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Summability Test for Multidimensional Singular Points

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Abstract. The closest singular point to the origin of the following power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is on the convergence disc. However, there is a difficulty when one tries to identify or pinpoint which point, particularly on the disc of convergence, is found to be a singular point. In 1965 J.P. King established tests using Taylor and Euler summability methods to identify singular points of power series. The goal, motivation, and objective behind this paper include, but are not limited to, the presentation of double sequence notions and the establishment of multidimensional analogs of summability tests presented in J.P. King's singular points for a double power series. The methodology in achieving this goal applied Hamilton's [2] methods to two dimensional tests for singularity.

1. INTRODUCTION

In 1965 J.P. King presented a test for identifying singular points [14] of a given power series by first defining the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and establishing a positive value radius of convergence. Without loss of generality, this radius of convergence of the defined series is considered to be 1. As stated by J. P. King [4], the circle of convergence for such a power series passes through this defined singular point nearest to the origin. He took advantage of this fact and used Euler and Taylor's summability methods to generate a natural test for singularity. To accomplish the already mentioned goals of this paper, we used both multidimensional Taylor and Euler summability methods and used appropriate notions of Pringsheim [12] convergence to present theorems such as the following definition which states: *A necessary and a sufficient condition that* $(z, w) = (1, 1)$ *be any singular point or a singularity of a particular function is stated as or by definition is the* *series* $f(z,w) = \sum_{m,n=0,0}^{\infty,\infty} a_{m,n} z^m w^n$ *is that:*

$$\boldsymbol{P} - \limsup_{k,l} \left| \sum_{m,n=0,0}^{k,l} \binom{k}{m}\binom{l}{n} r^m s^n (1-r)^{k-m}(1-s)^{l-n} a_{m,n} \right|^{\frac{1}{mn}} = 1$$

where $0 < r, s < 1$ *and* $\{a_{m,n}\}$ *are established to be a double sequence that is factor-able.*

This article aims to highlight J.P. King's original theorems and then state several associated and related definitions. The article then provides and proves the multidimensional theorems by extending J.P. King's theorems. It is noted that this paper has core connections to the following articles: Domain of Binomial Matrix in Some Spaces of Double Sequences [1], Some Special Legendre Mates of Spherical Legendre Curves [5] and An Application of Product Summability to Approximating the Conjugate Series of a Special Class of Signals [7].

2. Basic Results, Definitions, and Preliminaries

Consider the already mentioned or defined series (power series) $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that having a particular radius of convergence defined by,

$$R = |a_n|^{-1/n}.$$

Using the Cauchy-Taylor theorem, it is shown that the generalized circle of convergence of a power series passes through the singular point or singular points of an established function that is nearest to the origin. Let us consider the single Euler and Taylor summability methods, respectively.

Definition 2.1 (Powell and Shaw [11])**.** *Let* $r \in \mathbb{C} \setminus (0,1)$*, the Euler transform theorem of order 1 denoted by* (E,r)*- transform is the matrix* $[e^r_{n,m}]$ *where:*

$$e^r_{n,m} = \begin{cases} \binom{n}{m} r^m (1-r)^{n-m}, & \text{if} \quad m \leq n, \\ \\ 0, & \text{if} \quad m > n. \end{cases}$$

For r = 1, let

$$e^r_{n,m} = \begin{cases} 0, & \text{if} \quad m \neq n, \\ 1, & \text{if} \quad m = n. \end{cases}$$

Definition 2.2 (Powell and Shaw [11])**.** *Let* $r \in \mathbb{C} \setminus (0)$*, the Taylor transform* $T(r) = (c^r_{n,m})$ *where:*

$$c^r_{n,m} = \begin{cases} 0, & \text{if} \quad m < n, \\ \\ \binom{m}{n} r^{m-n} (1-r)^{n+1}, & \text{if} \quad m \geq n. \end{cases}$$

For r = 1, let

$$c^r_{n,m} = \begin{cases} 0, & \text{if} \quad m \neq n, \\ 1, & \text{if} \quad m = n. \end{cases}$$

Definition 2.3 (Stroud [13]). *A function $w = f(z)$ is considered to have some regular point or an analytic point at a point $z = z_0$, if it is defined as and also if it is single-valued, and also having derivative results at every point that is at and also points that are around z_0. Points in this region where $f(z)$ fail to be are not considered to be regular or analytic are called or classified as singular points or simply called singularities.*

Using these definitions J. P. King presented the following test for singular points.

Theorem 2.1 (J. P. King [4]). *A necessary and and also sufficient condition that $z = 1$ is said to be a singular point or a singularity of such function that is by definition the power series defined as $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is that:*

$$\limsup_n \left| \sum_{m=0}^{n} \binom{n}{m} r^m (1-r)^{n-m} a_m \right|^{\frac{1}{n}} = 1$$

where $0 < r < 1$.

Theorem 2.2 (J. P. King [4]). *A necessary condition and also a sufficient condition that $z = 1$ is said to be a singular point or a singularity of some function by definition is the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is that:*

$$\limsup_n \left| \sum_{m=0}^{n} \binom{n}{m} r^{n-m} (1-r)^{m+1} a_m \right|^{\frac{1}{n}} = 1$$

where $0 < r < 1$.

Throughout this paper, we will examine the following multidimensional power series:

$$f(z,w) = \sum_{m,n=0,0}^{\infty,\infty} a_{m,n} z^m w^n,$$

where $a_{m,n} = a_m a_n$. Following this notion of power series, we shall use the following definitions of limits to generate multidimensional tests, for double singular points.

Definition 2.4 (R. F. Patterson [10]). *A sequence is considered to be double sequenced and defined as $[x] = [x_{k,l}]$ will have the* Pringsheim limit *denoted as L, (and also denoted by P-$\lim x = L$), provided if some $\epsilon > 0$ then there will exist some $N \in \mathbb{N}$ where $|x_{k,l} - L| < \epsilon$ whenever the limits $k, l > N$. The value x is further described as* "P-convergent."

Definition 2.5 (R. F. Patterson [10]). *A double sequenced defined as $[x]$ is stated as bounded iff there exists some natural value M where $|x_{k,l}| < M$ for the limits k and l. Furthermore, any double sequence that is converging does not need to be a bounded sequence. Pringsheim established the subsequent definition in the early 1900s that an already defined double sequence $[x]$ is considered to be a definite divergent if that for an arbitrary value which is large $G > 0$, there exist two values that are naturally positive numbers n_1 and n_2,* *where* $|x_{k,l}| > G$ *for* $n \geq n_1$, $k \geq n_2$. *There is an equivalence relationship between this definition and the* $P - \lim |x| = \infty$.

Definition 2.6 (R. F. Patterson [8]). *A value* β *is furthermore classified as a Pringsheim limit point with a given multi (more specifically double) sequence* $[x] = [x_{n,k}]$ *if there also exists some sequence* $[y] = [y_{n,k}]$ *of* $[x_{n,k}]$ *having also a Pringsheim valued limit* $\beta : P - \lim y_{n,k} = \beta$.

Definition 2.7 (R. F. Patterson [8]). *The following multi-dimensional (double) sequence now defined as* $[y]$ *is considered to be a multi (more specifically double) subsequence of some already defined sequence* $[x]$ *provided that there exists two positively increasingly doubled indexed sequences* $[n_j]$ *and also* $[k_j]$ *where if the respective sequences* $[z_j] = [x_{n,k}]$, *and implying that* $[y]$ *is described below as:*

$$\begin{array}{cccc} z_1 & z_2 & z_5 & z_{10} \\ z_4 & z_3 & z_6 & - \\ z_9 & z_8 & z_7 & - \\ - & - & - & - \end{array}$$

Definition 2.8 (R. F. Patterson [8]). *An above defined multi (double) sequenced* $[x]$ *is said to diverge in the manner of Pringsheim (P-divergent) if given or provided the sequence* $[x]$ *never converges in any Pringsheim manner (P-convergent).*

Definition 2.9 (R. F. Patterson [9]). *Let* $[x] = [x_{k,l}]$ *be any defined multi (more specifically double) sequence of reals (that is, of reals) for every natural number n, and then let* $\alpha_n = \sup_n[x_{k,l} : k, l \geq n]$. *A value that is considered superior to some Pringsheim limit of the sequence* $[x]$ *is further defined or also stated below:*

(1) if $\alpha = +\infty$ *for every natural number* n, *implying* $P - \lim\sup[x] := +\infty$*;*

(2) if $\alpha < \infty$ *for some natural* n, *then the* $P - \lim\sup[x] := \inf_n[\alpha_n]$.

Equally, let $\beta_n = \inf_n[x_{k,l} : k, l \geq n]$ *then the inferior of the Pringsheim limit* $[x]$ *is by definition or also stated below:*

(1) if $\beta_n = -\infty$ *for each natural* n, *which implies* $P - \lim\inf[x] := -\infty$*;*

(2) if $\beta_n > -\infty$ *implying any natural* n, *which further implies* $P - \lim\inf[x] := \sup_n[\beta_n]$.

For double sequence, let us observe the following notions:

Definition 2.10 (Markushevich [6]). *By the Cartesian product of two sets* E_1 *and* E_2, *written by* $E_1 \times E_2$, *we mean the set of ordered pairs* (z, w), *where* $z \in E_1$*and* $w \in E_2$. *The pairs* (z, w) *are called points of* $E_1 \times E_2$.

Definition 2.11 (Markushevich [6]). *Consider G to be some given domain that is established in some z-plane and D some given domain that is established w-plane, and consider* $F(z, w)$ *be some single value continuous function that is also onto some four-dimensional* $G \times D$. *Then such function defined as* $F(z, w)$ *is therefore considered analytic upon* $G \times D$ *if* $F(z, w)$ *is further analytic on G for every fixed point* $w \in D$ *and analytic on D for every fixed* $z \in G$.

Corresponding to these definitions are the generalization or interpretation of Cauchy's integral and the Cauchy Theorem.

Definition 2.12 (Markushevich [6]). *Given some domain G in some z-plane and some domain D in some w-plane, let $F(z,w)$ be analytic on $G \times D$. Let C and Γ be closed rectifiable Jordan curves such that $\overline{IC} \subset G$ and $\overline{I\Gamma} \subset D$.Then,*

$$F(z,w) = -\frac{1}{4}\pi^2 \int_C d\zeta \int_\Gamma \frac{F(\zeta,\eta)}{(\zeta - z)(\eta - w)} d\eta,$$
$$F(z,w) = -\frac{1}{4}\pi^2 \int_\Gamma d\eta \int_C \frac{F(\zeta,\eta)}{(\zeta - z)(\eta - w)} d\zeta,$$

for every point (z,w) $\in I(C) \times I(\Gamma)$.

Definition 2.13 (Markushevich [6]). *Given some domain G in any z plane and any domain D in the w plane, let $F(z,w)$ be analytic on $G \times D$. Let (z_0, w_0) be an arbitrary point on $G \times D$ and let δ be a positive distance from z_0 to the defined boundary which is defined as G and the Δ distance from w_0 to the boundary D. There is a "double power series"*

$$f(z,w) = \sum_{m,n=0,0}^{\infty,\infty} A_{m,n}(z - z_0)^m (w - w_0)^n,$$

converging to $F(z,w)$ on the "four-dimensional disk" $K_1 \times K_2$, where $K_1 = \{z : |z - z_0| < \delta\}, K_2 = \{w : |w - w_0| < \Delta\},$ and

$$A_{m,n} = \frac{1}{m!n!} \frac{\partial^{m+n} F(z_0, w_0)}{\partial z^m \partial w^n}.$$

3. Main Results

Let us consider the double Euler and Taylor summability methods, respectively.

Definition 3.1 (H. J. Hamilton [2]). *Let $(r,s) \in \mathbf{C} \times \mathbf{C} \backslash (0,1) \times (0,1)$, the Euler transform theorem of order $(r,s) \in \ \backslash (E,r,s)$ - transform is the four dimensional matrix $[e^{r,s}_{m,n,k,l}]$ defined by:*

$$e^{r,s}_{m,n,k,l} = \begin{cases} \binom{m}{k}\binom{n}{l} r^k s^l (1-r)^{m-k}(1-s)^{n-l}, & \text{if} \quad k \leq m, l \leq n; \\ \\ 0, & \text{if} \quad otherwise. \end{cases}$$

Definition 3.2 (H. J. Hamilton [2]). *Let $(r,s) \in \mathbf{C} \times \mathbf{C} \setminus (0,0)$, a multi dimensional 4-D Taylor transform $T(r,s) = (c^{r,s}_{m,n,k,l})$ defined by:*

$$c^{r,s}_{m,n,k,l} = \begin{cases} 0, & \text{if} \quad otherwise; \\ \\ \binom{k}{m}\binom{l}{n} r^{k-m} s^{l-n} (1-r)^{m+1}(1-s)^{n+1}, & \text{if} \quad k \geq m, l \geq n. \end{cases}$$

The critical goals of this article present multidimensional analogs of Theorem 2.4 and Theorem 2.5. To that end, let us establish the theorem below:

Theorem 3.1. *The necessary condition is also sufficient that $(z, w) = (1, 1)$ be a singular point or a singularity of the function that is by definition the given series*

$$f(z, w) = \sum_{m,n=0,0}^{\infty,\infty} a_{m,n} z^m w^n$$

is that:

$$P - \limsup_{k,l} \left| \sum_{m,n=0,0}^{k,l} \binom{k}{m} \binom{l}{n} r^m s^n (1-r)^{k-m} (1-s)^{l-n} a_{m,n} \right|^{\frac{1}{mn}} = 1$$

where $0 < r, s < 1$ and $\{a_{m,n}\}$ is a factorable multi (more specifically, double) sequence.

Proof. We can again consider such function $F(\alpha, \beta)$ defined by:

$$F(\alpha, \beta) = \frac{1}{[(1-(1-r)\alpha)(1-(1-s)\beta)]} f\left(\frac{r\alpha}{(1-(1-r)\alpha)}, \frac{s\beta}{(1-(1-s)\beta)}\right)$$

which is analytic on the region:

$$D_{r,s} = \left\{(\alpha, \beta) : \left|\frac{r\alpha}{1-(1-r)\alpha}\right| < 1, \left|\frac{s\beta}{1-(1-s)\beta}\right| < 1\right\}.$$

Then the following cancellations are performed:

$$\begin{aligned}
F(\alpha, \beta) &= \frac{1}{[(1-(1-r)\alpha)(1-(1-s)\beta)]} f\left(\frac{r\alpha}{(1-(1-r)\alpha)}, \frac{s\beta}{(1-(1-s)\beta)}\right) \\
&= \frac{1}{[(1-(1-r)\alpha)][(1-(1-s)\beta)]} \sum_{m,n=0,0}^{\infty,\infty} a_{m,n} (r\alpha)^m (s\beta)^n \frac{1}{[(1-(1-r)\alpha)]^m [(1-(1-s)\beta)]^n} \\
&= \frac{1}{[(1-(1-r)\alpha)][(1-(1-s)\beta)]} \sum_{m,n=0,0}^{\infty,\infty} a_{m,n} (r\alpha)^m (s\beta)^n \frac{[(1-(1-r)\alpha)][(1-(1-s)\beta)]}{[(1-(1-r)\alpha)]^{m+1} [(1-(1-s)\beta)]^{n+1}} \\
&= \frac{[(1-(1-r)\alpha)][(1-(1-s)\beta)]}{[(1-(1-r)\alpha)(1-(1-s)\beta)]} \sum_{m,n=0,0}^{\infty,\infty} a_{m,n} (r\alpha)^m (s\beta)^n \frac{1}{[(1-(1-r)\alpha)]^{m+1} [(1-(1-s)\beta)]^{n+1}} \\
&= \sum_{m,n=0,0}^{\infty,\infty} a_{m,n} (r\alpha)^m (s\beta)^n \frac{1}{[(1-(1-r)\alpha)]^{m+1} [(1-(1-s)\beta)]^{n+1}},
\end{aligned}$$

provided that $(1-r)|\alpha| < 1, (1-s)|\beta| < 1$. The Taylor expansion grants the following

$$\begin{aligned} F(\alpha,\beta) &= \sum_{m,n=0,0}^{\infty,\infty} a_{m,n}(r\alpha)^m(s\beta)^n \sum_{k,l=m,n}^{\infty,\infty} \binom{k}{m}[(1-r)\alpha]^{k-m}\binom{l}{n}[(1-s)\beta]^{l-n} \\ &= \sum_{m,n=0,0}^{\infty,\infty} a_{m,n}(r\alpha)^m(s\beta)^n \sum_{k,l=m,n}^{\infty,\infty} \binom{k}{m}(1-r)^{k-m}\alpha^{k-m}\binom{l}{n}(1-s)^{l-n}\beta^{l-n} \\ &= \sum_{k,l=0,0}^{\infty,\infty} \alpha^k\beta^l \sum_{m,n=0,0}^{k,l} \binom{k}{m} a_{m,n}(1-r)^{k-m}r^m\binom{l}{n}(1-s)^{l-n}s^n \\ &= \sum_{k,l=0,0}^{\infty,\infty} \alpha^k\beta^l \sum_{m,n=0,0}^{k,l} \binom{k}{m}\binom{l}{n} a_{m,n}(1-r)^{k-m}r^m(1-s)^{l-n}s^n \\ &= \sum_{k,l=0,0}^{\infty,\infty} \alpha^k\beta^l \sum_{m,n=0,0}^{k,l} \binom{k}{m}\binom{l}{n}(1-r)^{k-m}r^m(1-s)^{l-n}s^n a_{m,n}. \end{aligned}$$

Thus letting

$$b_{m,n} = \binom{k}{m}\binom{l}{n}(1-r)^{k-m}r^m(1-s)^{l-n}s^n a_{m,n}$$

$$F(\alpha,\beta) = \sum_{m,n=0,0}^{\infty,\infty} \alpha^k\beta^n b_{m,n},$$

and observe that,

$$\limsup |b_{mn}|^{\frac{1}{mn}} = 1$$

. This concludes the proof. □

As a corollary to the above, please observe the below:

Corollary 3.2. *If the factorial double sequence $a_{m,n}$ is $E(r,s)$ summable for $0 < r, s < 1$ to a non-zero constant then $(z,w) = (1,1)$ is already described singular point of function below:*

$$F(z,w) = \sum_{m,n=0}^{\infty,\infty} a_{m,n}z^m w^n.$$

We will now establish the multidimensional analog to J. P. King's second theorem.

Theorem 3.3. *The necessary condition also sufficient that $(z,w) = (1,1)$ be a singular point or a singularity of the function by definition is the given series:*

$$f(z,w) = \sum_{m,n=0,0}^{\infty,\infty} a_{m,n}z^m w^n$$

is that:

$$P - \limsup_{k,l} \left| \sum_{m,n=0,0}^{k,l} \binom{k}{m} \binom{l}{n} r^{k-m} s^{l-n} (1-r)^{m+1} (1-s)^{n+1} a_{m,n} \right|^{\frac{1}{mn}} = 1$$

where $0 < r, s < 1$ *and* $\{a_{m,n}\}$ *is a factorable double sequence.*

Proof. Let us again consider the two-dimensional function

$$G(\alpha, \beta) = (1-r)(1-s) f[(r + (1-r)\alpha), (s + (1-s)\beta)],$$

which is analytic on the region

$$D_{r,s} = \{(\alpha, \beta) : |r + (1-r)\alpha| < 1, |s + (1-s)\beta| < 1\}.$$

Then the following reformulation is performed:

$$\begin{aligned} G(\alpha, \beta) &= (1-r)(1-s) f[(r + (1-r)\alpha), (s + (1-s)\beta)] \\ &= (1-r)(1-s) \sum_{m,n=0,0}^{\infty,\infty} a_{m,n} [(r + (1-r)\alpha)]^m [(s + (1-s)\beta)]^n, \end{aligned}$$

under the condition that $(1-r)|\alpha| < 1$ and $(1-s)|\beta| < 1$. The multidimensional Euler transformation grants us the following:

$$\begin{aligned} G(\alpha, \beta) &= (1-r)(1-s) \sum_{m,n=0,0}^{\infty,\infty} a_{m,n} \sum_{k,l=0,0}^{m,n} \binom{m}{k} r^{m-k} (1-r)^k \alpha^k \binom{n}{l} s^{n-l} (1-s)^l \beta^l \\ &= \sum_{k,l=0,0}^{\infty,\infty} \alpha^k \beta^l \sum_{m,n=k,l}^{\infty,\infty} \binom{m}{k} \binom{n}{l} r^{m-k} (1-r)^{k+1} s^{n-l} (1-s)^{l+1} a_{m,n}. \end{aligned}$$

Let

$$b_{k,l} = \sum_{m,n=k,l}^{\infty,\infty} \binom{m}{k} \binom{n}{l} r^{m-k} (1-r)^{k+1} s^{n-l} (1-s)^{l+1} a_{m,n}.$$

Thus $G(\alpha, \beta) = \sum_{k,l=0,0}^{\infty,\infty} \alpha^k \beta^l b_{k,l}$. Observe that

$$\limsup |b_{mn}|^{\frac{1}{mn}} = 1$$

This concludes the proof. □

4. Conclusion

A basket of singularities is a group of point and curve singularities. This paper gives us a natural tool for checking singularities of multidimensional functions within such a basket. Singularities are undefined points that can be unpredictable since that point is not differentiable. A real-life example lies in the analyses of stock price movements on the stock market. Examining stocks having large inconsistent fluctuations using tests for singularities, becomes possible when the fluctuations are characterized by a basket of singularities.

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