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Exponentiated Power Inverse Lomax Distribution with Applications.

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Abstract: The manuscript presents a novel distribution derived from the exponentiation technique called the Exponentiated Power Inverse Lomax Distribution (EPILD), which offers a generalized approach to the inverse Lomax distribution. To show the importance of the EPILD, we establish various mathematical properties including survival function, hazard rate, order statistics, entropy measures, r-th moment, and generating functions. To obtain the parameter estimates for the proposed model, we employed the maximum likelihood estimation(MLE) approach. The adaptability of the new model is evaluated by comparing it with well-known distributions using real-life data sets. In addition to real-life data sets, the flexibility of the model is also shown via a simulation study. We apply goodness of fit statistics, various model selection criteria, and graphical tools to examine the adequacy of the EPILD.

AMS (MOS) Subject Classification Codes:
Key Words: Inverse Lomax distribution, Exponentiation, Order Statistics, Maximum likelihood estimation, Moments, Entropy.

1. Introduction

Various fields such as insurance and engineering have already identified multiple useful probability distributions. Nevertheless, the generalization of these distributions remains an ongoing pursuit, prompting the creation of several fresh models that are more adaptable than the distributions employed as baseline models. The Lomax distribution has an inverse version referred to as the Inverse Lomax Distribution(ILD), which has been studied and proved to be particularly flexible in situations when the non-monotonicity of the failure rate has been realized. Many authors studied various forms of ILD like Bayesian analysis of inverse Lomax distribution using approximation techniques proposed by Jan and Ahmad, [10]. Using a two-component mixture inverse Lomax model, Rahman et al [15]. predicted future-ordered observations using a predictive model and a Bayesian framework. There have been numerous attempts to expand new families of distributions. For findings on this

subject, see β family of the distributions by [3], exponentiated technique for the Weibull lomax distribution by [7], power Lomax distribution introduced by[8], power transformation applied on Burr Type X Distribution by [20], new innovative techniques recently proposed by [12], by [16], a new family of KIES distribution by [2], mixture of two distributions see [19]. In this paper, our interest is to use the exponentiation technique presented by [17].
The exploration of the EPILD encompasses the following objectives: (i) deriving the EPILD model;(ii) The main factor supporting the construction of the exponentiated power inverse Lomax distribution is the fact that the hazard rate of the EPILD exhibits a variety of complex shapes, such as constant, increasing-decreasing, decreasing-increasing, etc, which avoids the restrictions of the inverse Lomax distribution. (iii) elucidating mathematical properties, including quantile function, raw and central moments, order statistics, reliability measures, generating functions and entropy measures; (iv) assessing the precision of maximum likelihood estimators through simulation studies; (v) demonstrating the potential and utility of the EPILD; (vi) serving as a primary alternative model to existing ones and for modeling real data; (vii) providing superior model fits compared to other classical models; and (viii) drawing empirical inferences from goodness-of-fit statistics and graphical tools.
The contents of the article are structured as follows. Section 2 describes the model, EPILD. In section 3, reliability analysis which includes survival function, hazard rate, and reverse hazard rate. Also, graphical plots of hazard rate functions are added. Section 4, encompasses the statistical characteristics of EPILD. In section 5, generating functions like the moment generating function, characteristics function, and cumulant functions of the EPLID are elaborated. Section 6, the random number generation via quantile function. In section 7 and section 8, order statistics and information measures are discussed. In section 9, we address the maximum likelihood estimation for the EPILD parameters. In Section 10, a simulation is carried out to evaluate the precision of the maximum likelihood estimators. In Section 11, we consider an application to elucidate the potentiality and utility of the EPILD model. In Section 12, we conclude the article.

If we assume that a random variable X follows the Inverse Lomax distribution(ILD) , then we can express its probability density function(pdf) and cumulative distribution function(cdf) as

$$f(x;\omega,\beta) = \frac{\omega\beta}{x^2}\left(1+\frac{\beta}{x}\right)^{-(1+\omega)} \; ; \; x>0,\omega>0,\beta>0 \tag{1. 1}$$

$$F(x;\omega,\beta) = \left(1+\frac{\beta}{x}\right)^{-\omega} \; ; \; x>0,\omega>0,\beta>0 \tag{1. 2}$$

The pdf and cdf of Power inverse Lomax distribution as studied by [6] are expressed as follows,

$$f(x;\beta,\omega,\tau) = \tau\omega\beta x^{\omega\tau-1}\left(\beta+x^{\tau}\right)^{-(\omega+1)} \; ; \; x>0,\beta>0,\omega>0,\tau>0 \tag{1. 3}$$

$$F(x;\beta,\omega,\tau) = \left(1+\frac{\beta}{x^{\tau}}\right)^{-\omega} \; ; \; x>0,\beta>0,\omega>0,\tau>0 \tag{1. 4}$$

2. Exponentiated Power Inverse Lomax Distribution (EPILD)

The EPILD is constructed from the power inverse Lomax distribution by using the exponentiation technique given as

$$F_{e(x)} = [F(x)]^{\alpha} \; ; \; \alpha > 0$$

Using the equation (1. 4),the cdf of the model is given by

$$F_e(x) = \left[\left(1 + \frac{\beta}{x^{\tau}}\right)^{-\omega}\right]^{\alpha} \tag{2. 5}$$

Differentiating equation (2. 5) will yield the pdf of EPILD given as

$$f_e(x) = \alpha\beta\tau\omega x^{-(\tau+1)} \left(1 + \frac{\beta}{x^{\tau}}\right)^{-(\omega\alpha+1)} \; ; \; x > 0, \alpha > 0, \beta > 0, \omega > 0, \tau > 0 \tag{2. 6}$$

Special cases: The important sub-cases of EPILD are presented in here

- For α =1, the EPILD model can be simplified to the three parametric Power inverse Lomax distribution.
- For τ = α =1 ,the EPILD model can be simplified to the two parametric inverse Lomax distribution.
- For τ = α =β=1 ,the EPILD model can be simplified to the one parametric inverse Lomax distribution

3. Reliability analysis of EPILD

3.1. **Survival function.** The survival function for EPILD is given as

$$S_{EPILD}(x) = 1 - \left[\left(1 + \frac{\beta}{x^{\tau}}\right)^{-\omega}\right]^{\alpha} \tag{3. 7}$$

3.2. **Hazard Rate.** Hazard function, force of mortality, and failure rate are other names for hazard rate which is given by

$$h(x; \alpha, \beta, \tau, \omega) = \frac{\alpha\beta\tau\omega x^{-(\tau+1)} \left(1 + \frac{\beta}{x^{\tau}}\right)^{-(\omega\alpha+1)}}{1 - \left[\left(1 + \frac{\beta}{x^{\tau}}\right)^{-\omega}\right]^{\alpha}} \tag{3. 8}$$

3.3. **Reverse Hazard function.** Using equation (2. 5) and (2. 6) , the reverse hazard function for the EPILD is obtained as

$$h_r(x; \alpha, \beta, \tau, \omega) = \frac{\alpha\beta\tau\omega x^{-(\tau+1)} \left(1 + \frac{\beta}{x^{\tau}}\right)^{-(\omega\alpha+1)}}{\left[\left(1 + \frac{\beta}{x^{\tau}}\right)^{-\omega}\right]^{\alpha}} \tag{3. 9}$$

3.4. **Plots of the EPILD Density and Hazard Rate Functions.** We generate plots of the probability density function (pdf), cumulative distributive function (cdf), and hazard rate function for various parameter values of the EPILD model. The probability density function of the EPILD exhibits diverse characteristics, including symmetrical, right-skewed, and left-skewed as shown in Figure 1. Additionally, the hazard rate function can illustrate shapes such as modified bathtub, decreasing, increasing, and increasing-decreasing-increasing, as depicted in Figure 2. Hence, the EPILD offers considerable flexibility and is well-suited for analyzing a wide range of datasets.

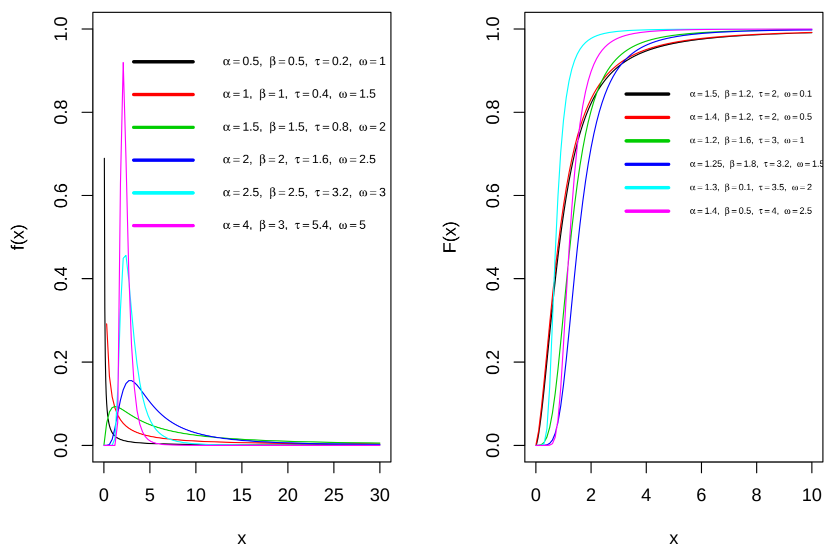

FIGURE 1. Graphs illustrating the pdf and cdf of EPILD are plotted for varying parameter values.

4. STATISTICAL CHARACTERISTICS OF EPILD

The relevant measures that are connected to the formulated model are discussed here.

4.1. **Raw Moments.** The r^{th}moment of the EPILD about origin $\mu_r^{'}$

$$\mu_r^{'} = E(x^r) = \int_0^\infty x^r f_e(x, \alpha, \beta, \tau, \omega)dx$$

$$\mu_r^{'} = \int_0^\infty x^r \alpha\beta\tau\omega x^{-(\tau+1)} \left(1 + \frac{\beta}{x^\theta}\right)^{-(\omega\alpha+1)} dx$$

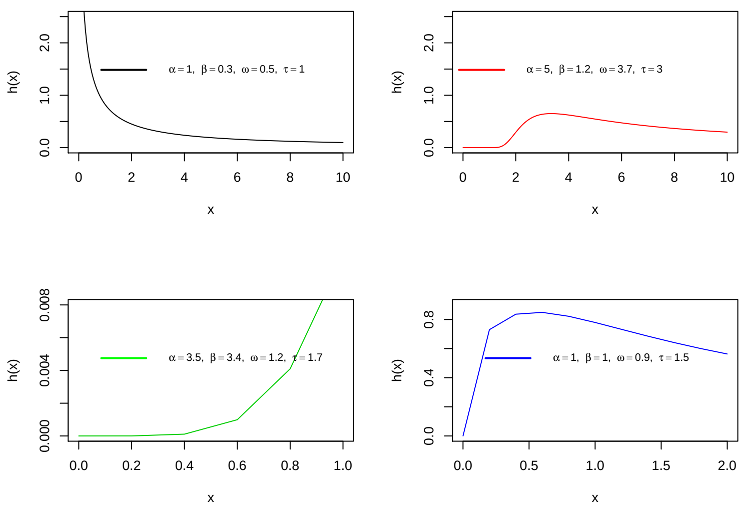

FIGURE 2. Multiple graphs depicting the Hazard rate function of EPILD are presented, with each plot displaying varying parameter values.

Put $\left(1+\frac{\beta}{x^{\tau}}\right)^{-\omega\alpha} = z$ and upon further simplification , we obtain the r^{th}moment of the EPILD about origin $\mu_r^{'}$ as

$$\mu_r^{'} = \omega\alpha\beta^{\frac{r}{\tau}} \times B\left(1-\frac{r}{\tau}, \lambda\alpha+\frac{r}{\tau}\right) \ ; \ r<\tau \tag{4. 10}$$

where $B\left(1-\frac{r}{\tau}, \omega\alpha+\frac{r}{\tau}\right)$ represents the beta functions of second type.

4.2. **Central Moments (EPILD).**

$$\mu_2 = \left[\omega\alpha\beta^{\frac{2}{\tau}} \times B\left(\frac{\tau-2}{\tau}, \frac{2+\alpha\lambda\tau}{\tau}\right)\right] - \left[\omega\alpha\beta^{\frac{1}{\tau}} \times B\left(\frac{\tau-1}{\tau}, \frac{1+\alpha\omega\tau}{\tau}\right)\right]^2 \tag{4. 11}$$

The equation (4. 11) represents the variance of EPILD.

$$\mu_3 = \omega\alpha\beta^{\frac{3}{\tau}} \times B\left(\frac{\tau-3}{\tau}, \frac{3+\alpha\omega\tau}{\tau}\right) - 3\omega\alpha\beta^{\frac{2}{\tau}} \times B\left(\frac{\tau-2}{\tau}, \frac{2+\alpha\omega\tau}{\tau}\right)\omega\alpha\beta^{\frac{1}{\tau}} \times B\left(\frac{\tau-1}{\tau}, \frac{1+\alpha\omega\tau}{\tau}\right) + 2\left(\omega\alpha\beta^{\frac{1}{\tau}} \times B\left(\frac{\tau-1}{\tau}, \frac{1+\alpha\omega\tau}{\tau}\right)\right)^3 \tag{4. 12}$$

$$\mu_4 = \omega\alpha\beta^{\frac{4}{\tau}} \times B\left(\frac{\tau-4}{\tau}, \frac{4+\alpha\omega\tau}{\tau}\right) - 4\omega\alpha\beta^{\frac{3}{\tau}} \times B\left(\frac{\tau-3}{\tau}, \frac{3+\alpha\omega\tau}{\tau}\right)\omega\alpha\beta^{\frac{1}{\tau}} \times B\left(\frac{\tau-1}{\tau}, \frac{1+\alpha\omega\tau}{\tau}\right)$$
$$+6\omega\alpha\beta^{\frac{2}{\tau}} \times B\left(\frac{\tau-2}{\tau}, \frac{2+\alpha\omega\tau}{\tau}\right)\left[\omega\alpha\beta^{\frac{1}{\tau}} \times B\left(\frac{\tau-1}{\tau}, \frac{1+\alpha\omega\tau}{\tau}\right)\right]^2$$
$$-3\left[\omega\alpha\beta^{\frac{1}{\tau}} \times B\left(\frac{\tau-1}{\tau}, \frac{1+\alpha\omega\tau}{\tau}\right)\right]^4 \tag{4. 13}$$

As a result, these equations may be used to calculate the skewness measure and kurtosis.

5. Generating functions of EPILD

The following theorem provides the MGF for the EPILD distribution.

Theorem 5.1. *We can obtain the moment generating function $M_X(t)$ for the EPILD distribution if we assume that X follows this distribution. The expression is*

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\omega\alpha\beta^{\frac{r}{\tau}} \times B\left(1-\frac{r}{\tau}, \omega\alpha+\frac{r}{\tau}\right) \tag{5. 14}$$

Proof: By definition

$$M_x(t) = \int_0^{\infty} e^{tx} f_e(x)dx$$

Using the series representation of e^{tx}, we have

$$\sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f_e(x;\alpha,\beta,\tau,\omega)dx$$

Using equation (4. 10) we obtain the M.G.F for EPILD as

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\omega\alpha\beta^{\frac{r}{\tau}} \times B\left(1-\frac{r}{\tau}, \omega\alpha+\frac{r}{\tau}\right) \tag{5. 15}$$

5.2. **Characteristic Function.** The characteristic function for EPILD distribution is given in the following theorem.

Theorem 5.3. *We can obtain the Characteristic Function for the EPILD distribution if we assume that X follows this distribution. The expression obtained is*

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!}\omega\alpha\beta^{\frac{r}{\tau}} \times B\left(1-\frac{r}{\tau}, \omega\alpha+\frac{r}{\tau}\right) \tag{5. 16}$$

Proof: The characteristic function for the EPILD can be obtained using the relation $\phi_X(t) = M_x(it)$

5.4. **Cumulant Function.** The cumulant function for the EPILD can be obtained using the relation $k_x(t) = \log M_x(t)$

$$k_x(t) = \log \left[\sum_{r=0}^{\infty} \frac{t^r}{r!} \omega\alpha\beta^{\frac{r}{\tau}} \times B\left(1 - \frac{r}{\tau}, \omega\alpha + \frac{r}{\tau}\right) \right] \tag{5. 17}$$

6. QUANTILES OF EPILD

Let X represent the random variable whose pdf is specified in equation (2. 6). By definition we have, $F\left[Q(u)\right] = u \implies Q(u) = F^{-1}(u)$
For the EPILD it is

$$Q(u) = \beta^{\frac{1}{\tau}} \left[u^{\frac{-1}{\omega\alpha}} - 1 \right]^{\frac{-1}{\tau}} \tag{6. 18}$$

Therefore, using the aforementioned calculation, one may get the quantiles of EPILD. Since it is generally known that standard measures of skewness and kurtosis have limitations when moments are absent for any distribution. To address these issues, the quantile measures can be used to study the examination of the variability of skewness and kurtosis.

7. ORDER STATISTICS OF EPILD

This section is dedicated to the order statistics related to the EPILD.Suppose we have a random sample $x_{(1)}, x_{(2)}, x_{(3)}, \ldots, x_{(n)}$ derived from the EPILD, with $X_{(t;n)}$ representing the t^{th} order statistics. Therefore, by definition

$$f_{(t;n)}(x) = \frac{n!}{(t-1)!(n-t)!} \left[F_e(x;\alpha,\beta,\tau,\omega)\right]^{t-1} \left[1 - F_e(x;\alpha,\beta,\tau,\omega)\right]^{n-t} f_e(x;\alpha,\beta,\tau,\omega) \tag{7. 19}$$

$$F_{(t;n)}(x) = \sum_{j=t}^{n} \binom{n}{j} \left[F_e(x;\alpha,\beta,\tau,\omega)\right]^{j} \left[1 - F_e(x;\alpha,\beta,\tau,\omega)\right]^{n-j} \tag{7. 20}$$

Using equation (2. 5) and equation (2. 6) in equation (7. 19) and equation (7. 20), the expressions of the t^{th} ordered statistics for the EPILD is derived as

$$f_{(t;n)}(x) = \frac{n!}{(t-1)!(n-t)!} \left[\left(1 + \frac{\beta}{x^\tau}\right)^{-\omega\alpha} \right]^{t-1} \times \left[1 - \left(1 + \frac{\beta}{x^\tau}\right)^{-\omega\alpha} \right]^{n-t} \alpha\beta\tau\omega x^{-(\tau+1)} \left(1 + \frac{\beta}{x^\tau}\right)^{-(\omega\alpha+1)} \tag{7. 21}$$

$$F_{(t;n)}(x) = \sum_{j=t}^{n} \binom{n}{j} \left[\left(1 + \frac{\beta}{x^\tau}\right)^{-\omega\alpha} \right]^{j} \left[1 - \left(1 + \frac{\beta}{x^\tau}\right)^{-\omega\alpha} \right]^{n-j} \tag{7. 22}$$

8. Information measure of EPILD

In this section we derive the expression for Renyi entropy of EPILD

Theorem 8.1. *The Renyi entropy for the EPILD is obtained as*

$$I_r(x) = \frac{1}{(1-\delta)} \ln \left[(\omega\alpha)^{\delta} \tau^{\delta-1} \beta^{\frac{1-\delta}{\tau}} B\left(\frac{\delta}{\tau} - \frac{1}{\tau} + \delta, \omega\alpha\delta + \frac{1}{\tau} - \frac{\delta}{\tau} \right) \right]$$

Proof: By definition, Renyi entropy is given as

$$I_r(x) = \frac{1}{(1-\delta)} \ln \int_0^{\infty} f(x)^{\delta} dx$$

Using the equation (2. 6) and substituting the $(1 + \frac{\beta}{x^{\tau}}) = t$ and upon further simplification, we obtain the Renyi entropy for the EPILD expressed as

$$I_r(x) = \frac{1}{(1-\delta)} \ln \left[(\omega\alpha)^{\delta} \tau^{\delta-1} \beta^{\frac{1-\delta}{\tau}} B\left(\frac{\delta}{\tau} - \frac{1}{\tau} + \delta, \omega\alpha\delta + \frac{1}{\tau} - \frac{\delta}{\tau} \right) \right] \tag{8. 23}$$

Theorem 8.2. *The Tsallis entropy for the EPILD is expressed as*

$$I_t(x) = \frac{1}{(\delta-1)} \left[1 - (\omega\alpha)^{\delta} \tau^{\delta-1} \beta^{\frac{1-\delta}{\tau}} B\left(\frac{\delta}{\tau} - \frac{1}{\tau} + \delta, \omega\alpha\delta + \frac{1}{\tau} - \frac{\delta}{\tau} \right) \right]$$

Proof: The Tsallis entropy for the EPILD distribution is defined as

$$I_t(x) = \frac{1}{(\delta-1)} \left[1 - \int_{-\infty}^{\infty} f_e(x)^{\delta} dx \right] \; ; \; \delta > 0, \delta \neq 1 \tag{8. 24}$$

using the equation (2. 6) and substituting the $(1+\frac{\beta}{x^{\tau}}) = t$ and upon further simplification, we obtain the Tsallis entropy for the EPILD expressed as

$$I_t(x) = \frac{1}{(\delta-1)} \left[1 - (\omega\alpha)^{\delta} \tau^{\delta-1} \beta^{\frac{1-\delta}{\tau}} B\left(\frac{\delta}{\tau} - \frac{1}{\tau} + \delta, \omega\alpha\delta + \frac{1}{\tau} - \frac{\delta}{\tau} \right) \right] \tag{8. 25}$$

9. Estimation of Parameters

9.1. **Maximum Likelihood Estimation(MLE).** Assuming a random sample of n observations say $x_1, x_2, x_3, \ldots, x_n$ taken from the EPILD and possessing the pdf given in (2. 6). Consequently, the logarithm likelihood function of EPILD is calculated for n observations as,

$$\log l = n\log\alpha + n\log\tau + n\log\omega + n\log\beta + (\omega\alpha\tau - 1)\sum_{i=1}^{n} \log(x_i) - (\omega\alpha + 1)\sum_{i=1}^{n} \log(\beta + x_i^{\tau}) \tag{9. 26}$$

Partially differentiating (9. 26), we have

$$\frac{\partial \log l}{\partial \tau} = \frac{n}{\tau} + (\omega\alpha)\sum_{i=1}^{n} \log(x_i) - (\omega\alpha + 1)\sum_{i=1}^{n} \frac{x_i^{\tau}\log(x_i)}{\beta + x_i^{\tau}} \tag{9. 27}$$

$$\frac{\partial \log l}{\partial \omega} = \frac{n}{\omega} + (\alpha\tau)\sum_{i=1}^{n}\log(x_i) - \alpha\sum_{i=1}^{n}\log(\beta + x_i^{\tau}) \tag{9. 28}$$

$$\frac{\partial \log l}{\partial \beta} = \frac{n}{\beta} - (\omega\alpha + 1)\sum_{i=1}^{n}\frac{1}{(\beta + x_i^{\tau})} \tag{9. 29}$$

$$\frac{\partial \log l}{\partial \alpha} = \frac{n}{\alpha} + (\omega\tau)\sum_{i=1}^{n}\log(x_i) - \omega\sum_{i=1}^{n}\log(\beta + x_i^{\tau}) \tag{9. 30}$$

The maximum likelihood estimators are given by the solution of the equation system given by the formulas in expressions (9. 27), (9. 28), (9. 29), and (9. 30) and made equal to zero. To solve these equations and estimate the parameters, we will use R software.

10. Simulation Illustration

This section examines the efficiency of the MLEs of EPILD. Using R Software, a simulation study was carried out to illustrate the behavior of MLEs while randomly generating different sample sizes from EPILD by using the quantile function expressed in equation (6. 18). There are 1000 iterations of the process. Various combinations of parameters are chosen as (3,1.5,1.5,2.4), (3.5,2, 1.5,2.4), and (3.5, 2, 1.7,2.6) . The MLE values and the empirical MSEs associated with each situation were calculated and presented in Table 1, Table 2, and Table 3 to report the research findings. The estimates are deemed stable and closely approximate the actual parameter values. Further, in all scenarios, the MSE drops as the sample size increases

TABLE 1. Simulation findings of the EPILD with parameter combination set as $(\alpha = 3, \beta = 1.5, \tau = 1.5, \omega = 2.4)$

Sample	MLE				MSE			
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\tau}$	$\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\tau}$	$\hat{\omega}$
30	3.891	1.166	1.404	3.476	2.250	0.174	0.0726	1.223
50	3.526	1.2913	1.583	3.051	0.782	0.083	0.0441	0.4644
100	3.364	1.426	1.56	2.855	0.376	0.012	0.017	0.2146
200	3.251	1.43	1.531	2.711	0.151	0.004	0.0060	0.1027
300	3.235	1.472	1.514	2.690	0.055	0.001	0.0015	0.092

TABLE 2. simulation findings of EPILD with parameter combination set as $(\alpha = 3.5, \beta = 2, \tau = 1.5, \omega = 2.4)$

Sample	MLE				MSE			
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\tau}$	$\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\tau}$	$\hat{\omega}$
30	4.5174	9.386	1.467	4.0866	11.982	54.650	0.0943	2.9344
50	4.138	3.175	1.495	3.366	3.324	1.262	0.0691	1.003
100	3.293	3.116	1.498	2.120	0.2085	1.254	0.0084	0.0864
200	3.3602	2.449	1.5075	2.2075	0.1070	0.2063	0.0047	0.0416
300	3.601	2.007	1.5021	2.555	0.1046	0.0010	0.0027	0.0252

TABLE 3. Simulation findings of EPILD model with parameter combination set as $(\alpha = 3.5, \beta = 2, \tau = 1.7, \omega = 2.6)$

Sample	MLE				MSE			
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\tau}$	$\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\tau}$	$\hat{\omega}$
30	4.2821	5.566	1.8653	3.9585	9.3597	12.8048	0.1521	1.9305
50	4.2321	4.9457	1.7935	3.7059	4.349	8.8145	0.1372	1.360
100	4.0217	2.5845	1.753	3.296	3.0118	0.3546	0.0157	0.497
200	3.724	2.577	1.722	2.896	0.427	0.260	0.001	0.098
300	3.514	2.4039	1.719	2.635	0.382	0.169	0.007	0.007

11. APPLICATION

Using two actual data sets from the environmental and medical fields, we demonstrate the utility of the EPILD distribution in this section.

- **Data set First** : [9] collected the first set of data and was also presented by [13]. It comprises of thirty consecutive readings for March precipitation in Minneapolis/St. Paul (in inches).
- **Data set Second** : The data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported first by [11]

The application of the suggested model to actual data sets is highlighted here. This section demonstrates the significance and superiority of EPILD by utilizing two actual data sets. To check the superiority and the flexibility of EPILD, we compare it with some known well competitive models which are mentioned as ILD(Inverse Lomax Distribution) and LD(Lomax Distribution). Performance comparing methods used to select the best distribution among compared models. These criteria select the superior distribution as the one with the lowest values of Akaike information criteria (AIC), HannanQuinn information criteria (HQIC), and Akaike information criteria Corrected (AICC) . The model with the highest *p*- value and the lowest KS value is regarded as the superior and best-fitted model.

TABLE 4. MLE's of EPILD and compared distributions with corresponding standard error (given in parenthesis) for data set first.

Model	$\hat{\beta}$	$\hat{\omega}$	$\hat{\alpha}$	$\hat{\tau}$
EPILD	11.088	0.708	0.7411	3.646
	(21.863)	(39.278)	(53.2041)	(1.383)
ILD	98.742	0.011		
	(146.489)	(0.017)		
LD	65305.779	109371.804		
	(11863.828)	(303.255)		

TABLE 5. Comparison of EPILD and compared distributions (data set first)

Model	-2$\hat{l}$	AIC	AICC	HQIC	K-S	p-value
EPILD	**77.232**	**85.232**	**86.832**	**87.02547**	**0.064**	**0.9997**
ILD	92.693	96.693	97.137	97.589	0.255	0.0503
LD	90.949	94.949	95.393	95.845	0.2352	0.0722

TABLE 6. MLE's of EPILD and compared distributions with corresponding standard error (given in parenthesis) for data set second.

Model	$\hat{\beta}$	$\hat{\omega}$	$\hat{\alpha}$	$\hat{\tau}$
EPILD	124.997	0.600	0.976	2.172
	(137.077)	(39.055)	(63.515)	(0.322)
ILD	2.002	2.462		
	(0.631)	(0.593)		
LD	13.865	120.342		
	(15.168)	(140.674)		

TABLE 7. Comparison of EPILD and compared distributions (data set second)

Model	-2$\hat{l}$	AIC	AICC	HQIC	K-S	p-value
EPILD	**819.3958**	**827.3958**	**827.7210**	**832.031**	**0.0322**	**0.9994**
ILD	849.351	853.3514	853.447	855.6690	0.1184	0.0549
LD	827.665	831.6658	831.7618	833.9834	0.0967	0.1821

Results shown in Table 5 and 7 reveals that EPILD is having a smallest value of AIC ,HQIC and AICC as compared to other competing models (Lomax and Inverse Lomax)and thus outperforms the models of inverse Lomax distribution and Lomax distribution.The results are further supported by Figures (3), (4) ,(5) ,(6) and (7).

12. Conclusion

The primary contribution of this manuscript is the suggestion of an adaptable generalization of the inverse Lomax distribution that may be used in place of the classical inverse Lomax in many situations. In this regard, we employ the exponentiation approach and provide the EPILD , a novel probability model. We explain some of its important features and derive parameters through a reasonably effective estimation method. The incorporation of two actual data sets serves as a practical example of how EPILD might be applied. The suggested model's performance in comparison to other well-established models is evaluated using the goodness of fit measure. The results obtained demonstrate a positive outcome, indicating that the EPILD model outperforms other competing models for the provided data sets.

Declaration of Competing Interest
The authors declare no conflict of interest.

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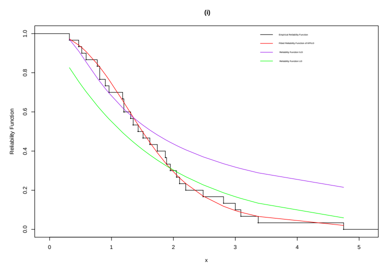

FIGURE 3. (i)For the first data set, the fitted EPILD reliability function and the empirical reliability function along with competitive models (Inverse Lomax distribution(ILD) and Lomax distribution(LD) are presented.

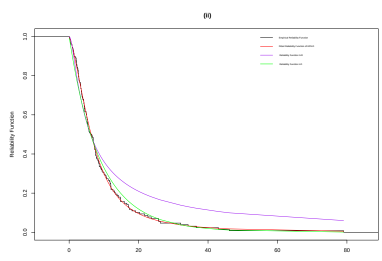

FIGURE 4. (ii)For the second data set, the fitted EPILD reliability function and the empirical reliability function along with competitive models(Inverse Lomax distribution(ILD) and Lomax distribution(LD) are presented.

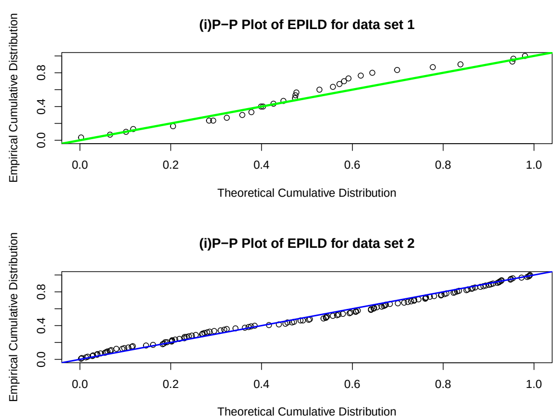

FIGURE 5. P-P Plot of the EPILD model for the both data set.

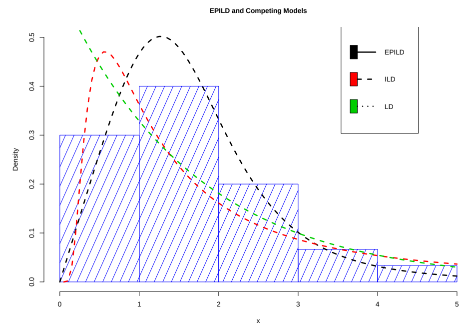

FIGURE 6. The first data set is represented through a histogram and a EPILD fitted model along with competing models

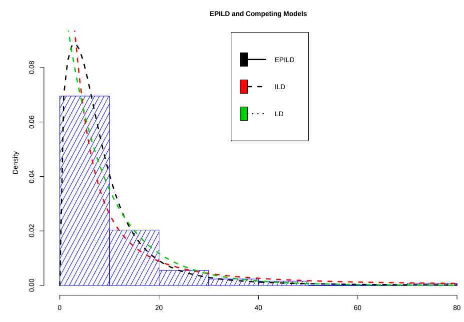

FIGURE 7. The second data set is represented through a histogram and a EPILD fitted model along with competing models

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