal{B}\mathcal{E}'(\omega) = M\mathcal{E}(\omega) + N\mathcal{E}(\omega - \varrho)$ is well posed.
$[D_2]$: $\Phi : \mathcal{J} \times \mathcal{J} \times \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}$ fulfills the Caratheodory assumptions and there exists constant $\mathcal{L}_\Phi > 0$ provided

$$\|\Phi(\omega, \varepsilon, z_1, z_2) - \Phi(\omega, \varepsilon, z_1', z_2')\| \leq \sum_{i=1}^{2} \mathcal{L}_\Phi \|z_i - z_i'\|,$$

for every $z_i, z_i' \in \mathcal{Z}$.
$[D_3]$: $\Upsilon_k \in \mathcal{C}(\mathcal{J}, \mathcal{Z}) \to \mathcal{Z}$, $k \in \mathbb{N}_1^m$, there exists constants $\mathcal{L}_{\Upsilon_k} > 0$ provided

$$\|\Upsilon_k(\mathcal{E}(\omega_k)) - \Upsilon_k(\mathcal{E}(\omega_k^{'}))\| \leq \mathcal{L}_{\Upsilon_k}\|\omega_k - \omega_k^{'}\|, \tag{4. 8}$$

for every $\omega_k,\ \omega_k^{'}$.
$[D_4]$: $\left(\mathcal{S}\mathcal{M}\mathcal{L}_\Phi \tau^2(\mathcal{L}_x + \mathcal{L}_z) + \sum_{k=1}^{m} \mathcal{M}\mathcal{L}_{\Upsilon_k}\right) < 1$ is satisfied.
Now, we study that the nonsingular impulsive delay integro differential problem (2. 1) has unique solution.

Theorem 4.1. *System (2. 1) has unique solution $\mathcal{E} \in \mathcal{PC}(\mathcal{J}, \mathcal{Z})$ provided the assertions $[D_1] - [D_4]$ with assertion (3. 4) are satisfied.*

Proof. Let $\mathcal{G}:\mathcal{PC}(\mathcal{J},\mathcal{Z})\to\mathcal{PC}(\mathcal{J},\mathcal{Z})$ be defined as:

$$\begin{aligned}(\mathcal{G}\mathcal{E})(\omega) \quad = \quad & Z(\omega+\varrho)\phi(-\varrho)+\mathcal{B}^{-1}\int_{-\varrho}^{0} Z(\omega-\varepsilon)[\mathcal{B}\dot{\phi}(\varepsilon)-M\phi(\varepsilon)]d\varepsilon \\ & +\mathcal{B}^{-1}\int_{0}^{\omega} Z(\omega-\varepsilon)\bigg(\Phi\bigg(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\mathcal{E}(u))du,\int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\mathcal{E}(u-\varrho))du\bigg)\bigg)d\varepsilon \\ & +\sum_{0\le k\le m} Z(\omega-\omega_k)\Upsilon_k(\mathcal{E}(\omega_k)).\end{aligned}$$

Now, for every $\mathcal{E},\mathcal{H}\in\mathcal{PC}(\mathcal{J},\mathcal{Z})$, one can see

$$\begin{aligned} & \|(\mathcal{G}\mathcal{E})(\omega)-(\mathcal{G}\mathcal{H})(\omega)\| \\ \le \quad & \|\mathcal{B}^{-1}\|\int_{0}^{\omega}\|Z(\omega-\varepsilon)\| \\ & \times\bigg\|\Phi\bigg(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\mathcal{E}(u))du,\int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\mathcal{E}(u-\varrho))du\bigg) \\ & -\Phi\bigg(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\mathcal{H}(u))du,\int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\mathcal{H}(u-\varrho))du\bigg)\bigg\|d\varepsilon \\ & +\sum_{0\le k\le m}\|Z(\omega-\omega_k)\|\|\Upsilon_k(\mathcal{E}(\omega_k))-\Upsilon_k(\mathcal{H}(\omega_k))\| \\ \le \quad & \mathcal{S}\int_{0}^{\omega}\mathcal{M}\mathcal{L}_{\Phi}\|\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\mathcal{E}(u))du-\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\mathcal{H}(u))du\|d\varepsilon \\ & +\mathcal{S}\int_{0}^{\omega}\mathcal{M}\mathcal{L}_{\Phi}\|\int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\mathcal{E}(u-\varrho))du-\int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\mathcal{H}(u-\varrho))du\|d\varepsilon \\ & +\sum_{0\le k\le m}\mathcal{M}\mathcal{L}_{\Upsilon_k}\|\mathcal{E}(\omega_k)-\mathcal{H}(\omega_k)\| \\ \le \quad & \mathcal{S}\int_{0}^{\omega}\mathcal{M}\mathcal{L}_{\Phi}\mathcal{L}_x\bigg(\int_{\varepsilon_0}^{\varepsilon}\|\mathcal{E}(u)-\mathcal{H}(u)\|du\bigg)d\varepsilon \\ & +\mathcal{S}\int_{0}^{\omega}\mathcal{M}\mathcal{L}_{\Phi}\mathcal{L}_z\bigg(\int_{\varepsilon_0}^{\varepsilon}\|\mathcal{E}(u-\varrho)-\mathcal{H}(u-\varrho)\|du\bigg)d\varepsilon \\ & +\sum_{0\le k\le m}\mathcal{M}\mathcal{L}_{\Upsilon_k}\|\mathcal{E}(\omega_k)-\mathcal{H}(\omega_k)\| \\ \le \quad & \mathcal{S}\int_{0}^{\omega}\mathcal{M}\mathcal{L}_{\Phi}\mathcal{L}_x\bigg(\int_{\varepsilon_0}^{\varepsilon}\|\mathcal{E}-\mathcal{H}\|_{\mathcal{PC}}du\bigg)d\varepsilon\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{S}\int_0^{\omega}\mathcal{M}\mathcal{L}_{\Phi}\mathcal{L}_z\bigg(\int_{\varepsilon_0}^{\varepsilon}\|\mathcal{E}-\mathcal{H}\|_{\mathcal{PC}}du\bigg)d\varepsilon\\
&+\sum_{0\le k\le m}\mathcal{M}\mathcal{L}_{\Upsilon_k}\|\mathcal{E}-\mathcal{H}\|_{\mathcal{PC}}\\
\le\;& \mathcal{S}\int_0^{\omega}\mathcal{M}\mathcal{L}_{\Phi}\mathcal{L}_x\bigg(\varepsilon\|\mathcal{E}-\mathcal{H}\|_{\mathcal{PC}}\bigg)d\varepsilon\\
&+\mathcal{S}\int_0^{\omega}\mathcal{M}\mathcal{L}_{\Phi}\mathcal{L}_z\bigg(\varepsilon\|\mathcal{E}-\mathcal{H}\|_{\mathcal{PC}}\bigg)d\varepsilon\\
&+\sum_{0\le k\le m}\mathcal{M}\mathcal{L}_{\Upsilon_k}\|\mathcal{E}-\mathcal{H}\|_{\mathcal{PC}}\\
\le\;& \mathcal{S}\mathcal{M}\mathcal{L}_{\Phi}\mathcal{L}_x\tau^2\|\mathcal{E}-\mathcal{H}\|_{\mathcal{PC}}+\mathcal{S}\mathcal{M}\mathcal{L}_{\Phi}\mathcal{L}_z\tau^2\|\mathcal{E}-\mathcal{H}\|_{\mathcal{PC}}+\sum_{k=1}^{m}\mathcal{M}\mathcal{L}_{\Upsilon_k}\|\mathcal{E}-\mathcal{H}\|_{\mathcal{PC}}\\
=\;& \bigg(\mathcal{S}\mathcal{M}\mathcal{L}_{\Phi}\tau^2(\mathcal{L}_x+\mathcal{L}_z)+\sum_{k=1}^{m}\mathcal{M}\mathcal{L}_{\Upsilon_k}\bigg)\|\mathcal{E}-\mathcal{H}\|_{\mathcal{PC}}\\
<\;& \|\mathcal{E}-\mathcal{H}\|_{\mathcal{PC}},
\end{aligned}
$$

where τ is the length of the interval $\mathcal{J}$. Implies, $\mathcal{G}$ is contractive map w. r. t $\|.\|_{\mathcal{PC}}$. Hence, utilizing contraction theorem, the map $\mathcal{G}$ has a unique fixed point which is clearly solution of the problem (2. 1). □

5. β–HURS ON A COMPACT INTERVAL

To examine β–HURS of problem (2. 1) on a compact interval, we establish some other assertions along with aforementioned assertions (D_1), (D_2) and (D_3). The assertions are:
$[D_2^*]$: The mapping $f:\mathcal{J}\times\mathcal{J}\times\mathcal{Z}\times\mathcal{Z}\to\mathcal{Z}$ which holds Caratheodory assertions and there exits mappings $\mathcal{L}_f\in\mathcal{C}(\mathcal{J},\mathcal{Z})$ such that

$$\|f(\omega,\varepsilon,\varrho_1,\varrho_2)-f(\omega,\varepsilon,\varrho_1',\varrho_2')\|\le\sum_{i=1}^{2}\mathcal{L}_f(\omega)\|\varrho_i-\varrho_i'\|,$$

for every $\omega,\varepsilon\in\mathcal{J}$ and $\varrho_i,\varrho_i'\in\mathcal{Z},\ i=1,2.$
$[D_5]$: There exists $\varphi\in\mathcal{PC}(\mathcal{J},\mathcal{Z})$ a non decreasing map along with $\varphi(\omega)\ge 0$ and η_φ is a positive constant provided

$$\int_0^{\omega}\varphi(\varepsilon)d\varepsilon\le\eta_\varphi\varphi(\omega),\ \forall\,\omega\in\mathcal{J}.$$

Utilizing inequality (3. 3) and above- mentioned assertions, we study the following result.

Theorem 5.1. *System (2. 1) has β–HURS w. r. t $(\psi^\beta,\varphi^\beta)$ on a compact interval provided $[D_1],[D_2^*]$ along with $[D_3]-[D_5]$ and (3. 4) are satisfied.*

Proof. The nonsingular impulsive and delay integro differential system of the form

$$\begin{cases} \mathcal{B}\mathcal{E}'(\omega) = M\mathcal{E}(\omega) + N\mathcal{E}(\omega-\varrho) + \Phi\left(\omega,\varepsilon,\int_{\omega_0}^{\omega} x(\omega,\varepsilon,\mathcal{E}(\varepsilon))d\varepsilon, \int_{\omega_0}^{\omega} z(\omega,\varepsilon,\mathcal{E}(\varepsilon-\varrho))d\varepsilon\right) \\ +f(\omega), \omega,\ \varepsilon \in [0,\tau],\ \omega,\ \varepsilon \geq 0, \varrho \geq 0, \\ \mathcal{E}(\omega) = \phi(\omega), -\varrho \leq \omega \leq 0, \\ \Delta\mathcal{E}(\omega_k) = \mathcal{E}(\omega_k^+) - \mathcal{E}(\omega_k) = \Upsilon_k(\mathcal{E}(\omega_k)),\ k \in \mathbb{N}_1^m. \end{cases}$$

has a unique solution

$$\mathcal{E}(\omega) = \begin{cases} Z(\omega+\varrho)\phi(-\varrho) + \mathcal{B}^{-1}\int\limits_{-\varrho}^{0} Z(\omega-\varepsilon)[\mathcal{B}\dot{\phi}(\varepsilon) - M\phi(\varepsilon)]d\varepsilon \\ +\mathcal{B}^{-1}\int\limits_{0}^{\omega} Z(\omega-\varepsilon)\left(\Phi\left(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\mathcal{E}(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\mathcal{E}(u-\varrho))du\right) + f(\omega)\right)d\varepsilon \\ +\sum_{0<k<m} Z(\omega-\omega_k)\Upsilon_k(\mathcal{E}(\omega_k)), \end{cases}$$

where $Z(\omega) = e^{M\mathcal{B}^{-1}\omega} e_{\varrho}^{N_1\mathcal{B}^{-1}(\omega-\varrho)},\ N_1 = e^{-M\mathcal{B}^{-1}\varrho}B$ and $M\mathcal{B} = \mathcal{B}M$, $MN = NM$, $N\mathcal{B} = \mathcal{B}N$.

Let Θ be the solution of the inequality (3. 3). Then, for every $\omega, \varepsilon \geq 0$, we can obtain that,

$$\begin{aligned} & \|\Theta(\omega) - Z(\omega+\varrho)\phi(-\varrho) - \mathcal{B}^{-1}\int\limits_{-\varrho}^{0} Z(\omega-\varepsilon)[\mathcal{B}\dot{\phi}(\varepsilon) - M\phi(\varepsilon)]d\varepsilon \\ & -\mathcal{B}^{-1}\int\limits_{0}^{\omega} Z(\omega-\varepsilon)\Phi\left(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\mathcal{E}(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\mathcal{E}(u-\varrho))du\right)d\varepsilon \\ & -\sum_{0<k<m} Z(\omega-\omega_k)\Upsilon_k(\mathcal{E}(\omega_k))\| \\ \leq\ & \mathcal{S}\int\limits_{0}^{\omega} \epsilon\mathcal{M}\varphi(\varepsilon)d\varepsilon + m\epsilon\mathcal{M}\psi \\ \leq\ & \epsilon\mathcal{M}\left(m\psi + \mathcal{S}\int\limits_{0}^{\omega}\varphi(\varepsilon)d\varepsilon\right) \\ \leq\ & \epsilon\mathcal{M}(m+\eta_{\varphi})(\mathcal{S}\varphi(\omega)+\psi). \end{aligned}$$

Therefore, for every $\omega, \varepsilon \in (\omega_k, \omega_{k+1}]$, one can see

$$\begin{aligned}
& \|\Theta(\omega) - \mathcal{E}(\omega)\|^\beta \\
= \; & \|\Theta(\omega) - Z(\omega+\varrho)\phi(-\varrho) - \mathcal{B}^{-1}\int_{-\varrho}^{0} Z(\omega-\varepsilon)[\mathcal{B}\dot{\phi}(\varepsilon) - M\phi(\varepsilon)]d\varepsilon \\
& -\mathcal{B}^{-1}\int_{0}^{\omega} Z(\omega-\varepsilon) \times \Phi\Big(\omega, \varepsilon, \int_{\varepsilon_0}^{\varepsilon} x(\varepsilon, u, \mathcal{E}(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon, u, \mathcal{E}(u-\varrho))du\Big)d\varepsilon \\
& - \sum_{0<k<m} Z(\omega-\omega_k)\Upsilon_k(\mathcal{E}(\omega_k))\|^\beta \\
= \; & \|\Theta(\omega) - Z(\omega+\varrho)\phi(-\varrho) - \mathcal{B}^{-1}\int_{-\varrho}^{0} Z(\omega-\varepsilon)[\mathcal{B}\dot{\phi}(\varepsilon) - M\phi(\varepsilon)]d\varepsilon \\
& -\mathcal{B}^{-1}\int_{0}^{\omega} Z(\omega-\varepsilon)\Phi\Big(\omega, \varepsilon, \int_{\varepsilon_0}^{\varepsilon} x(\varepsilon, u, \mathcal{E}(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon, u, \mathcal{E}(u-\varrho))du\Big)d\varepsilon \\
& +\mathcal{B}^{-1}\int_{0}^{\varepsilon} Z(\omega-\varepsilon)\Phi\Big(\omega, \varepsilon, \int_{\varepsilon_0}^{\varepsilon} x(\varepsilon, u, \mathcal{E}(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon, u, \mathcal{E}(u-\varrho))du\Big)d\varepsilon \\
& -\mathcal{B}^{-1}\int_{0}^{\varepsilon} Z(\omega-\varepsilon)\Phi\Big(\omega, \varepsilon, \int_{\varepsilon_0}^{\varepsilon} x(\varepsilon, u, \mathcal{E}(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon, u, \mathcal{E}(u-\varrho))du\Big)d\varepsilon \\
& - \sum_{0<k<m} Z(\omega-\omega_k)\Upsilon_k(\mathcal{E}(\omega_k)) + \sum_{0<k<m} Z(\omega-\omega_k)\Upsilon_k(\mathcal{E}(\omega_k)) \\
& - \sum_{0<k<m} Z(\omega-\omega_k)\Upsilon_k(\mathcal{E}(\omega_k))\|^\beta \\
\leq \; & \Big(\|\Theta(\omega) - Z(\omega+\varrho)\phi(-\varrho) - \mathcal{B}^{-1}\int_{-\varrho}^{0} Z(\omega-\varepsilon)[\mathcal{B}\dot{\phi}(\varepsilon) - M\phi(\varepsilon)]d\varepsilon \\
& -\mathcal{B}^{-1}\int_{0}^{\omega} Z(\omega-\varepsilon)\Phi\Big(\omega, \varepsilon, \int_{\varepsilon_0}^{\varepsilon} x(\varepsilon, u, \mathrm{Z}(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon, u, \mathrm{Z}(u-\varrho))du\Big)d\varepsilon \\
& - \sum_{0<k<m} Z(\omega-\omega_k)\Upsilon_k(\mathcal{E}(\omega_k))\|\Big)^\beta \\
& +\Big(\|\mathcal{B}^{-1}\int_{0}^{\omega} Z(\omega-\varepsilon)\Big(\Phi\big(\omega, \varepsilon, \int_{\varepsilon_0}^{\varepsilon} x(\varepsilon, u, \mathrm{Z}(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon, u, \mathrm{Z}(u-\varrho))du\big) \\
& -\Phi\big(\omega, \varepsilon, \int_{\varepsilon_0}^{\varepsilon} x(\varepsilon, u, \mathcal{E}(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon, u, \mathcal{E}(u-\varrho))du\big)\|\Big)^\beta d\varepsilon \\
& +\Big(\sum_{0<k<m} \|Z(\omega-\omega_k)\Upsilon_k(\mathrm{Z}(\omega_k)) - Z(\omega-\omega_k)\Upsilon_k(\mathcal{E}(\omega_k))\|\Big)^\beta
\end{aligned}$$

$$
\begin{aligned}
\leq\quad & \Big(\epsilon\mathcal{M}(m+\eta_{\varphi})(\mathcal{S}\varphi(\omega)+\psi)\Big)^{\beta}+\Big(\|\mathcal{B}^{-1}\|\int\limits_{0}^{\omega}\|Z(\omega-\varepsilon)\| \\
& \times\big|\big|\Phi\big(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon}x(\varepsilon,u,\mathrm{Z}(u))du,\int_{\varepsilon_0}^{\varepsilon}z(\varepsilon,u,\mathrm{Z}(u-\varrho))du\big) \\
& -\Phi\big(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon}x(\varepsilon,u,\mathcal{E}(u))du,\int_{\varepsilon_0}^{\varepsilon}z(\varepsilon,u,\mathcal{E}(u-\varrho))du\big)\big|\big|d\varsigma\Big)^{\beta} \\
& +\Big(\sum_{k=1}^{m}\|Z(\omega-\omega_k)\Upsilon_k(\mathrm{Z}(\omega_k))-Z(\omega-\omega_k)\Upsilon_k(\mathcal{E}(\omega_k))\|\Big)^{\beta},
\end{aligned}
$$

where

$$
\begin{aligned}
& \int\limits_{0}^{\omega}\|Z(\omega-\varepsilon)\|\big|\big|\Phi\big(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon}x(\varepsilon,u,\mathrm{Z}(u))du,\int_{\varepsilon_0}^{\varepsilon}z(\varepsilon,u,\mathrm{Z}(u-\varrho))du\big) \\
& -\Phi\big(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon}x(\varepsilon,u,\mathrm{Z}(u))du,\int_{\varepsilon_0}^{\varepsilon}z(\varepsilon,u,\mathrm{Z}(u-\varrho))du\big)\big|\big|d\varsigma \\
\leq\quad & \int\limits_{0}^{\omega}\mathcal{M}\mathcal{L}_{\Phi}(\varepsilon)\mathcal{L}_{x}(\varepsilon)\|\Theta(\varepsilon)-\mathcal{E}(\varepsilon)\|d\varepsilon+\int\limits_{0}^{\omega}\mathcal{M}\mathcal{L}_{\Phi}(\varepsilon)\mathcal{L}_{z}(\varepsilon)\|\Theta(\varepsilon)-\mathcal{E}(\varepsilon)\|d\varepsilon.
\end{aligned}
$$

And

$$
\sum_{k=1}^{m}\|Z(\omega-\omega_k)\Upsilon_k(\mathrm{Z}(\omega_k))-Z(\omega-\omega_k)\Upsilon_k(\mathcal{E}(\omega_k))\|\leq\mathcal{M}\sum_{k=1}^{m}\mathcal{L}_{\Upsilon_k}\|\mathrm{Z}(\omega_k)-\mathcal{E}(\omega_k)\|
$$

Thus,

$$
\begin{aligned}
& \|\Theta(\omega)-\mathcal{E}(\omega)\| \\
\leq\quad & 3^{\frac{1}{\beta}-1}\Bigg(\Big(\epsilon\mathcal{M}(m+\eta_{\varphi})(\mathcal{S}\varphi(\omega)+\psi)\Big) \\
& +\Big(\mathcal{S}\int\limits_{0}^{\omega}\mathcal{M}\mathcal{L}_{\Phi}(\varepsilon)\mathcal{L}_{x}(\varepsilon)\|\Theta(\varepsilon)-\mathcal{E}(\varepsilon)\|d\varepsilon+\mathcal{S}\int\limits_{0}^{\omega}\mathcal{M}\mathcal{L}_{\Phi}(\varepsilon)\mathcal{L}_{z}(\varepsilon)\|\Theta(\varepsilon)-\mathcal{E}(\varepsilon)\|d\varepsilon\Big) \\
& +\mathcal{M}\sum_{k=1}^{m}\mathcal{L}_{\Upsilon_k}\|\mathrm{Z}(\omega_k)-\mathcal{E}(\omega_k)\|\Bigg).
\end{aligned}
$$

By utilizing the relation

$$
(p+q+r)^{\gamma}\leq 3^{\gamma-1}(p^{\gamma}+q^{\gamma}+r^{\gamma}),\ \text{where } p,\,q,\,r\geq 0,\ \text{and } \gamma>1.
$$

Taking

$$
\mathcal{L}_{\Phi}(\mathcal{L}_p+\mathcal{L}_r)=\mathcal{L}_{\Phi,p,r}.
$$

And setting, $\mathcal{L}_\Upsilon = \max\{\mathcal{L}_{\Upsilon_1}, \mathcal{L}_{\Upsilon_2}, \ldots, \mathcal{L}_{\Upsilon_m}\}$. By using Grönwall's Lemma 3.8, we get,

$$\begin{aligned} & \|\Theta(\omega) - \mathcal{E}(\omega)\| \\ \leq \quad & 3^{\frac{1}{\beta}-1}\Bigg(\Big(\epsilon\mathcal{M}(m+\eta_\varphi)(\mathcal{S}\varphi(\omega)+\psi)\Big)\Bigg)\Bigg(1+3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_\Upsilon\Bigg)^k \\ & \exp\Big(3^{\frac{1}{\beta}-1}\mathcal{S}\int_0^\omega \mathcal{M}\mathcal{L}_{\Phi,p,r}(\varepsilon)d\varepsilon\Big). \end{aligned}$$

Therefore,

$$\begin{aligned} & \|\Theta(\omega) - \mathcal{E}(\omega)\|^\beta \\ \leq \quad & 3^{1-\beta}\Big(\epsilon\mathcal{M}(m+\eta_\varphi)(\mathcal{S}\varphi(\omega)+\psi)\Big)^\beta \\ & \Bigg(1+3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_\Upsilon\Bigg)^{k\beta} \exp\Big(3^{\frac{1}{\beta}-1}\mathcal{S}\mathcal{M}\int_0^\omega \mathcal{L}_{\Phi,p,r}(\varepsilon)d\varepsilon\Big)^\beta \\ \leq \quad & 3^{1-\beta}\Big(\epsilon\mathcal{M}(m+\eta_\varphi)\Big)^\beta\Big(\mathcal{S}\varphi(\omega)+\psi\Big)^\beta \\ & \Bigg(1+3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_\Upsilon\Bigg)^{k\beta} \exp\Big(3^{\frac{1}{\beta}-1}\beta\mathcal{S}\mathcal{M}\int_0^\omega \mathcal{L}_{\Phi,p,r}(\varepsilon)d\varepsilon\Big) \\ \leq \quad & \mathcal{K}_{\Phi,\mathcal{S},\mathcal{M},\varphi},\psi\epsilon^\beta\Big(\varphi^\beta(\omega)+\psi^\beta(\omega)\Big), \end{aligned}$$

utilizing the relation $(p+q)^r \leq (p^r+q^r),\ p,\ q \geq 0,$ for any $r \in (0,1]$, implies,

$$\mathcal{K}_{\Phi,\mathcal{S},\mathcal{M},\varphi,\psi} = 3^{1-\beta}\Big(\mathcal{M}(m+\eta_\varphi)\Big)^\beta\Bigg(1+3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_\Upsilon\Bigg)^{m\beta} \exp\Big(3^{\frac{1}{\beta}-1}\beta\mathcal{S}\mathcal{M}\int_0^\tau \mathcal{L}_{\Phi,p,r}(\varepsilon)d\varepsilon\Big).$$

From the above estimates, we conclude that the system (2. 1) has β–HURS on compact interval w. r. t $(\psi^\beta, \varphi^\beta)$. □

6. β–HURS ON AN UNBOUNDED INTERVAL

In this part, we examine β–HURS on an unbounded interval. To achieve the desired result, we introduce some additional assertions.

$[D_0]$: Let $\{Z(\omega-\varepsilon) : \omega \geq \varepsilon \geq 0\}$ be the family of exponentially stable operators, that is we can find $\mathcal{M} \geq 1$ and $\kappa < 0$, such that

$$\|Z(\omega-\varepsilon)\| \leq \mathcal{M}e^{\kappa(\omega-\varepsilon)},\ \omega > \varepsilon \geq 0.$$

$[D_6]$: $\Phi \in \mathcal{C}(\Re^+ \times \Re^+ \times \mathcal{Z} \times \mathcal{Z}, \mathcal{Z})$ and there exists a map $\mathcal{L}_\Phi \in \mathcal{C}(\Re^+, \mathcal{Z})$ satisfying

$$\|\Phi(\omega,\varepsilon,\varrho_1,\varrho_2) - \Phi(\omega,\varepsilon,\varrho_1',\varrho_2')\| \leq \sum_{i=1}^2 \mathcal{L}_\Phi(\omega)\|\varrho_i - \varrho_i'\|,$$

for every $\omega, \varepsilon \in \Re^+$ and $\varrho, \varrho' \in \mathcal{Z}$. Also, we assume that

$$\int_0^\omega \mathcal{L}_{\Phi,p,r}(\varepsilon)d\varepsilon \leq \kappa_\Phi \omega + \zeta_\Phi,$$

for each $\omega, \varepsilon \geq 0$, $\kappa_\Phi, \zeta_\Phi \geq 0$, and $3^{\frac{1}{3}-1}\mathcal{S}\mathcal{M}_\Phi + \kappa < 0$ for $\beta \in (0,1)$.
$[D_7]$: $\Upsilon_k : \mathcal{Z} \to \mathcal{Z}$ and there exists a constant $\mathcal{L}_{\Upsilon_k} > 0$, so that

$$\|\Upsilon_k(\mathcal{E}(\omega)) - \Upsilon_k(\mathcal{E}^*(\omega))\| \leq \mathcal{L}_{\Upsilon_k}\|\mathcal{E} - \mathcal{E}^*\|,$$

for every $\omega \in \Re^+$ and $\mathcal{E}, \mathcal{E}' \in \mathcal{Z}$. Furthermore, we assume that

$$\mathcal{L}_\Upsilon := 3^{\frac{1}{\beta}-1}\mathcal{Z}\mathcal{M} \sup_{k\in M} \sum_{i=1}^{k} \mathcal{L}_{\Upsilon_i} < \infty.$$

$[D_8]$: A map $\varphi \in \mathcal{PC}(\Re^+, \mathcal{Z})$ and $\eta_\varphi > 0$ a constant, provided

$$\int_0^\omega e^{\kappa(\omega-\varepsilon)}\varphi(\varepsilon)d\varepsilon \leq \eta_\varphi \varphi(\omega), \text{for each } \omega \in \Re^+.$$

$[D_9]$: Put

$$\mathcal{M}_1 := \sup_{k\in\mathbb{N}_1^m} \sum_{i=1}^{k} e^{\kappa(\varepsilon_k-\omega_i)+3^{\frac{1}{\beta}-1}\mathcal{S}\mathcal{M}\kappa_\Phi\varepsilon_k} + e^{\kappa\varepsilon_k+3^{\frac{1}{\beta}-1}\mathcal{S}\mathcal{M}\kappa_\Phi\varepsilon_k}.$$

In addition, for the case $\mathbb{N}_1^m = \mathbb{N}$, we suppose $\mathcal{M}_1 < \infty$.
Now, utilizing inequality (3. 3) along with above mentioned assertions we prove our second result.

Theorem 6.1. *System (2. 1) has β–HURS w. r. t $(\psi^\beta, \varphi^\beta)$ on an unbounded interval provided $[D_0]$, $[D_1]$ and $[D_6] - [D_9]$ are satisfied.*

Proof. The nonsingular impulsive and delay integro differential system of the form

$$\begin{cases} \mathcal{B}\mathbb{U}'(\omega) = M\mathbb{U}(\omega) + N\mathbb{U}(\omega - \varepsilon) \\ +\Phi\big(\omega, \varepsilon, \int_{\varepsilon_0}^{\varepsilon} x(\omega, \varepsilon, \mathbb{U}(\varepsilon))d\varepsilon, \int_{\varepsilon_0}^{\varepsilon} z(\omega, \varepsilon, \mathbb{U}(\varepsilon - \varrho))d\varepsilon\big) + f(\omega), \\ \omega,\ \varepsilon \geq 0, \varrho \geq 0, \\ \mathbb{U}(\varepsilon) = \phi(\varepsilon), -\varrho \leq \varepsilon \leq 0, \\ \Delta\mathbb{U}(\omega_k) = \mathbb{U}(\omega_k^+) - \mathbb{U}(\omega_k) = \Upsilon_k(\mathbb{U}(\omega_k)),\ k \in \mathbb{N}_1^m := \overline{1,m}, \end{cases}$$

has a unique solution

$$\mathbb{U}(\omega) = \begin{cases} Z(\omega + \varrho)\phi(-\varrho) + \mathcal{B}^{-1}\int_{-\varrho}^{0} Z(\omega - \varepsilon)[\mathcal{B}\dot{\phi}(\varepsilon) - M\phi(\varepsilon)]d\varepsilon \\ +\mathcal{B}^{-1}\int_0^\omega Z(\omega - \varepsilon)\Phi\big(\omega, \varepsilon, \int_{\varepsilon_0}^{\varepsilon} x(\omega, \varepsilon, \mathbb{U}(\varepsilon))d\varepsilon, \int_{\varepsilon_0}^{\varepsilon} z(\omega, \varepsilon, \mathbb{U}(\varepsilon - \varrho))du\big)d\varepsilon \\ + \sum_{0<k<m} Z(\omega - \omega_k)\Upsilon_k(\mathbb{U}(\omega_k)). \end{cases} \tag{6. 9}$$

Let Θ satisfy (3. 3), then for every $\omega, \varepsilon \in (\omega_k, \omega_{k+1}]$, we obtain that

$$
\begin{aligned}
& \|\Theta(\omega) - Z(\omega+\varrho)\phi(-\varrho) - \mathcal{B}^{-1}\int\limits_{-\varrho}^{0} Z(\omega-\varepsilon)[\mathcal{B}\dot{\phi}(\varepsilon) - M\phi(\varepsilon)]d\varepsilon \\
& -\mathcal{B}^{-1}\int\limits_{0}^{\omega} Z(\omega-\varepsilon)\Phi\big(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\Theta(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\Theta(u-\varrho))du\big)d\varepsilon \\
& - \sum_{0<k<m} Z(\omega-\omega_k)\Upsilon_k(\Theta(\omega_k))\| \\
\leq\ & \sum_{k=1}^{k} \|Z(\omega-\omega_k)\|\|\hbar_k\| + \|\mathcal{B}^{-1}\|\int_0^{\omega} \|Z(\omega-\varepsilon)\|\big|f(\varepsilon)\big|d\varepsilon \\
\leq\ & \sum_{k=1}^{k} \mathcal{M}e^{(\omega-\omega_k)}\epsilon\psi + \mathcal{S}\int\limits_{0}^{\omega} \mathcal{M}e^{\kappa(\omega-\varepsilon)}\epsilon\varphi(\varepsilon)d\varepsilon.
\end{aligned}
$$

Thus, for every $\omega, \varepsilon \in (\omega_k, \omega_{k+1}]$, we get that

$$
\begin{aligned}
& \|\Theta(\omega) - \mathbb{U}(\omega)\|^{\beta} = \|\Theta(\omega) - Z(\omega+\varrho)\phi(-\varrho) - \mathcal{B}^{-1}\int\limits_{-\varrho}^{0} Z(\omega-\varepsilon)[\mathcal{B}\dot{\phi}(\varepsilon) - M\phi(\varepsilon)]d\varepsilon \\
& -\mathcal{B}^{-1}\int\limits_{0}^{\omega} Z(\omega-\varepsilon)\Phi\big(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\mathbb{U}(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\mathbb{U}(u-\varrho))du\big)d\varepsilon \\
& - \sum_{0<k<m} Z(\omega-\omega_k)\Upsilon_k(\mathbb{U}(\omega_k))\|^{\beta} \\
=\ & \|\Theta(\omega) - Z(\omega+\varrho)\phi(-\varrho) - \mathcal{B}^{-1}\int\limits_{-\varrho}^{0} Z(\omega-\varepsilon)[\mathcal{B}\dot{\phi}(\varepsilon) - M\phi(\varepsilon)]d\varepsilon \\
& -\mathcal{B}^{-1}\int\limits_{0}^{\omega} Z(\omega-\varepsilon)\Phi\big(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\mathbb{U}(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\mathbb{U}(u-\varrho))du\big)d\varepsilon \\
& +\mathcal{B}^{-1}\int\limits_{0}^{\omega} Z(\omega-\varepsilon)\Phi\big(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\Theta(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\Theta(u-\varrho))du\big)d\varepsilon \\
& -\mathcal{B}^{-1}\int\limits_{0}^{\omega} Z(\omega-\varepsilon)\Phi\big(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\Theta(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\Theta(u-\varrho))du\big)d\varepsilon \\
& - \sum_{0<k<m} Z(\omega-\omega_k)\Upsilon_k(\mathbb{U}(\omega_k)) + \sum_{0<k<m} Z(\omega-\omega_k)\Upsilon_k(\Theta(\omega_k)) \\
& - \sum_{0<k<m} Z(\omega-\omega_k)\Upsilon_k(\Theta(\omega_k))\|^{\beta}
\end{aligned}
$$

$$
\begin{aligned}
\leq \quad & \Bigg(\|\Theta(\omega) - Z(\omega+\varrho)\phi(-\varrho) - \mathcal{B}^{-1}\int\limits_{-\varrho}^{0} Z(\omega-\varepsilon)[\mathcal{B}\dot{\phi}(\varepsilon) - M\phi(\varepsilon)]d\varepsilon \\
& -\mathcal{B}^{-1}\int\limits_{0}^{\omega} Z(\omega-\varepsilon)\Phi\big(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\Theta(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\Theta(u-\varrho))du\big)d\varepsilon \\
& - \sum_{0<k<m} Z(\omega-\omega_k)\Upsilon_k(\Theta(\omega_k))\| \Bigg)^{\beta} \\
& + \Bigg(\|\mathcal{B}^{-1}\int\limits_{0}^{\omega} Z(\omega-\varepsilon)\Bigg(\Phi\big(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\Theta(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\Theta(u-\varrho))du\big) \\
& -\Phi\big(\omega,\varepsilon,\int_{\varepsilon_0}^{\varepsilon} x(\varepsilon,u,\mathbb{U}(u))du, \int_{\varepsilon_0}^{\varepsilon} z(\varepsilon,u,\mathbb{U}(u-\varrho))du\Bigg)d\varepsilon\| \Bigg)^{\beta} \\
& + \Bigg(\sum_{0<k<m} \|Z(\omega-\omega_k)\Upsilon_k(\Theta(\omega_k)) - \sum_{0<k<m} Z(\omega-\omega_k)\Upsilon_k(\mathbb{U}(\omega_k))\| \Bigg)^{\beta} \\
\leq \quad & \Big(\sum_{k=1}^{m} e^{m(\omega-\omega_k)}\mathcal{M}\epsilon\psi + \mathcal{S}\int\limits_{0}^{\omega} \mathcal{M}e^{\kappa(\omega-\varepsilon)}\epsilon\varphi(\varepsilon)d\varepsilon \Big)^{\beta} \\
& + \Big(\mathcal{S}\int\limits_{0}^{\omega} \mathcal{M}e^{\kappa(\varepsilon-\varsigma)}\mathcal{L}_{\Phi,x,z}(\varepsilon)\|\Theta(\varepsilon) - \mathbb{U}(\varepsilon)\|d\varepsilon \Big)^{\beta} \\
& + \sum_{0<k<m} \Bigg(\mathcal{M}\mathcal{L}_{\Upsilon_k}\|\Theta(\omega_k) - \mathbb{U}(\omega_k)\| \Bigg)^{\beta} \ where \ \mathcal{L}_{\Phi,x,z} = \mathcal{L}_{\Phi}(\mathcal{L}_x + \mathcal{L}_z) \\
\leq \quad & \Big(\sum_{k=1}^{m} e^{m(\omega-\omega_k)}\mathcal{M}\epsilon\psi + \mathcal{S}\int\limits_{0}^{\omega} \mathcal{M}e^{\kappa(\omega-\varepsilon)}\epsilon\varphi(\varepsilon)d\varepsilon \Big)^{\beta} \\
& + \Big(\mathcal{S}\int\limits_{0}^{\omega} \mathcal{M}e^{\kappa(\omega-\varepsilon)}\mathcal{L}_{\Phi,x,z}(\varsigma)\|\Theta(\varepsilon) - \mathbb{U}(\varepsilon)\|d\varepsilon \Big)^{\beta} \\
& + \sum_{0<k<m} \Bigg(\mathcal{M}\mathcal{L}_{\Upsilon_k}\|\Theta(\omega_k) - \mathbb{U}(\omega_k)\| \Bigg)^{\beta}.
\end{aligned}
$$

Setting $\bar{\Theta}(\omega) := e^{-\kappa\omega}\Theta(\omega)$, $\bar{\mathbb{U}}(\omega) := e^{-\kappa\omega}\mathbb{U}(\omega)$, we have

$$\begin{aligned}\|\bar{\Theta}(\omega)-\bar{\mathbb{U}}(\omega)\|^{\beta} \quad &\leq \quad \Big(\sum_{k=1}^{m}\mathcal{M}e^{-m\omega_k}\epsilon\psi+\mathcal{S}\int\limits_{0}^{\omega}\mathcal{M}e^{-\kappa\varepsilon}\epsilon\varphi(\varepsilon)d\varepsilon\Big)^{\beta}\\ &+\Big(\mathcal{S}\int\limits_{0}^{\omega}\mathcal{M}\mathcal{L}_{\Phi,x,z}(\varepsilon)\|\bar{\Theta}(\varepsilon)-\bar{\mathbb{U}}(\varepsilon)\|d\varsigma\Big)^{\beta}\\ &+\sum_{k=1}^{m}\Big(\mathcal{M}\mathcal{L}_{\Upsilon_k}\|\bar{\Theta}(\omega_k)-\bar{\mathbb{U}}(\omega_k)\|\Big)^{\beta},\end{aligned}$$

utilizing

$$\Big(p+q+r\Big)^{\gamma}\leq 3^{\gamma-1}\Big(p^{\gamma}+q^{\gamma}+r^{\gamma}\Big), \text{provided } p,\, q,\, r\geq 0,\ \gamma>1,$$

we get that

$$\begin{aligned}\|\bar{\Theta}(\omega)-\bar{\mathbb{U}}(\omega)\| \quad &\leq \quad 3^{\frac{1}{\beta}-1}\Big(\mathcal{M}\sum_{k=1}^{m}e^{-m\omega_k}\epsilon\psi+\mathcal{S}\mathcal{M}\int\limits_{0}^{\omega}e^{-\kappa\varepsilon}\epsilon\varphi(\varepsilon)d\varepsilon\Big)\\ &+3^{\frac{1}{\beta}-1}\mathcal{S}\int\limits_{0}^{\omega}\mathcal{M}\mathcal{L}_{\Phi,p,r}(\varepsilon)\|\bar{\Theta}(\varepsilon)-\bar{\mathbb{U}}(\varepsilon)\|d\varepsilon\\ &+3^{\frac{1}{\beta}-1}\sum_{k=1}^{m}\mathcal{L}_{\Upsilon_k}\mathcal{M}\|\bar{\Theta}(\varepsilon_k)-\bar{\mathbb{U}}(\varepsilon_k)\|.\end{aligned}$$

By using Lemma 3.8, we obtain

$$\begin{aligned}\|\bar{\Theta}(\omega)-\bar{\mathbb{U}}(\omega)\| \quad &\leq \quad 3^{\frac{1}{\beta}-1}\Big(\mathcal{M}\sum_{k=1}^{m}e^{-m\omega_k}\epsilon\psi+\mathcal{S}\mathcal{M}\int\limits_{0}^{\omega}e^{-\kappa\varepsilon}\epsilon\varphi(\varepsilon)d\varepsilon\Big)\mathcal{L}_{\Upsilon_k}\\ &\times\exp\Big(3^{\frac{1}{\beta}-1}\mathcal{S}\mathcal{M}\int\limits_{0}^{\omega}\mathcal{L}_{\Phi,p,r}(\varepsilon)d\varepsilon\Big),\end{aligned}$$

by resubmitting the values, we have

$$\begin{aligned}\|\Theta(\omega)-\mathbb{U}(\omega)\| &\leq 3^{\frac{1}{\beta}-1}e^{\kappa\varepsilon}\Big(\mathcal{M}\sum_{k=1}^{m}e^{m(\omega-\omega_k)}\epsilon\psi+\mathcal{S}\mathcal{M}\int_0^{\omega}e^{\kappa(\omega-\varepsilon)}\epsilon\varphi(\varepsilon)d\varepsilon\Big)\mathcal{L}_{\Upsilon_k}\\ &\quad\times\exp\Big(3^{\frac{1}{\beta}-1}\mathcal{S}\mathcal{M}\int_0^{\omega}\mathcal{L}_{\Phi,p,r}(\varepsilon)d\varepsilon\Big)\\ &\leq 3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_{\Upsilon}\epsilon\Big(\sum_{i=1}^{k}e^{\kappa(\omega-\omega_k)+3^{\frac{1}{\beta}-1}\mathcal{S}\mathcal{M}(\kappa_\Phi\omega+\zeta_\Phi)}\psi\\ &\quad+e^{\kappa\omega+3^{\frac{1}{\beta}-1}\mathcal{S}\mathcal{M}(\kappa_\Phi\omega+\zeta_\Phi)}+\int_0^{\omega}e^{\kappa\omega+3^{\frac{1}{\beta}-1}\mathcal{S}\mathcal{M}(\kappa_\Phi\omega+\zeta_\omega)}\varphi(\varepsilon)d\varepsilon\Big)\\ &\leq 3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_{\Upsilon}\epsilon e^{3^{\frac{1}{3}-1}\mathcal{S}\mathcal{M}\zeta_\Phi}\big(\mathcal{M}_1+\eta_\varphi\big)\big(\varphi(\omega)+\psi\big),\end{aligned}$$

which implies,

$$\|\Theta(\omega)-\mathbb{U}(\omega)\|^{\beta} \leq \mathcal{K}_{\Phi,\mathcal{S},\mathcal{M},\varphi,\psi,\beta}\epsilon^{\beta}\eta_\varphi\big(\varphi^{\beta}(\varepsilon)+\psi^{\beta}\big),$$

where

$$\mathcal{K}_{\Phi,\mathcal{S},\mathcal{M},\varphi,\psi,\beta}:=3^{\frac{1}{\beta}-1}(\mathcal{S}\mathcal{M}\mathcal{L}_{\Upsilon})^{\beta}\Big(e^{3^{\frac{1}{\beta}-1}\mathcal{S}\mathcal{M}\zeta_\omega}(\mathcal{M}_1+\eta_\varphi)\Big)^{\beta}.$$

Hence the system (2. 1) has β–HURS on an unbounded interval w. r. t $(\psi^{\beta},\varphi^{\beta})$. □

EXAMPLE

Assume the nonsingular impulsive delay integro differential system of the form,

$$\begin{cases}\mathcal{B}\mathrm{F}'(\omega)=M\mathrm{F}(\omega)+N\mathrm{F}(\omega-0.2)+\Phi\big(\omega,\varepsilon,\int_{\omega_0}^{\omega}x(\omega,\varepsilon,\mathrm{F}(\varepsilon))d\varepsilon,\int_{\omega_0}^{\omega}z(\omega,\varepsilon,\mathrm{F}(\varepsilon-0.2))d\varepsilon\big),\\ \mathrm{F}(0)=1,\ \omega,\varepsilon\in[0,4],\\ \mathrm{F}(\varepsilon)=\phi(0.0199,0.2186)^{\top},-0.2\leq\varepsilon\leq 0,\\ \Delta\mathrm{F}(\omega_k)=\mathrm{F}(\omega_k)-\mathrm{F}(\omega_k),\ k\in\mathbb{N}_1^3,\end{cases} \tag{6. 10}$$

and the associated inequality

$$\begin{cases}|\mathcal{B}\mathrm{F}'(\omega)-M\mathrm{F}(\omega)-N\mathrm{F}(\omega-0.2)\\ -\Phi\big(\omega,\varepsilon,\int_{\omega_0}^{\omega}x(\omega,\varepsilon,\mathrm{F}(\varepsilon))d\varepsilon,\int_{\omega_0}^{\omega}z(\omega,\varepsilon,\mathrm{F}(\varepsilon-0.2))d\varepsilon\big)|\leq 1,\omega,\varepsilon\in[0,4],\\ \mathrm{F}(\varepsilon)=\phi(0.2,0.3)^{\top},-0.2\leq\varepsilon\leq 0,\\ \Delta\mathrm{F}(\omega_k)=\mathrm{F}(\omega_k)-\mathrm{F}(\omega_k),\ k\in\mathbb{N}_1^3,\end{cases} \tag{6. 11}$$

where $\varrho=0.2$. Let $\mathfrak{D}_1^*=\int_{\omega_0}^{\omega}x(\omega,\varepsilon,\mathrm{F}(\varepsilon))d\varepsilon$, $\mathfrak{D}_2^*=\int_{\omega_0}^{\omega}x(\omega,\varepsilon,\mathrm{F}(\varepsilon-\varrho))d\varepsilon$. Taking

$$M=\begin{bmatrix}-3.3 & 0.6\\ 0 & -3.15\end{bmatrix},N=\begin{bmatrix}1.5 & 1.2\\ 0 & 1.8\end{bmatrix},\mathcal{B}=\begin{bmatrix}1.5 & 0\\ 0 & 1.5\end{bmatrix},$$

$$\Phi\big(\omega,\varepsilon,\mathfrak{D}_1^*,\mathfrak{D}_2^*\big)=\begin{bmatrix}\int_0^{\omega}0.2\mathrm{F}_1(\varepsilon-0.2)cos(\varepsilon)d\varepsilon+0.2\int_0^{\omega}sin(\varepsilon)\frac{d}{d\varepsilon}\mathrm{F}_1(\varepsilon-0.2)d\varepsilon\\ \int_0^{\omega}0.15\mathrm{F}_2(\varepsilon-0.2)cos(\varepsilon)d\varepsilon+0.15\int_0^{\omega}sin(\varepsilon)\frac{d}{d\varepsilon}\mathrm{F}_2(\varepsilon-0.2)d\varepsilon\end{bmatrix}$$

$$\Phi\big(\omega,\varepsilon,\mathfrak{D}_1^*,\mathfrak{D}_2^*\big)=\begin{bmatrix}0.2\mathrm{F}_1(\omega-0.2)\sin\omega\\0.15\mathrm{F}_2(\omega-0.2)\sin\omega\end{bmatrix}$$

$and\ \phi(\omega)=cos(\omega+\pi/2)$, clearly $\cos(0)=1$.
Note that

$$MN=\begin{bmatrix}-4.95&-2.88\\0&-5.67\end{bmatrix}=NM, M\mathcal{B}=\begin{bmatrix}-4.95&0.9\\0&-4.725\end{bmatrix}=\mathcal{B}M,$$

$$N\mathcal{B}=\begin{bmatrix}-4.95&0.9\\0&-4.725\end{bmatrix}=\mathcal{B}N,\mathcal{B}^{-1}=\begin{bmatrix}0.6667&0\\0&0.6667\end{bmatrix},$$

$$N_1\mathcal{B}^{-1}=\begin{bmatrix}1.5527&1.0946\\0&1.8264\end{bmatrix},M\mathcal{B}^{-1}=\begin{bmatrix}1.5527&1.0946\\0&1.8264\end{bmatrix}.$$

If $\mathrm{F}\in C([0,4],\mathfrak{R})$ satisfies (6. 11), then there exists $f\in C([0,4],\mathfrak{R})$ where $f(\omega)=\cos(\pi/2+\omega)$ such that $f(\omega)\le 1$. So, we have

$$\begin{cases}\mathcal{B}\mathrm{F}'(\omega)=M\mathrm{F}(\omega)+N\mathrm{F}(\omega-0.2)\\+\Phi\big(\omega,\varepsilon,\int_{\omega_0}^{\omega}x(\omega,\varepsilon,\mathrm{F}(\varepsilon))d\varepsilon,\int_{\omega_0}^{\omega}z(\omega,\varepsilon,\mathrm{F}(\varepsilon-0.2))d\varepsilon\big)+f(\omega),\\\mathrm{F}(0)=1,\ \omega,\varepsilon\in[0,4],\\\mathrm{F}(\varepsilon)=\phi(0.0199,0.2186)^{\top},-0.2\le\varepsilon\le 0,\\\Delta\mathrm{F}(\omega_k)=\mathrm{F}(\omega_k)-\mathrm{F}(\omega_k)=\Upsilon_k(\mathrm{F}(\omega_k)),\ k\in\mathbb{N}_1^3,\end{cases}$$

and the solution of (6. 10)

$$\begin{aligned}\mathrm{F}(\omega)&=Z(\omega+\varrho)\phi(-\varrho)+\mathcal{B}^{-1}\int_{-\varrho}^{0}Z(\omega-\varepsilon)[\mathcal{B}\dot{\phi}(\varepsilon)-M\phi(\varepsilon)]d\varepsilon\\&+\mathcal{B}^{-1}\int_0^t Z(\omega-\varepsilon)\Phi\big(\omega,\varepsilon,\int_{\omega_0}^{\omega}x(\omega,\varepsilon,\mathrm{F}(\varepsilon))d\varepsilon,\int_{\omega_0}^{\omega}z(\omega,\varepsilon,\mathrm{F}(\varepsilon-0.2))d\varepsilon\big)d\varsigma\\&+\sum_{k=1}^{0}Z(\omega-\omega_k)\Upsilon_k(\mathrm{F}(\omega_k)),\end{aligned}$$

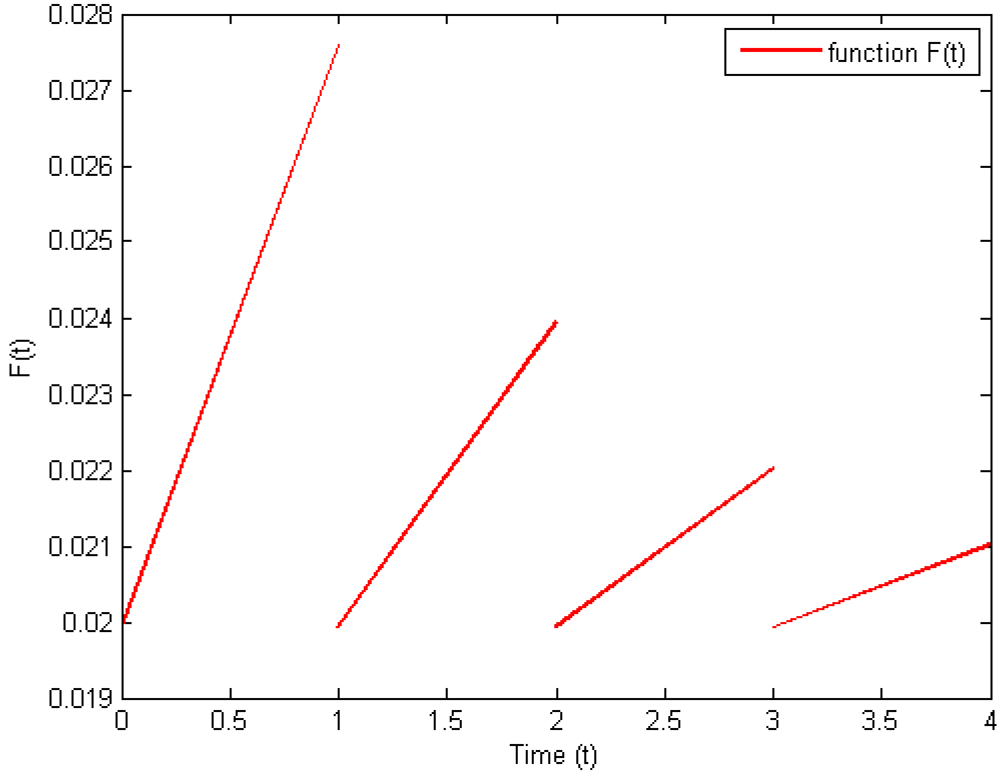
where $Z(\omega)=e^{M\mathcal{B}^{-1}\omega}e_{\varrho}^{N_1\mathcal{B}^{-1}(\varepsilon-\varrho)},\ N_1=e^{-M\mathcal{B}^{-1}\varrho}B$ and $M\mathcal{B}=\mathcal{B}M$, $MN=NM$, $N\mathcal{B}=\mathcal{B}N$ are used.
From Remark 2.3 of [20], the above solution becomes,

$$\mathrm{F}(\omega)=Z(\omega+\varrho)\phi(0)+B\int_{-\varrho}^{0}Z(\omega-\varrho-\varepsilon)\phi(\varepsilon)d\varepsilon+\sum_{k=1}^{2}Z(\omega-\omega_k)\Upsilon_k(\mathrm{F}(\omega_k)).$$

Utilizing Matlab, we see,

$$\mathrm{F}(\omega)=\begin{pmatrix}c_{11}\\c_{21}\end{pmatrix},$$

Figure 1: Graph of the function F(t).



where

$$c_{11}=2\left(\sin\frac{1}{10}\right)^2\left(1\Big/\left(\exp\left(\frac{1748192912675783}{5629499534213120}\right)\exp\left(\frac{3643934411889949\,\omega}{5629499534213120}\right)\right)+1\right),$$

$$c_{21}=2\left(\sin\frac{1}{10}\right)^2\left((1\Big/\left(\exp\left(\frac{4112583296853779}{11258999068426240}\right)\exp\left(\frac{3080981559426209\,\omega}{11258999068426240}\right)\right)+1\right).$$

Taking $f(\omega)=cos(\omega+\frac{\pi}{2})$, we have

$$\mathrm{F}^*(\omega) \quad = \quad Z(\omega+\varrho)\phi(0)+B\int_{-\varrho}^{0} Z(\omega-\varrho-\varepsilon)\phi(\varepsilon)d\varepsilon+\left[\cos\left(\omega+\frac{\pi}{2}\right)\ \cos\left(\omega+\frac{\pi}{2}\right)\right]^t.$$

Using Matlab,

$$\mathrm{F}^*(\omega)=\begin{pmatrix} d_{11}\\ d_{21}\end{pmatrix},$$

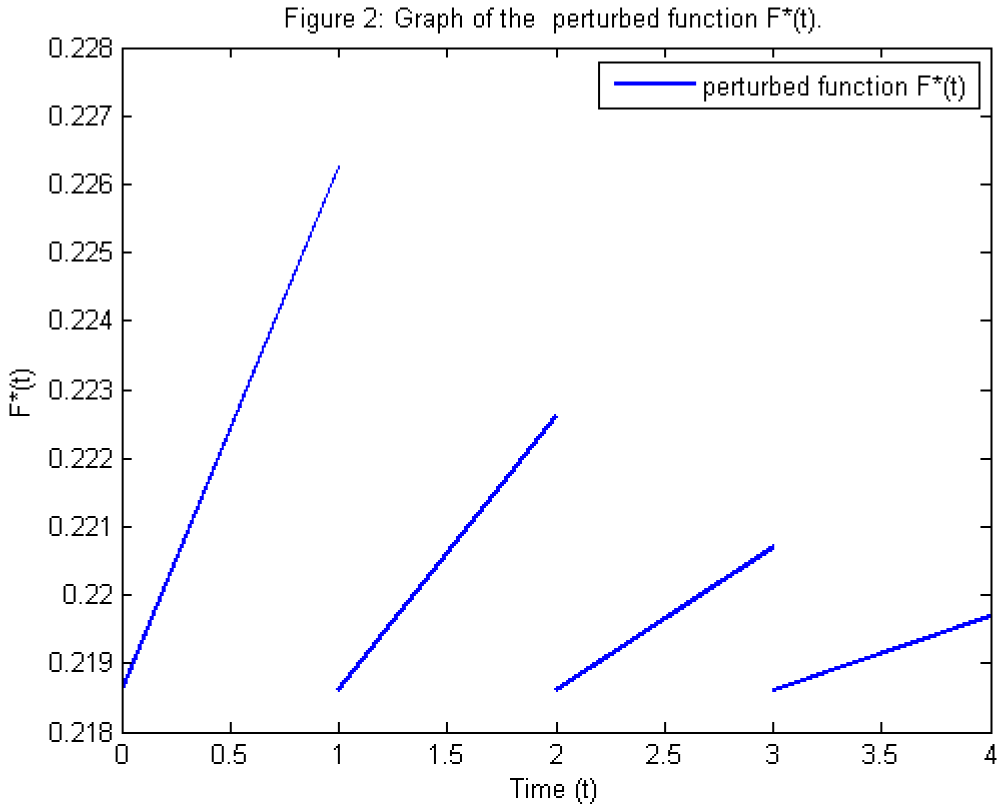


Figure 2: Graph of the perturbed function F*(t).
perturbed function F*(t)
F*(t)
Time (t)
0.228
0.227
0.226
0.225
0.224
0.223
0.222
0.221
0.22
0.219
0.218
0
0.5
1
1.5
2
2.5
3
3.5
4

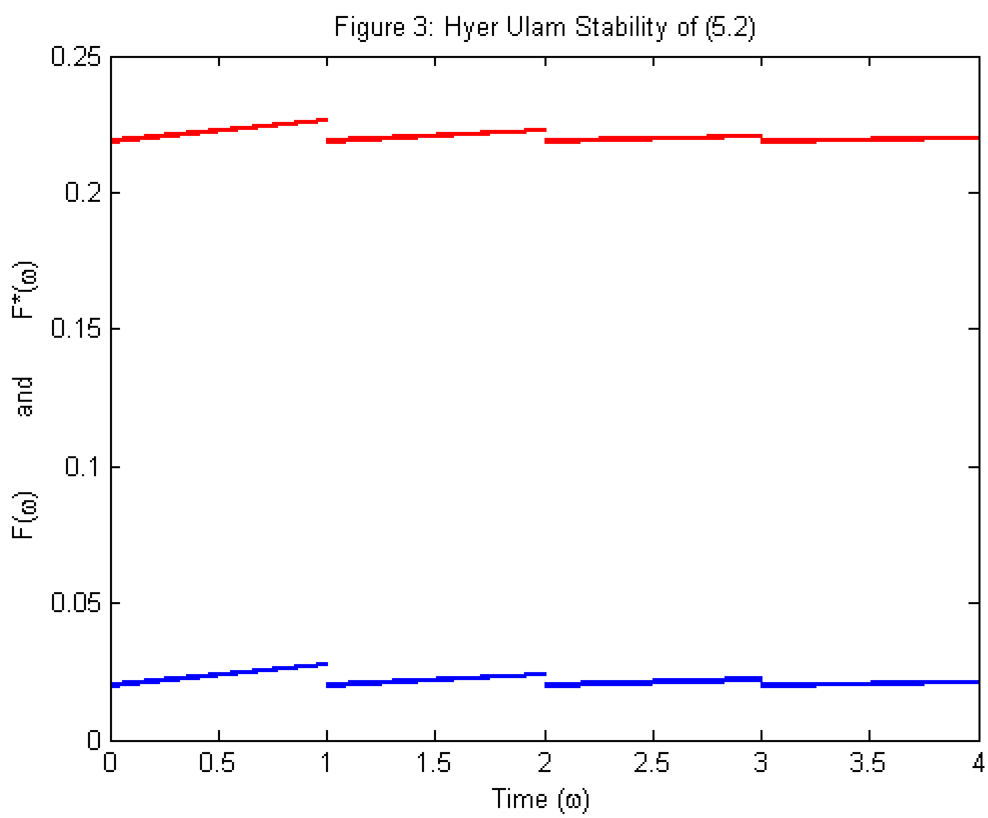


Figure 3: Hyer Ulam Stability of (5.2)
F(ω) and F*(ω)
Time (ω)
0.25
0.2
0.15
0.1
0.05
0
0
0.5
1
1.5
2
2.5
3
3.5
4

where,

$$\begin{cases} d_{11} = \sin\frac{1}{5} + 2\sin\left(\frac{1}{10}\right)^2 + \left(2\sin\left(1/10\right)^2\right) \Big/ \left(exp\left(\frac{1748192912675783}{5629499534213120}\right) \exp\left(\frac{3643934411889949\ \omega}{5629499534213120}\right)\right), \\ d_{21} = \sin\frac{1}{5} + 2\sin\left(\frac{1}{10}\right)^2 + \left(2\sin\left(\frac{1}{10}\right)^2\right) \Big/ \left(\exp\left(\frac{4112583296853779}{11258999068426240}\right) \exp\left(\frac{3080981559426209\ \omega}{11258999068426240}\right)\right), \end{cases}$$

we see that $\left[\epsilon\ \epsilon\right]^t = \left[0.1987\ 0.1987\right]^t$. Based on our theoretical result, Eq.(6. 10) has a unique solution in $C([0,4],\mathfrak{R})$ and is Hyers–Ulam stable on $[0,4]$.

• In Figure 1, we sketched the exact solution (represented by the red line) of the nonsingular impulsive delay integro differential Eq. (6. 10) on time interval $[0,4]$ with impulsive effect.

• In Figure 2, we sketched the perturbed solution (represented by the blue line) of the nonsingular impulsive delay integro differential Eq. (6. 10) on time interval $[0,4]$ with impulsive effect.

• In Figure 3, we sketched the graph of the exact solution and the perturbed solution which shows β–Hyers–Ulam–Rassias stability for the nonsingular impulsive delay integro differential equation on the time interval $[0,4]$.

CONCLUSION

Many researchers expressed their passion and interests in the notion of impulsive dynamic problems in the last few decades. Particularly, to examine Ulam's stability of dynamic problems, various kinds of assertions were utilized in the form of inequalities. We examined β–HURS of semilinear nonautonomous impulsive and delayed integro dynamic problems using fixed point method. Firstly, the investigations are carried out on compact intervals and then the results are extended to unbounded intervals. We utilize Gronwall's type inequality as a main tool to achieve our desired objectives. At the end, we provide an example along with graphical representation to verify the applicability of the reported results.

CONFLICTS OF INTEREST:

The authors declare no conflict of interest.

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