Punjab University Journal of Mathematics (2024), 56(10), 615-649
https://doi.org/10.52280/pujm.2024.56(10)03

A Generalized Fractional Integral Transform for Solutions of Fractional Burger's Equation

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Received: 28 August, 2024 / Accepted: 02 May, 2025 / Published online:20 May, 2025

Abstract. This paper introduces a novel fractional-order integral transform within the field of fractional calculus and applies it to the solution of fractional burger's equation with different fractional differential operator. In this study, we apply the newly proposed transform to several fractional differential, including Caputo, Caputo-Fabrizio, Riemann-Liouville, New Fractional Derivative and Atangana-Baleanu operators in both the Riemann-Liouville and Caputo senses. For varying values of $\phi^{\alpha}(s)$, $\psi(s)$ and $\gamma(t)$, the over 200 existing integral transforms and fractional integral transforms can be considered special cases of the proposed transform when applied to the aforementioned derivatives. This suggests the versatility and applicability of our newly introduced fractional-order integral transform within the broader context of fractional calculus, engineering and physics. The analytical solution of Fractional order Viscous Burger's equation with different differential and integral operators are also discussed.

AMS (MOS) Subject Classification Codes: 31B10, 44A10, 26A33.
Key Words: Integral transform; Fractional derivatives; Fractional integrals; Caputo; Caputo-Fabrizio; Riemann-Liouville; Atangana-Baleanu; Mittag Leffler function.

1. Introduction

The credit of being the father of classical calculus goes to two famous mathematician known as Isaac Newton and Gottfried Wilhelm Leibniz. But Leibniz is also a pioneer of fractional calculus (FC). On September 30, 1965, he wrote a letter to L'Hospital and asked about the half order derivative, which led to the formation of FC. However, other than fractional order the field can converge to complex order also. Initially, FC faced some challenges in its promotion due to lack of knowledge about its practical application in real world's problems. Even Leibniz himself doesn't propose any work on fractional calculus.[34] But with the passage of time, mathematician keep working on it and made many fractional differential and integral operators along with formation of functions and integral transformations. After that in 19th century along with many other researcher, Leibniz, Eular, Fourier, Laplace, Liouville, Riemann, Grunwald, Letnikov, and Hamdard are few names who worked on FC and presented their definitions.[36] Liouville was the one who give two definition, but the most important paper was given by Riemann who came after Liouville. The bulk of scientific domains have seen FC use during the 20th century. In 20th century, not only FC field progressed rather first conference and book on it is published where many uses of FC came in view using some famous differential operators like Riemann-Liouville, Caputo, Caputo-Fabrizio and many others.

The RL derivative,[41] which includes both Lacroix's and Liouville's formulas, is a generalization of Cauchy's integral function. The Riemann-Liouville fractional derivative has the drawback that it does not have a fixed base point; instead, the base point varies from function to function. The RL derivative converges to Lacroix's formula when the base point is zero and to Liouville's when the base point (lower bound) is zero [6]. The Riemann-Liouville fractional derivative was improved by Caputo in 1967, and he then provided a new fractional derivative with a single exponential-type kernel. The only shortcoming of the Caputo derivative is the singularity issue.

In 2015, Caputo and Fabrizio altered the Caputo derivative's singularity problem. They provided a brand-new definition devoid of a single (exponential type) kernel [16]. Atangana and Baleanu further alter the exponential kernel by applying the non-singular and non-local derivatives of the one-parameter Mittag-Leffler functions. Both Riemann-Liouville and Caputo senses of the Atangana-Baleanu derivative are provided in [6, 34]. Jassim and Hussein further updated the Liouville-Caputo derivative by presenting a New Fractional Derivative (NFD) without a singular kernel and with order $\alpha \geq 0$. They employed an exponential-type kernel, and this NFD has an advantage over the other derivative in that it converges to classical calculus more quickly [16].

Calculus, often known as the mathematics of change, is the study of change and is concerned with limits, differentiation, and integration. In the classical calculus, the derivative has an important geometric interpretation; namely, it is associated with the concept of tangent, in opposition to what occurs in the case of FC.[36] The 300-year-old subject of fractional calculus can be considered as an old as well as, as a novel field because it serves as a testing ground for various transformations and differential and integral operators and due to its inadequacy of a exact limit boundary that gives the exact results of the differential or integral problem just like classical calculus.

The FC era spans the 18th and 19th centuries. Numerous transformations of the Laplace type are performed, which is helpful in solving integral and differential equations of various types and orders [34]. These transforms offer an alternative to integration by simplifying complex equations into straightforward algebraic ones. In this study, a novel fractional-order integral transform is suggested. The bulk of scientific domains have seen FC use during the 20th century. FC has applications outside of mathematics, physics, and engineering, including in the areas of medical, engineering, physics, economics, demographics, finance, signal processing, artificial intelligence, robotics and electrical circuits [42, 6, 16, 10].
The new fractional integral operator introduced by [49]. This operator is helpful for relationship between weighted extended Cebysev version and Polya-Szego type inequalities. The iterative Elzaki transform method was introduced by [40] for the solution of nonlinear fractional Fishers model. The analytical solution of Burgers fluid with a fractional derivatives model analyzed through a rotating annulus [17].

Integral transformation is the easiest, simplest, and most effective mathematical technique and plays an important role in modeling real-world problems. They are used in a wide range of fields like medical and other sciences,engineering, technology, commerce, finance, economics, etc., and are mostly used to convert complex differential equations into easily solvable algebraic equations. Likewise, integral transforms are widely used to solve mathematical models like cancer models, tumor growth models, chemical reaction models, decay problems, health science models, and traffic flow models.

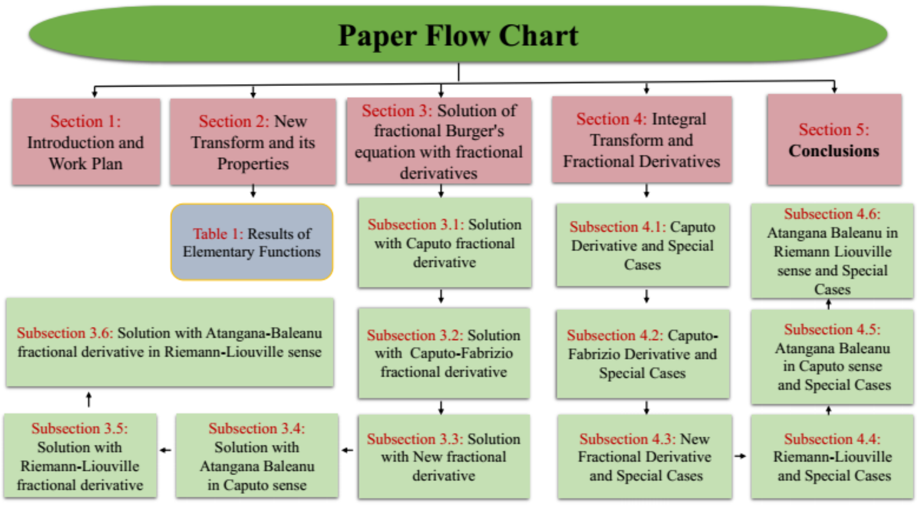

FIGURE 1. Represents the flow chart of our contributions and paper structure.

In this study, we provide a generalized integral transform that takes into account both fractional and integer orders. The main characteristic of this transform is that it may be used as a special instance of our suggested method for almost all existing integral transforms belonging to the Laplace family of integral transforms. We demonstrate how our transform is applied to fractional derivatives. We also go over the specific instances that our suggested transforms produce in the form of derivatives of already-existing transforms. Moreover, we border its use to fractional viscous Burger's equation by applying it with six well known fractional differential operators. Burger's equation attract the attention of researchers for many years and has its application in many field and mathematical models like Traffic flow, fluid mechanics, nonlinear wave propagation, gas dynamic, heat conduction, financial mathematics, mathematical physics, applied sciences and engineering.

1.1. **Motivation.** In the domain of fractional calculus and integral transforms, existing integral transforms such as the Laplace, Fourier, and Sumudu transforms have shown significant utility. However, none of these provide a unified framework capable of simultaneously addressing both integer-order and fractional-order transforms across various fractional operators. This lack of generality limits their applicability, especially when dealing with complex mathematical models involving diverse types of fractional derivatives.

To address these limitations, we propose a highly generalized integral transform derived from the Laplace transform family. This new transform offers a unified structure that encompasses and extends over 200 known fractional differential transforms.

The main contributions and highlights of our work are as follows:

- Our proposed transform encapsulates a wide range of fractional differential transforms as special cases, depending on the choice of kernel functions $\phi^{\alpha}(s), \gamma(t)$ and $\psi(s)$.
- We demonstrate the application of the transform in solving the fractional viscous Burgers' equation using various well-established fractional derivatives.
- We have extensively applied these proposed transforms to fractional viscous Burger's equation using well-known fractional derivatives.
- The transform is successfully applied to fractional operators such as Caputo, Caputo-Fabrizio, Riemann-Liouville, Atangana-Baleanu (in both Riemann-Liouville and Caputo senses), and the New Fractional Derivative.
- Our approach provides a consistent and unified methodology for analyzing fractional models, extending the versatility and applicability of integral transforms in the field.

The contributions of our work and the structure of our paper are depicted in a flow chart, illustrated in Figure 1. This visualization provides a concise overview of the main aspects of our research and the organization of the paper.

2. NEW INTEGRAL TRANSFORM

In this section, we introduce the most comprehensive integral transform, encompassing nearly all existing integral transforms within the Laplace transform family as well as fractional-order integral transforms.

Let $f(t)$ be an integrable function defined for all $t \geq 0, \gamma(t)$ is a polynomial, $\phi^{\alpha}(s) \neq 0$ and $\psi(s)$ is a positive real function. Then, the new integral transform is defined as follows [48]:

$$S\{f(t); s\} \;=\; F(s) \;= \phi^{\alpha}(s) \int_{0}^{\infty} \gamma(t) e^{-\psi(s)t} \, f(t) dt. \tag{2. 1}$$

Provided that the integral exists for some $\psi(s)$ and the inverse of the transform is the original function $f(t) = S^{-1}\{F(s)\}$. After applying the proposed transform to some functions, results are summarized in Table 1.

TABLE 1. *Shows the results of some functions after applying the new transform.*

Function $f(t) = S^{-1}F(s)$	New integral transform $F(s) = S\{f(t); s\}$
c	$c\frac{\phi^{\alpha}(s)}{\psi(s)}$
$lf(t)$	$lF(s)$
t^n	$\Gamma(n+1)\frac{\phi^{\alpha}(s)}{\psi^{n+1}(s)}$
e^{xt}	$\frac{\phi^{\alpha}(s)}{\psi(s)-x}$
$f'(t)$	$\psi(s)F(s) - \phi^{\alpha}(s)f(0)$
$f^{(n)}$	$\psi^n(s)F(s) - \phi^{\alpha}(s)\sum_{k=0}^{n-1}\psi^{n-k-1}(s)f^k(0)$
$\int_0^t f(w)dw$	$\frac{1}{\psi(s)}F(s)$
$[xf_1(t) + yf_2(t)]$	$[xF_1(s) + yF_2(s)]$
$f_1(t) * f_2(t)$	$\frac{1}{\phi^{\alpha}(s)}F_1(s)F_2(s)$

3. THE SOLUTION OF FRACTIONAL VISCOUS BURGER'S EQUATION WITH FRACTIONAL DIFFERENTIAL OPERATORS

In this section, the solution of fractional Burger's equation with different fractional differential operator is presented. The proposed linear transform in combination with adomian decomposition method is used to get the exact solution.

3.1. **The solution of Burger's equation with Caputo fractional operator.** We consider the fractional viscous Burgers' equation, where the viscosity coefficient is assumed to be 1 for simplicity. The model is defined over a one-dimensional spatial domain and does not include any external force terms. The function $y(x,t)$ is assumed to be sufficiently smooth (i.e., continuously differentiable where necessary) to permit the application of both analytical and numerical methods. The equation is given by [16]:

$$ {}_0D_t^{\alpha}y(x,t) + yy_x \quad = \quad y_{xx} \quad y(x,0) = x, \tag{3. 2}$$

where y is velocity field, t is the time and x is the spatial coordinates. Now, consider equation (3. 2) with Caputo fractional derivative, it becomes

$$ {}_0^CD_t^{\alpha}y(x,t) + yy_x = y_{xx} \quad y(x,0) = x. \tag{3. 3}$$

The Caputo fractional derivative for $n = 1$ is

$$ {}_0^CD_t^{\alpha}y(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-y)^{-\alpha}f'(t). \tag{3. 4}$$

After putting (3. 4) in (3. 3) and applying the proposed transform

$$ S\Big\{\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-y)^{-\alpha}y'(x,t)\Big\} - S\{yy_x\} = S\{y_{xx}\}.$$

By using the integral and transform properties, it becomes

$$\frac{1}{\Gamma(1-\alpha)} S\{(t)^{-\alpha} y'(x,t)\} = S\{y_{xx} - y y_x\}.$$

Using properties from Table 1 simplifies as,

$$\frac{1}{\Gamma(1-\alpha)\phi^{\alpha}(s)} \Big[\frac{\phi^{\alpha}(s)\Gamma(1-\alpha)}{\psi^{1-\alpha}(s)} \{\psi(s) S\{y(x,t)\} - \phi^{\alpha}(s) u(x,0)\} \Big] = S\{y_{xx} - y y_x\}.$$

After using IC, the above equation takes the form

$$S\{y(x,t)\} \quad = \quad \frac{\phi^{\alpha}(s)}{\psi(s)} x + \frac{1}{\psi^{\alpha}(s)} S\{y_{xx} - y y_x\}.$$

After applying the inverse transform on both sides

$$y(x,t) = x S^{-1}\Big\{ \frac{\phi^{\alpha}(s)}{\psi(s)} \Big\} + S^{-1}\Big\{ \frac{1}{\psi^{\alpha}(s)} S\{y_{xx} - y y_x\} \Big\}.$$

Using properties of inverse transform

$$y(x,t) = x + S^{-1}\Big\{ \frac{1}{\psi^{\alpha}(s)} S\{y_{xx} - y y_x\} \Big\}.$$

Using decomposition series we have, $y(x,t) = \sum_{n=0}^{\infty} y_n(x,t)$, where nonlinear term can be decomposed as,

$$N y(x,t) = A_n(y) = A_n(y_0 + y_1 + y_2, ..., y_n) = \frac{1}{n!} \frac{d^n}{d\lambda^n} [N(\sum_{k=0}^{\infty} \lambda^k y_k)]_{\lambda=0}, n = 0, 1, 2, \ldots$$

So the equation takes the form

$$y_0(x,t) + \sum_{n=1}^{\infty} y_n(x,t) \quad = \quad x + S^{-1}\Big\{ \frac{1}{\psi^{\alpha}(s)} S\{\sum_{n=0}^{\infty} y_{nxx} - \sum_{n=0}^{\infty} A_n(y)\} \Big\}.$$

This implies

$$y_0(x,t) = x.$$

So, a recursive relation is defined as

$$y_{k+1}(x,t) = S^{-1}\Big\{ \frac{1}{\psi^{\alpha}(s)} S\{y_{kxx} - A_k(y)\} \Big\}, \quad k \geq 0, \tag{3. 5}$$

For $k = 1$, put $k = 0$ in (3. 5), we have

$$y_1(x,t) = S^{-1}\Big\{ \frac{1}{\psi^{\alpha}(s)} S\{y_{0xx} - y_0 y_0(x)\} \Big\}.$$

After putting the value of $y_0(x,t) = x$, in above equation, we have

$$y_1(x,t) \quad = \quad -x \frac{t^{\alpha}}{\Gamma(\alpha+1)}.$$

For $k = 2$, put $k = 1$ in (3. 5), we have

$$y_2(x,t) \quad = \quad S^{-1}\Big\{ \frac{1}{\psi^{\alpha}(s)} S\{y_{1xx} - (y_0 y_{1x} + y_{0x} y_1)\} \Big\}.$$

After simplifying the above equation

$$y_2(x,t) \quad = \quad 2x\frac{t^{\alpha}}{\Gamma(2\alpha+1)}.$$

The solution of the equation (3. 2) with Caputo fractional derivative is

$$y_C(x,t) = x\Big(1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2\frac{t^{\alpha}}{\Gamma(2\alpha+1)} + \dots\Big). \tag{3. 6}$$

3.2. **The solution of Burger's equation with Caputo-Fabrizio fractional operator.** Here, we present the the solution of Burger's equation (3. 2) with Caputo-Fabrizio fractional derivative

$${}_0^{CFC}D_t^{\alpha}y(x,t) + yy_x = y_{xx}, \quad y(x,0) = x. \tag{3. 7}$$

After apply transform on (3. 7), it bocomes

$$S\{{}_0^{CFC}D_t^{\alpha}y(x,t)\} + S\{yy_x\} = S\{y_{xx}\}, \quad y(x,0) = x.$$

By using IC and after simplification

$$\frac{L(\alpha)}{(1-\alpha)\psi(s)+\alpha}\Big\{\psi(s)S\{y(x,t)\} - \phi^{\alpha}(s)x\Big\} \quad = \quad S\{y_{xx} - yy_x\},$$

where $L(\alpha)$ is a normalization function, such that $L(0) = L(1) = 1$.

$$\psi(s)S\{y(x,t)\} - \phi^{\alpha}(s)x \quad = \quad ((1-\alpha)\psi(s)+\alpha)S\{y_{xx} - yy_x\}.$$

This implies

$$S\{y(x,t)\} \quad = \quad \frac{\phi^{\alpha}(s)}{\psi(s)}x + \frac{(1-\alpha)\psi(s)+\alpha}{\psi(s)}S\{y_{xx} - yy_x\}.$$

After applying the inverse transform

$$y(x,t) \quad = \quad x + S^{-1}\Big\{\frac{(1-\alpha)\psi(s)+\alpha}{\psi(s)}S\{y_{xx} - yy_x\}\Big\}.$$

Using decomposition series we have

$$y_0(x,t) + \sum_{n=1}^{\infty} y_n(x,t) \quad = \quad x + S^{-1}\Big\{\frac{(1-\alpha)\psi(s)+\alpha}{\psi(s)}S\{\sum_{n=0}^{\infty} y_{nxx} - \sum_{n=0}^{\infty} A_n(y)\}\Big\}.$$

This implies

$$y_0(x,t) = x.$$

A recursive relation is defined as

$$y_{k+1}(x,t) = S^{-1}\Big\{\frac{(1-\alpha)\psi(s)+\alpha}{\psi(s)}S\{y_{kxx} - A_k(y)\}\Big\}, \quad k \geq 0, \tag{3. 8}$$

For $k = 1$, put $k = 0$ in (3. 8), we have

$$y_1(x,t) = S^{-1}\Big\{\frac{(1-\alpha)\psi(s)+\alpha}{\psi(s)}S\{y_{0xx} - y_0y_{0x}\}\Big\}.$$

After putting the value of $y_0(x,t) = x$, in above equation, we have

$$y_1(x,t) \quad = \quad -x(1-\alpha+\alpha t).$$

For $k = 2$, put $k = 1$ in (3. 8), we have

$$y_2(x,t) \quad = \quad S^{-1}\Big\{\frac{(1-\alpha)\psi(s)+\alpha}{\psi(s)}S\{y_{1xx}-(y_0y_{1x}+y_{0x}y_1)\}\Big\}.$$

After simplification

$$y_2(x,t) \quad = \quad x(2-4\alpha+2\alpha^2+(4\alpha-4\alpha^2)t+\alpha^2t^2).$$

The solution of the equation (3. 2) with Caputo-Fabrizio fractional operator is

$$y_{CFC}(x,t) \quad = \quad x((2-3\alpha+2\alpha^2)+(3\alpha-4\alpha^2)t+\alpha^2t^2+\ldots). \qquad (3.\ 9)$$

3.3. **The solution of Burger's equation with new fractional operator.** Here, we present the the solution of Burger's equation (3. 2) with new fractional derivative

$$ {}_a^M D_t^{\alpha} y(x,t) + yy_x = y_{xx}, \quad y(x,0) = x, \qquad (3.\ 10)$$

The new fractional derivative for $n = 1$ is

$${}_a^M D_t^{\alpha} y(x,t) = M_\alpha \int_a^t y'(x,t)e^{-M_\alpha(\tau-t)}dt, \qquad 0<\alpha\le 1, \qquad (3.\ 11)$$

where M_α is the function of α, and $M_\alpha = \Gamma^2(1-\alpha)$, such that $\lim_{\alpha\longrightarrow n} M_\alpha = \infty$. After putting (3. 11) in (3. 10) and apply transform, it becomes

$$S\Big\{M_\alpha \int_a^t y'(x,t)e^{-M_\alpha(\tau-t)}dt\Big\} + S\{yy_x\} = S\{y_{xx}\}.$$

After applying the properties of integral and simplify

$$\Gamma^2(1-\alpha)S\{y'(x,t)e^{-\Gamma^2(1-\alpha)(t)}\} = S\{y_{xx}-yy_x\}.$$

Using properties from Table1, results is

$$\Gamma^2(1-\alpha)\frac{1}{\phi^\alpha(s)}\Big\{\Big(\psi(s)S\{y(x,t)\}-\phi^\alpha(s)y(x,0)\Big)\frac{\phi^\alpha(s)}{\psi(s)+\Gamma^2(1-\alpha)}\Big\} = S\{y_{xx}-yy_x\},$$

After using the IC and simplify

$$\frac{\Gamma^2(1-\alpha)}{\psi(s)+\Gamma^2(1-\alpha)}\Big\{\psi(s)S\{y(x,t)\}-\phi^\alpha(s)x\Big\} \quad = \quad S\{y_{xx}-yy_x\}.$$

This implies

$$S\{y(x,t)\} \quad = \quad \frac{\phi^\alpha(s)}{\psi(s)}x + \frac{1}{\psi(s)}\frac{\psi(s)+\Gamma^2(1-\alpha)}{\Gamma^2(1-\alpha)}S\{y_{xx}-yy_x\}.$$

After applying the inverse transform

$$y(x,t) \quad = \quad x + S^{-1}\Big\{\frac{\psi(s)+\Gamma^2(1-\alpha)}{\psi(s)\Gamma^2(1-\alpha)}S\{y_{xx}-yy_x\}\Big\}.$$

By using decomposition series, we get

$$\Sigma_{n=0}^{\infty}y_n(x,t) \quad = \quad x + S^{-1}\Big\{\frac{\psi(s)+\Gamma^2(1-\alpha)}{\psi(s)\Gamma^2(1-\alpha)}S\{\Sigma_{n=0}^{\infty}y_{nxx}-\Sigma_{n=0}^{\infty}A_n(y)\}\Big\},$$

$$y_0(x,t)+\Sigma_{n=1}^{\infty}y_n(x,t) \quad = \quad x + S^{-1}\Big\{\frac{\psi(s)+\Gamma^2(1-\alpha)}{\psi(s)\Gamma^2(1-\alpha)}S\{\Sigma_{n=0}^{\infty}y_{nxx}-\Sigma_{n=0}^{\infty}A_n(y)\}\Big\}.$$

This implies

$$y_0(x,t) = x.$$

A recursive relation is defined as

$$y_{k+1}(x,t) = x + S^{-1}\Big\{\frac{\psi(s)+\Gamma^2(1-\alpha)}{\psi(s)\Gamma^2(1-\alpha)} S\{y_{kxx} - A_k(y)\}\Big\}, \quad k \geq 0, \tag{3. 12}$$

For $k = 1$, put $k = 0$ in (3. 12), we have

$$y_1(x,t) = S^{-1}\Big\{\frac{\psi(s)+\Gamma^2(1-\alpha)}{\psi(s)\Gamma^2(1-\alpha)} S\{y_{0xx} - y_0 y_{0x}\}\Big\}.$$

After putting the value of $y_0(x,t) = x$, in above equation, we have

$$y_1(x,t) \quad = \quad -x\Big(\frac{1}{\Gamma^2(1-\alpha)} + t\Big).$$

For $k = 2$, put $k = 1$ in (3. 12), we have

$$y_2(x,t) \quad = \quad S^{-1}\Big\{\frac{\psi(s)+\Gamma^2(1-\alpha)}{\psi(s)\Gamma^2(1-\alpha)} S\{y_{1xx} - (y_0 y_{1x} + y_{0x} y_1)\}\Big\}.$$

After simplification, we get

$$y_2(x,t) \quad = \quad x\Big(\frac{2}{\Gamma^2(1-\alpha)} + 4\frac{t}{\Gamma^2(1-\alpha)} + t^2\Big).$$

The solution of the equation (3. 2) with new fractional operator is

$$y_{NFD}(x,t) \quad = \quad x\Big(1 + \frac{1}{\Gamma^2(1-\alpha)} + \Big(\frac{4}{\Gamma^2(1-\alpha)} - 1\Big)t + t^2 + \dots\Big). \tag{3. 13}$$

3.4. **The solution of Burger's equation with Atangana-Baleanu fractional derivative in Caputo sense.** Now, consider equation (3. 2) with Atangana-Baleanu fractional derivative in Caputo sense, then equation become

$${}_0^{ABC}D_t^{\alpha} y(x,t) + y y_x = y_{xx}, \quad y(x,0) = x, \tag{3. 14}$$

After applying the proposed transform on (3. 14), we have

$$S\{{}_0^{ABC}D_t^{\alpha} y(x,t)\} + S\{y y_x\} = S\{y_{xx}\}.$$

After using the IC and simplify

$$\frac{L(\alpha)}{(1-\alpha)\psi^{\alpha}(s)+\alpha}\Big(\psi^{\alpha}(s) S\{y(x,t)\} - \phi^{\alpha}(s)\psi^{\alpha-1} y(x,0)\Big) \quad = \quad S\{y_{xx} - y y_x\}.$$

where $L(\alpha)$ is a normalization function, such that $L(0) = L(1) = 1$. After simplification we have

$$S\{y(x,t)\} \quad = \quad x + \Big(1 - \alpha + \frac{\alpha}{\psi^{\alpha}(s)}\Big) S\{y_{xx} - y y_x\}.$$

After applying the inverse transform

$$y(x,t) \quad = \quad x + S^{-1}\Big\{\Big(1 - \alpha + \frac{\alpha}{\psi^{\alpha}(s)}\Big) S\{y_{xx} - y y_x\}\Big\}.$$

Using decomposition series, we get

$$\begin{aligned}\Sigma_{n=0}^{\infty} y_n(x,t) &= x + S^{-1}\Big\{\Big(1-\alpha+\frac{\alpha}{\psi^{\alpha}(s)}\Big)S\{\Sigma_{n=0}^{\infty} y_{nxx} - \Sigma_{n=0}^{\infty} A_n(y)\}\Big\},\\ y_0(x,t)+\Sigma_{n=1}^{\infty} y_n(x,t) &= x + S^{-1}\Big\{\Big(1-\alpha+\frac{\alpha}{\psi^{\alpha}(s)}\Big)S\{\Sigma_{n=0}^{\infty} y_{nxx} - \Sigma_{n=0}^{\infty} A_n(y)\}\Big\}.\end{aligned}$$

This implies

$$y_0(x,t) = x.$$

A recursive relation is defined as

$$y_{k+1}(x,t) = S^{-1}\Big\{\Big(1-\alpha+\frac{\alpha}{\psi^{\alpha}(s)}\Big)S\{y_{kxx} - A_k(y)\}\Big\}, \quad k \geq 0. \tag{3. 15}$$

For $k=1$, put $k=0$ in (3. 15), we have

$$y_1(x,t) = S^{-1}\Big\{\Big(1-\alpha+\frac{\alpha}{\psi^{\alpha}(s)}\Big)S\{y_{0xx} - y_0 y_{0x}\}\Big\}.$$

After putting the value of $y_0(x,t)=x$, in above equation, we have

$$y_1(x,t) = -x\Big(1-\alpha+\alpha\frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big).$$

For $k=2$, put $k=1$ in (3. 15), we have

$$y_2(x,t) = S^{-1}\Big\{\Big((1-\alpha)+\frac{\alpha}{\psi^{\alpha}(s)}\Big)S\{y_{1xx} - (y_0 y_{1x} + y_{0x} y_1)\}\Big\}.$$

After simplification, we have

$$y_2(x,t) = x\Big(2-4\alpha+2\alpha^2+(4\alpha-4\alpha^2)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\alpha^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big).$$

The solution of the equation (3. 2) with Atangana-Baleanu fractional derivative in Caputo sense is

$$y_{ABC}(x,t) = x\Big((2-3\alpha+2\alpha^2)+(3\alpha-4\alpha^2)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\alpha^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\Big) \tag{3. 16}$$

When $\alpha=1$, the approximate solution of equations (3. 6),(3. 9),(3. 13), and (3. 16) is

$$y(x,t) = x(1-t+t^2+...).$$

The exact solution of the Equations (3. 6),(3. 9),(3. 13), and (3. 16) is

$$y(x,t) = \frac{x}{1+t}.$$

3.5. **The solution of Burger's equation with Riemann-Liouville fractional derivative.** Now, consider equation (3. 2) with Riemann-Liouville fractional derivative, it becomes

$${}_0^{RL}D_t^{\alpha} y(x,t) + y y_x = y_{xx}, \quad y(x,0) = x. \tag{3. 17}$$

Consider $n=1$ in (3. 17), with lower base $a=0$, equation takes the form

$${}_0^{RL}D_t^{\alpha} y(t) = \frac{d}{dx}\{{}_0I_t^{1-\alpha} y(t)\},$$

and

$$\frac{d}{dx}\{{}_0I_t^{1-\alpha}y(t)\} = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^\infty (t-T)^{-\alpha}y(x,T)dT.$$

This implies

$${}_0^{RL}D_t^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^\infty (t-T)^{-\alpha}y(x,T)dT, \quad (3.\ 18)$$

After putting (3. 18) in (3. 17), and applying the proposed transform

$$\frac{1}{\Gamma(1-\alpha)}S\Big\{\frac{d}{dt}\int_0^\infty (t-T)^{-\alpha}y(x,T)dT\Big\} = S\{y_{xx} - yy_x\}.$$

Using integral property, we have

$$S\Big\{\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\Big((t)^{-\alpha}y(x,t)\Big)\Big\} = S\{y_{xx} - yy_x\}, \quad (3.\ 19)$$

Using properties of transform from Table1, simplifies as

$$\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\Big((t)^{-\alpha}y(x,t)\Big) = S\{y_{xx} - yy_x\}.$$

This implies

$$S\{y(x,t)\} = \frac{1}{\psi^\alpha(s)}S\{y_{xx} - yy_x\}.$$

After applying the inverse transform, we have

$$y(x,t) = S^{-1}\Big\{\frac{1}{\psi^\alpha(s)}S\{y_{xx} - yy_x\}\Big\}.$$

Using decomposition series, we get

$$\Sigma_{n=0}^\infty y_n(x,t) = S^{-1}\Big\{\frac{1}{\psi^\alpha(s)}S\{\Sigma_{n=0}^\infty y_{nxx} - \Sigma_{n=0}^\infty A_n(y)\}\Big\},$$

$$y_0(x,t) + \Sigma_{n=1}^\infty y_n(x,t) = S^{-1}\Big\{\frac{1}{\psi^\alpha(s)}S\{\Sigma_{n=0}^\infty y_{nxx} - \Sigma_{n=0}^\infty A_n(y)\}\Big\}.$$

This implies

$$y_0(x,t) = 0.$$

Similarly, we have

$$y_1(x,t) = y_2(x,t) = y_3(x,t) = ... = 0. \quad (3.\ 20)$$

3.6. **The solution of Burger's equation with Atangana-Baleanu fractional derivative in Riemann-Liouville sense.** Now, consider equation (3. 2) with Atangana-Baleanu fractional derivative in Riemann-Liouville sense, it becomes

$${}_0^{ABR}D_t^\alpha y(x,t) + yy_x = y_{xx}, \quad y(x,0) = x, \quad (3.\ 21)$$

After applying the transform on (3. 21)

$$\frac{L(\alpha)}{(1-\alpha)\psi^\alpha(s)+\alpha}\Big(\psi^\alpha(s)S\{y(x,t)\}\Big) = S\{y_{xx} - yy_x\},$$

where $L(\alpha)$ is a normalization function, such that $L(0) = L(1) = 1$.

$$\frac{1}{(1-\alpha)\psi^{\alpha}(s)+\alpha}\Big(\psi^{\alpha}(s)S\{y(x,t)\}\Big) = S\{y_{xx} - yy_x\}.$$

After simplification, we have

$$S\{y(x,t)\} \quad = \quad \Big((1-\alpha) + \frac{\alpha}{\psi^{\alpha}(s)}\Big)S\{y_{xx} - yy_x\}.$$

After applying the inverse transform, the above equation takes the form

$$S\{y(x,t)\} = S^{-1}\Big\{\Big((1-\alpha) + \frac{\alpha}{\psi^{\alpha}(s)}\Big)S\{y_{xx} - yy_x\}\Big\}.$$

Using decomposition series, we get

$$\begin{aligned}\Sigma_{n=0}^{\infty} y_n(x,t) \quad &= \quad S^{-1}\Big\{\Big((1-\alpha) + \frac{\alpha}{\psi^{\alpha}(s)}\Big)S\{\Sigma_{n=0}^{\infty} y_{nxx} - \Sigma_{n=0}^{\infty} A_n(y)\}\Big\},\\ y_0(x,t) + \Sigma_{n=1}^{\infty} y_n(x,t) \quad &= \quad S^{-1}\Big\{\Big((1-\alpha) + \frac{\alpha}{\psi^{\alpha}(s)}\Big)S\{\Sigma_{n=0}^{\infty} y_{nxx} - \Sigma_{n=0}^{\infty} A_n(y)\}\Big\}.\end{aligned}$$

This implies

$$y_0(x,t) = 0.$$

Similarly, we have

$$y_1(x,t) = y_2(x,t) = y_3(x,t) = ... = 0. \tag{3. 22}$$

Hence the approximate and exact solution of equations (3. 20) and (3. 22) is 0.

4. Integral Transform and Fractional Derivatives

In this section, we will present the applications of the proposed transform on different types of fractional derivatives, such as Caputo, Caputo-Fabrizio, Riemann-Liouville, New Fractional Derivative, Atangana-Baleanu in the Caputo sense, and Atangana-Baleanu in the Riemann-Liouville sense. We also present the special cases of the proposed transform.

4.1. **Proposed Integral Transform on Caputo Derivative.** The Caputo derivative is the most common fractional derivative and was introduced by Michele Caputo [7] in 1967. The general form of the Caputo fractional derivative is defined as

$${}_{a}^{C}D_t^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-T)^{n-\alpha-1} f^{(n)}(T). \tag{4. 23}$$

where n is the smallest integer, $n > \alpha$, $\alpha \geq 0$, is a real number, $a \in (-\infty, t),\ t > y,\ y \in H^n(a,b)$ and $a > b$. Here, we will apply the proposed transform to the Caputo fractional derivative and derive the general formula.

Theorem 4.2. *If ${}_{0}^{C}D_t^{\alpha} y(t)$ is a Caputo derivative, then the proposed transform on the Caputo derivative is $\psi^{\alpha}(s)S\{y(t)\} - \phi^{\alpha}(s)\sum_{k=0}^{n-1}\psi^{\alpha-k-1}(s)y^k(0)$.*

Proof. After applying the proposed transform (2. 1) on the Caputo derivative defined in (4. 23) and by using the properties from Table 1, we get

$$\begin{aligned} S\{{}_0^C D_t^\alpha y(t)\} &= \frac{1}{\Gamma(n-\alpha)}\frac{1}{\phi^\alpha(s)}\Big[\Big(\frac{\Gamma(n-\alpha)\phi^\alpha(s)}{\psi^{n-\alpha}(s)}\Big)\Big(\psi^n(s)Y(s) \\ &\quad -\phi^\alpha(s)\sum_{k=0}^{n-1}\psi^{n-k-1}(s)y^k(0)\Big)\Big]. \end{aligned}$$

After simplification, we get

$$\begin{aligned} S\{{}_0^C D_t^\alpha y(t)\} &= \frac{1}{\psi^{n-\alpha}(s)}\Big\{\psi^n(s)Y(s)-\phi^\alpha(s)\sum_{k=0}^{n-1}\psi^{n-k-1}(s)y^k(0)\Big\}, \\ S\{{}_0^C D_t^\alpha y(t)\} &= \psi^{\alpha-n+n}(s)Y(s)-\phi^\alpha(s)\sum_{k=0}^{n-1}\psi^{\alpha-n+n-k-1}(s)y^k(0). \end{aligned}$$

This can be written as

$$S\{{}_0^C D_t^\alpha y(t)\} = \psi^\alpha(s)S\{y(t)\}-\phi^\alpha(s)\sum_{k=0}^{n-1}\psi^{\alpha-k-1}(s)y^k(0). \quad (4.\ 24)$$

This is the general form after applying the integral transform to the Caputo fractional derivative. □

Proposition 4.3. *If $\phi^\alpha(s)=\frac{1}{s}$ and $\psi(s)=\frac{1}{s}$ in equation (4. 24), then the proposed transform on the Caputo derivative converges to $s^{-\alpha}S\{y(t)\}-\sum_{k=0}^{n-1}s^{k-\alpha}y^k(0)$, where $S\{y(t)\}$, is Sumudu transform.*

Proof. We can easily verify the above result using $\phi^\alpha(s)=\frac{1}{s}$ and $\psi(s)=\frac{1}{s}$ in equation (4. 24)

$$\begin{aligned} S\{{}_0^C D_t^\alpha y(t)\} &= \Big(\frac{1}{s}\Big)^\alpha S\{y(t)\}-\frac{1}{s}\sum_{k=0}^{n-1}\Big(\frac{1}{s}\Big)^{\alpha-k-1}(s)y^k(0), \\ S\{{}_0^C D_t^\alpha y(t)\} &= s^{-\alpha}S\{y(t)\}-\sum_{k=0}^{n-1}\Big(\frac{1}{s}\Big)^{\alpha-k}(s)y^k(0). \end{aligned}$$

Or equivalently,

$$S\{{}_0^C D_t^\alpha y(t)\} = s^{-\alpha}S\{y(t)\}-\sum_{k=0}^{n-1}s^{k-\alpha}(s)y^k(0). \quad (4.\ 25)$$

Equation(4. 25) is equivalent to the results of Caputo derivative after applying Sumudu transform. □

4.3.1. ***Special cases of proposed transform via Caputo derivative***. Here, we will present the special cases of the proposed integral transform via the Caputo derivative. These special cases are generated from the result of the proposed transform (4. 24) for different values of $\phi^\alpha(s)$, $\psi(s)$ and $\gamma(t)=1$. These cases are,

(1) For $\phi^\alpha(s) = 1$ and $\psi(s) = s$, transform gives $s^\alpha L\{y(t)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1} y^k(0)$, where $L\{y(t)\}$ is Laplace transform [39].
(2) For $\phi^\alpha(s) = \frac{1}{v}$ and $\psi(s) = v$, new transform gives $v^\alpha A\{y(t)\} - \sum_{k=0}^{n-1} v^{\alpha-k-2} y^k(0)$, where $A\{y(t)\}$ is Aboodh transform [2].
(3) For $\phi^\alpha(s) = p^2$ and $\psi(s) = \frac{1}{p}$, yields $p^{-\alpha} A\{y(t)\} - \sum_{k=0}^{n-1} p^{k-\alpha+3} y^k(0)$, where $A\{y(t)\}$ is Anuj transform [23].
(4) For $\phi^\alpha(s) = v$ and $\psi(s) = \frac{1}{v}$, produces $v^{-\alpha} E\{y(t)\} - \sum_{k=0}^{n-1} v^{k-\alpha+2} y^k(0)$, where $E\{y(t)\}$ is Elzaki transform [11].
(5) For $\phi^\alpha(s) = \frac{1}{\varphi(s)}$ and $\psi(s) = \varphi^2(s)$, it gives $\varphi^{2\alpha}(s) EF\{y(t)\} - \sum_{k=0}^{n-1} \varphi^{2\alpha-2k-3}(s) y^k(0)$, where $EF\{y(t)\}$ is Emad-Falih transform [22].
(6) For $\phi^\alpha(s) = \frac{1}{u}$ and $\psi(s) = \frac{1}{u^2}$, transform gives $u^{-2\alpha} T\{y(t)\} - \sum_{k=0}^{n-1} u^{2k-2\alpha+1} y^k(0)$, where $T\{y(t)\}$ is Tarig transform [12].
(7) For $\phi^\alpha(s) = s^m$ and $\psi(s) = s^n$, transform corresponding to $s^{n\alpha} G\{y(t)\} - \sum_{k=0}^{n-1} s^{n\alpha-nk+m-n} y^k(0)$, where $G\{y(t)\}$ is G-transform[4].
(8) For $\phi^\alpha(s) = \frac{s}{v}$ and $\psi(s) = \frac{s}{v} i$, transform results in the $\left(\frac{si}{v}\right)^\alpha HY\{y(t)\} - \frac{s}{v} \sum_{k=0}^{n-1} \left(\frac{si}{v}\right)^{\alpha-k-1} y^k(0)$, where $HY\{y(t)\}$ is HY transform [37].
(9) For $\phi^\alpha(s) = p(s)$ and $\psi(s) = q(s)$, transform results in the $q^\alpha(s) J\{y(t)\} - p(s) \sum_{k=0}^{n-1} q^{\alpha-k-1}(s) y^k(0)$, where $J\{y(t)\}$ is Jafari transform [15].
(10) For $\phi^\alpha(s) = 1$ and $\psi(s) = \frac{1}{v}$, transform yields the $v^{-\alpha} K\{y(t)\} - \sum_{k=0}^{n-1} v^{k-\alpha+1} y^k(0)$, where $K\{y(t)\}$ is Kamal transform [43].
(11) For $\phi^\alpha(s) = 1$ and $\psi(s) = \frac{1}{v^2}$, new transform gives $v^{-2\alpha} KF\{y(t)\} - \sum_{k=0}^{n-1} v^{2k-2\alpha+2} y^k(0)$, where $KF\{y(t)\}$ is Kashuri-Fundo transform [19].
(12) For $\phi^\alpha(s) = s^3$ and $\psi(s) = \frac{1}{s^2}$, transform corresponding to $s^{-2\alpha} KT\{y(t)\} - \sum_{k=0}^{n-1} s^{2k-2\alpha+5} y^k(0)$, where $KT\{y(t)\}$ is Kharrat-Toma transform[21].
(13) For $\phi^\alpha(s) = p$ and $\psi(s) = \frac{1}{p}$, transform results in the $p^{-\alpha} ET\{y(t)\} - \sum_{k=0}^{n-1} p^{k-\alpha+2} y^k(0)$, where $ET\{y(t)\}$ is Elzaki-Tarig transform [32].
(14) For $\phi^\alpha(s) = v$ and $\psi(s) = v^\alpha$, transform yields the $v^{2\alpha} KU\{y(t)\} - \sum_{k=0}^{n-1} v^{\alpha-\alpha k+1} y^k(0)$, where $KU\{y(t)\}$ is Kushare transform [25].
(15) For $\phi^\alpha(s) = v$ and $\psi(s) = v$, transform results in the $v^\alpha MG\{y(t)\} - \sum_{k=0}^{n-1} v^{\alpha-k} y^k(0)$, where $MG\{y(t)\}$ is Mahgoub transform [27].
(16) For $\phi^\alpha(s) = v^2$ and $\psi(s) = v$, transform yields the $v^\alpha M\{y(t)\} - \sum_{k=0}^{n-1} v^{\alpha-k+1} y^k(0)$, where $M\{y(t)\}$ is Mohand transform [28].
(17) For $\phi^\alpha(s) = \frac{1}{u}$ and $\psi(s) = \frac{s}{u}$, transform corresponding to $\left(\frac{s}{u}\right)^\alpha RG\{y(t)\} - \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{u^{\alpha-k}} y^k(0)$, where $RG\{y(t)\}$ is Ramdan-Group transform [45].
(18) For $\phi^\alpha(s) = \frac{1}{u}$ and $\psi(s) = \frac{s}{u}$, transform results in $\left(\frac{s}{u}\right)^\alpha N\{y(t)\} - \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{u^{\alpha-k}} y^k(0)$, where $N\{y(t)\}$ is N-transform [20].

(19) For $\phi^\alpha(s) = \frac{\sigma}{\varepsilon}$ and $\psi(s) = \frac{\sigma}{\varepsilon}$, our new transform gives $\left(\frac{\sigma}{\varepsilon}\right)^\alpha R\{y(t)\} - \sum_{k=0}^{n-1} \left(\frac{\sigma}{\varepsilon}\right)^{\alpha-k} y^k(0)$, where $R\{y(t)\}$ is Rishi transform [24].

(20) For $\phi^\alpha(s) = \frac{1}{v^\beta}$ and $\psi(s) = v^\alpha$, transform corresponding to $v^{2\alpha}S\{y(t)\} - \sum_{k=0}^{n-1} v^{\alpha-\alpha k-\beta} y^k(0)$, where $S\{y(t)\}$ is Sadik transform [44].

(21) For $\phi^\alpha(s) = \frac{1}{v^2}$ and $\psi(s) = \frac{1}{v}$, transform gives $v^{-\alpha}Sa\{y(t)\} - \sum_{k=0}^{n-1} v^{k-\alpha-1} y^k(0)$, where $Sa\{y(t)\}$ is Sawi transform [29].

(22) For $\phi^\alpha(s) = 1$ and $\psi(s) = \frac{s}{u}$, transform corresponding to $\left(\frac{s}{u}\right)^\alpha Sh\{y(t)\} - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{\alpha-k-1} y^k(0)$, where $Sh\{y(t)\}$ is Shehu transform [30].

(23) For $\phi^\alpha(s) = \frac{1}{v}$ and $\psi(s) = v^\alpha$, transform yields $v^{2\alpha}So\{y(t)\} - \sum_{k=0}^{n-1} v^{\alpha-\alpha k-1} y^k(0)$, where $So\{y(t)\}$ is Soham transform [38].

(24) For $\phi^\alpha(s) = s$ and $\psi(s) = s$, our new transform gives $s^\alpha LC\{y(t)\} - \sum_{k=0}^{n-1} s^{\alpha-k} y^k(0)$, where $LC\{y(t)\}$ is Laplace-Carson transform [31].

(25) For $\phi^\alpha(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$, our new transform corresponding to $s^{-\alpha}S\{y(t)\} - \sum_{k=0}^{n-1} s^{k-\alpha} y^k(0)$, where $S\{y(t)\}$ is Samudu transform [47].

(26) For $\phi^\alpha(s) = s$ and $\psi(s) = s^2$, our transform yields $s^{2\alpha}AJ\{y(t)\} - \sum_{k=0}^{n-1} s^{2\alpha-2k-1} y^k(0)$, where $AJ\{y(t)\}$ is Pourreza transform [3].

(27) For $\phi^\alpha(s) = 1$ and $\psi(s) = s^{\frac{1}{\alpha}}$, transform corresponding to $sAL\{y(t)\} - \sum_{k=0}^{n-1} s^{1-\frac{k}{\alpha}-\frac{1}{\alpha}} y^k(0)$, where $AL\{y(t)\}$ is α-Laplace transform [35].

(28) For $\phi^\alpha(s) = \frac{s}{u}$ and $\psi(s) = \frac{s}{u}$, transform gives $\left(\frac{s}{u}\right)^\alpha Na\{y(t)\} - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{\alpha-k} y^k(0)$, where $Na\{y(t)\}$ is Natural transform [20].

(29) For $\phi^\alpha(s) = \frac{1}{q^3}$ and $\psi(s) = q$, new transform corresponds to $q^\alpha Gu\{y(t)\} - \sum_{k=0}^{n-1} q^{\alpha-k-4} y^k(0)$, where $Gu\{y(t)\}$ is Gupta transform [13].

(30) For $\phi^\alpha(s) = p^5$ and $\psi(s) = p$, our transform produce the result $p^\alpha DV\{y(t)\} - \sum_{k=0}^{n-1} p^{\alpha-k+4} y^k(0)$, where $DV\{y(t)\}$ is Dinesh-Verma transform [46].

(31) For $\phi^\alpha(s) = s$ and $\psi(s) = s$, transform results in $s^\alpha Ra\{y(t)\} - \sum_{k=0}^{n-1} s^{\alpha-k} y^k(0)$, where $Ra\{y(t)\}$ is Raj transform [18].

(32) For $\phi^\alpha(s) = \frac{1}{u}$ and $\psi(s) = \frac{1}{us}$, new transform gives $(us)^{-\alpha}AS\{y(t)\} - \sum_{k=0}^{n-1} \frac{u^{k-\alpha}}{s^{\alpha-k-1}} y^k(0)$, where $AS\{y(t)\}$ is Abaoub-Skheam transform [1].

(33) For $\phi^\alpha(s) = u$ and $\psi(s) = u$, our new transform yields $u^\alpha JT\{y(t)\} - \sum_{k=0}^{n-1} u^{\alpha-k} y^k(0)$, where $JT\{y(t)\}$ is Jabber-Tawfiq transform [14].

(34) For $\phi^\alpha(s) = 1$ and $\psi(s) = \frac{1}{u}$, new transform results in $u^{-\alpha}Y\{y(t)\} - \sum_{k=0}^{n-1} u^{k-\alpha+1} y^k(0)$, where $Y\{y(t)\}$ is Yang transform [9].

4.4. **Proposed Integral Transform on Caputo-Fabrizio derivative in Caputo sense.** In 2015, Caputo and Fabrizio [8] introduced new fractional derivative called the Caputo-Fabrizio fractional derivative. Let $y \in H^1(a,b),\ a < b,\ \alpha \in (0,1]$, then the Caputo-Fabrizio fractional derivative is defined as

$$ {}_{a}^{CFC}D_t^{\alpha} y(t) = \frac{L(\alpha)}{1-\alpha}\int_a^t y'(t)e^{(\frac{-\alpha(t-T)}{1-\alpha})}dT, \tag{4. 26}$$

where $L(\alpha)$ is a normalization function, such that $L(0) = L(1) = 1$.

Theorem 4.5. *If ${}_{0}^{CFC}D_t^{\alpha} y(t)$ is a Caputo-Fabrizio derivative in the Caputo sense, then the proposed transform on the Caputo-Fabrizio derivative is $\frac{L(\alpha)}{(1-\alpha)\psi(s)+\alpha}(\psi(s)S\{y(t)\} - \phi^{\alpha}(s)y(0))$.*

Proof. After applying the proposed transform (2. 1) on the Caputo-Fabrizio derivative defined in (4. 26) and using properties from Table 1, we have

$$ S\{{}_{0}^{CFC}D_t^{\alpha} y(t)\} \;=\; \frac{L(\alpha)}{(1-\alpha)\phi^{\alpha}(s)}\Big[\Big\{\frac{\phi^{\alpha}(s)}{\psi(s)+(\frac{\alpha}{1-\alpha})}\Big\}\Big\{\psi(s)Y(s) - \phi^{\alpha}(s)y(0)\Big\}\Big].$$

After simplification

$$ S\{{}_{0}^{CFC}D_t^{\alpha} y(t)\} \;=\; \frac{L(\alpha)}{(1-\alpha)}\Big\{\frac{1}{\frac{(1-\alpha)\psi(s)+\alpha}{1-\alpha}}\Big\}\Big\{\psi(s)Y(s) - \phi^{\alpha}(s)y(0)\Big\}.$$

This implies that

$$ S\{{}_{0}^{CFC}D_t^{\alpha} y(t)\} \;=\; \frac{L(\alpha)}{(1-\alpha)\psi(s)+\alpha}\Big\{\psi(s)S\{y(t)\} - \phi^{\alpha}(s)y(0)\Big\}. \tag{4. 27}$$

After using the integral transform on the Caputo-Fabrizio derivative, this is the general form. □

Proposition 4.6. *If $\phi^{\alpha}(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$ in equation (4. 27), then the proposed transform on the Caputo-Fabrizio derivative converges to $\frac{L(\alpha)}{(1-\alpha)+s\alpha}\Big(S\{y(t)\} - y(0)\Big)$, where $S\{y(t)\}$, is Sumudu transform.*

Proof. We can easily verify the above result using $\phi^{\alpha}(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$ in equation (4. 27)

$$ S\{{}_{0}^{CFC}D_t^{\alpha} y(t)\} \;=\; \frac{L(\alpha)}{(1-\alpha)(\frac{1}{s})+\alpha}\Big\{\frac{1}{s}S\{y(t)\} - \frac{1}{s}y(0)\Big\},$$

$$ S\{{}_{0}^{CFC}D_t^{\alpha} y(t)\} \;=\; \frac{sL(\alpha)}{(1-\alpha)+s\alpha}\frac{1}{s}\Big\{S\{y(t)\} - y(0)\Big\}.$$

Equivalently,

$$ S\{{}_{0}^{CFC}D_t^{\alpha} y(t)\} \;=\; \frac{L(\alpha)}{(1-\alpha)+s\alpha}\Big\{S\{y(t)\} - y(0)\Big\}. \tag{4. 28}$$

Equation(4. 28) is equivalent to the results of Caputo-Fabrizio derivative after applying Sumudu transform. □

4.6.1. ***Special cases of proposed transform via Caputo-Fabrizio derivative***. We will show that the existing integral transforms via Caputo-Fabrizio are the special cases of the result of the proposed transform (4. 27) for different values of $\phi^{\alpha}(s), \psi(s)$ and $\gamma(t) = 1$. These cases are given as

(1) For $\phi^{\alpha}(s) = 1$ and $\psi(s) = s$, transform gives $\frac{L(\alpha)}{(1-\alpha)s+\alpha}\Big(sL\{y(t)\} - y(0)\Big)$, where L{y(t)} is Laplace transform [39].
(2) For $\phi^{\alpha}(s) = \frac{1}{v}$ and $\psi(s) = v$, new transform gives $\frac{L(\alpha)}{(1-\alpha)v+\alpha}\Big(vA\{y(t)\} - v^{-1}y(0)\Big)$, where A{y(t)} is Aboodh transform [2].
(3) For $\phi^{\alpha}(s) = p^2$ and $\psi(s) = \frac{1}{p}$, yields $\frac{L(\alpha)}{(1-\alpha)+p\alpha}\Big(A\{y(t)\} - p^3 y(0)\Big)$, where A{y(t)} is Anuj transform [23].
(4) For $\phi^{\alpha}(s) = v$ and $\psi(s) = \frac{1}{v}$, produces $\frac{L(\alpha)}{(1-\alpha)+v\alpha}\Big(E\{y(t)\} - v^2 y(0)\Big)$ where E{y(t)} is Elzaki transform [11].
(5) For $\phi^{\alpha}(s) = \frac{1}{\varphi(s)}$ and $\psi(s) = \varphi^2(s)$, it gives $\frac{L(\alpha)}{(1-\alpha)\varphi^2(s)+\alpha}\Big(\varphi^2(s)EF\{y(t)\} - \varphi^{-1}(s)y(0)\Big)$ where EF{y(t)} is Emad-Falih transform [22].
(6) For $\phi^{\alpha}(s) = \frac{1}{u}$ and $\psi(s) = \frac{1}{u^2}$, transform gives $\frac{L(\alpha)}{(1-\alpha)+u^2\alpha}\Big(T\{y(t)\} - uy(0)\Big)$ where T{y(t)} is Tarig transform [12].
(7) For $\phi^{\alpha}(s) = s^m$ and $\psi(s) = s^n$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)s^n+\alpha}\Big(s^n G\{y(t)\} - s^m y(0)\Big)$ where G{y(t)} is G-transform [4].
(8) For $\phi^{\alpha}(s) = \frac{s}{v}$ and $\psi(s) = \frac{s}{v}i$, transform results in the $\frac{L(\alpha)}{(1-\alpha)si+v\alpha}\Big(siHY\{y(t)\} - sy(0)\Big)$ where HY{y(t)} is HY-transform [37].
(9) For $\phi^{\alpha}(s) = p(s)$ and $\psi(s) = q(s)$, transform results in the $\frac{L(\alpha)}{(1-\alpha)q(s)+\alpha}\Big(q(s)J\{y(t)\} - p(s)y(0)\Big)$, where J{y(t)} is Jafari transform [15].
(10) For $\phi^{\alpha}(s) = 1$ and $\psi(s) = \frac{1}{v}$, transform yields the $\frac{L(\alpha)}{(1-\alpha)+v\alpha}\Big(K\{y(t)\} - vy(0)\Big)$, where K{y(t)} is Kamal transform [43].
(11) For $\phi^{\alpha}(s) = 1$ and $\psi(s) = \frac{1}{v^2}$, new transform gives $\frac{L(\alpha)}{(1-\alpha)v^{-2}+\alpha}\Big(v^{-2}KF\{y(t)\} - y(0)\Big)$, where KF{y(t)} is Kushari-Fundo transform [19].
(12) For $\phi^{\alpha}(s) = s^3$ and $\psi(s) = \frac{1}{s^2}$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)+s^2\alpha}\Big(KT\{y(t)\} - s^5 y(0)\Big)$, where KT{y(t)} is Kharrat-Toma transform [21].
(13) For $\phi^{\alpha}(s) = p$ and $\psi(s) = \frac{1}{p}$, transform results in the $\frac{L(\alpha)}{(1-\alpha)+p\alpha}\Big(ET\{y(t)\} - p^2 y(0)\Big)$, where ET{y(t)} is Elzaki-Tarig transform [32].
(14) For $\phi^{\alpha}(s) = v$ and $\psi(s) = v^{\alpha}$, transform yields the $\frac{L(\alpha)}{(1-\alpha)v^{\alpha}+\alpha}\Big(v^{\alpha}KU\{y(t)\} - vy(0)\Big)$, where KU{y(t)} is Kushare transform [25].

(15) For $\phi^\alpha(s) = v$ and $\psi(s) = v$, transform results in the $\frac{L(\alpha)}{(1-\alpha)v+\alpha}\Big(vMG\{y(t)\} - vy(0)\Big)$, where MG{y(t)} is Mahgoub transform [27].

(16) For $\phi^\alpha(s) = v^2$ and $\psi(s) = v$, transform yields the $\frac{L(\alpha)}{(1-\alpha)v+\alpha}\Big(vM\{y(t)\} - v^2y(0)\Big)$, where M{y(t)} is Mohand transform [28].

(17) For $\phi^\alpha(s) = \frac{1}{u}$ and $\psi(s) = \frac{s}{u}$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)s+u\alpha}\Big(sRG\{y(t)\} - y(0)\Big)$, where RG{y(t)} is Ramdan Group transform [45].

(18) For $\phi^\alpha(s) = \frac{1}{u}$ and $\psi(s) = \frac{s}{u}$, transform results in $\frac{L(\alpha)}{(1-\alpha)s+u\alpha}\Big(sN\{y(t)\} - y(0)\Big)$, where N{y(t)} is N-transform [20].

(19) For $\phi^\alpha(s) = \frac{\sigma}{\varepsilon}$ and $\psi(s) = \frac{\sigma}{\varepsilon}$, our new transform gives $\frac{L(\alpha)}{(1-\alpha)\sigma+\varepsilon\alpha}\Big(\sigma R\{y(t)\} - \sigma y(0)\Big)$ where R{y(t)} is Rishi transform [24].

(20) For $\phi^\alpha(s) = \frac{1}{v^\alpha}$ and $\psi(s) = v^\alpha$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)v^\alpha+\alpha}\Big(v^\alpha S\{y(t)\} - v^{-\alpha}y(0)\Big)$, where S{y(t)} is Sadik transform [44].

(21) For $\phi^\alpha(s) = \frac{1}{v^2}$ and $\psi(s) = \frac{1}{v}$, transform gives $\frac{L(\alpha)}{(1-\alpha)v+\alpha v^2}\Big(vSa\{y(t)\} - y(0)\Big)$ where Sa{y(t)} is Sawi transform [29].

(22) For $\phi^\alpha(s) = 1$ and $\psi(s) = \frac{s}{u}$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)s+u\alpha}\Big(sSh\{y(t)\} - uy(0)\Big)$ where Sh{y(t)}, is Shehu transform [30].

(23) For $\phi^\alpha(s) = \frac{1}{v}$ and $\psi(s) = v^\alpha$, transform yields $\frac{L(\alpha)}{(1-\alpha)v^{\alpha+1}+v\alpha}\Big(v^{\alpha+1}So\{y(t)\} - y(0)\Big)$, where So{y(t)} is Soham transform [38].

(24) For $\phi^\alpha(s) = s$ and $\psi(s) = s$, our new transform gives $\frac{L(\alpha)}{(1-\alpha)s+\alpha}\Big(sLC\{y(t)\} - sy(0)\Big)$, where LC{y(t)} is Laplace-Carson transform [31].

(25) For $\phi^\alpha(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$, our new transform corresponding to $\frac{L(\alpha)}{(1-\alpha)+s\alpha}\Big(S\{y(t)\} - y(0)\Big)$, where S{y(t)} is Samudu transform [47].

(26) For $\phi^\alpha(s) = s$ and $\psi(s) = s^2$, our transform yields $\frac{L(\alpha)}{(1-\alpha)s^2+\alpha}\Big(s^2AJ\{y(t)\} - sy(0)\Big)$, where AJ{y(t)} is Pourreza transform [3].

(27) For $\phi^\alpha(s) = 1$ and $\psi(s) = s^{\frac{1}{\alpha}}$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)s^{\frac{1}{\alpha}}+\alpha}\Big(s^{\frac{1}{\alpha}}AL\{y(t)\} - y(0)\Big)$, where AL{y(t)} is α-Laplace transform [35].

(28) For $\phi^\alpha(s) = \frac{s}{u}$ and $\psi(s) = \frac{s}{u}$, transform gives $\frac{L(\alpha)}{(1-\alpha)s+u\alpha}\Big(sNa\{y(t)\} - sy(0)\Big)$, where Na{y(t)} is Natural transform [20].

(29) For $\phi^\alpha(s) = \frac{1}{q^3}$ and $\psi(s) = q$, new transform corresponds to $\frac{L(\alpha)}{(1-\alpha)q+\alpha}\Big(qG\{y(t)\} - q^{-3}y(0)\Big)$, where G{y(t)} is Gupta transform [13].

(30) For $\phi^{\alpha}(s) = p^5$ and $\psi(s) = p$, our transform produce the result $\frac{L(\alpha)}{(1-\alpha)p+\alpha}\Big(pDV\{y(t)\} - p^5 y(0)\Big)$, where DV{y(t)} is Dinesh-Verma transform [46].

(31) For $\phi^{\alpha}(s) = s$ and $\psi(s) = s$, transform results in $\frac{L(\alpha)}{(1-\alpha)s+\alpha}\Big(sRa\{y(t)\} - sy(0)\Big)$ where Ra{y(t)} is Raj transform [18].

(32) For $\phi^{\alpha}(s) = \frac{1}{u}$ and $\psi(s) = \frac{1}{us}$, new transform gives $\frac{L(\alpha)}{(1-\alpha)+us\alpha}\Big(AS\{y(t)\} - sy(0)\Big)$, where AS{y(t)} is Abaoub-Skheam transform [1].

(33) For $\phi^{\alpha}(s) = u$ and $\psi(s) = u$, our new transform yields $\frac{L(\alpha)}{(1-\alpha)u+\alpha}\Big(uJT\{y(t)\} - uy(0)\Big)$, where JT{y(t)} is Jabber-Tawfiq transform [14].

(34) For $\phi^{\alpha}(s) = 1$ and $\psi(s) = \frac{1}{u}$, new transform results in $\frac{L(\alpha)}{(1-\alpha)+u\alpha}\Big(Y\{y(t)\} - uy(0)\Big)$, where Y{y(t)} is Yang transform [9].

4.7. **Proposed Integral Transform on Riemann-Liouville derivative.** In 1832, Rieman and Liouville [41] presented the fractional derivative. The Riemann-Liouville derivative is defined as

$$ {}_a^{RL}D_t^{\alpha} y(t) = \frac{d^n}{dt^n}\{{}_aI_t^{n-\alpha} y(t)\}. \tag{4. 29}$$

Theorem 4.8. *If ${}_0^{RL}D_t^{\alpha} y(t)$ is the Riemann-Liouville derivative, the proposed transform on the Riemann-Liouville derivative gives $\psi^{\alpha}(s)S\{y(t)\} - \phi^{\alpha}(s)\Sigma_{k=0}^{n-1}\psi^k(s)D^{\alpha-k-1}(0)$.*

Proof. Let, $S\{y^n(t)\} = \psi^n(s)Y(s) - \phi^{\alpha}(s)\Sigma_{k=0}^{n-1}\psi^k(s)y^{n-k-1}(0)$, $y(s) = \{{}_0I_t^{n-\alpha}\}y(t)$ while $Y(s) = S\{y(t)\}$ then, with the proposed transform (2. 1) on the Riemann-Liouville derivative defined in (4. 29), we have

$$S\Big\{\frac{d^n}{dx^n}\{{}_0I_t^{n-\alpha} y(t)\}\Big\} = \psi^n(s)S\{{}_0I_t^{n-\alpha} y(t)\} - \phi^{\alpha}(s)\Sigma_{k=0}^{n-1}\psi^k(s)\frac{d^{n-k-1}}{dt^{n-k-1}}\{{}_0I_t^{n-\alpha} y(0)\}.$$

This implies

$$S\Big\{\frac{d^n}{dt^n}\{{}_0I_t^{n-\alpha} y(t)\}\Big\} = \psi^n(s)\{\psi^{\alpha-n}(s)Y(s)\} - \phi^{\alpha}(s)\Sigma_{k=0}^{n-1}\psi^k(s)\frac{d^{n-k-1}}{dt^{n-k-1}}D^{\alpha-n}y(0).$$

This can be written as

$$S\Big\{\frac{d^n}{dt^n}\{{}_0I_t^{n-\alpha} y(t)\}\Big\} = \psi^{\alpha}(s)S\{y(t)\} - \phi^{\alpha}(s)\Sigma_{k=0}^{n-1}\psi^k(s)D^{\alpha-k-1}(0). \tag{4. 30}$$

When the Riemann-Liouville derivative is transformed integrally, this is the general form that results. □

Proposition 4.9. *If $\phi^{\alpha}(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$ in equation (4. 30), then the proposed transform on the Riemann-Liouville fractional derivative converges to $s^{-\alpha}S\{y(t)\} - \Sigma_{k=0}^{n-1}s^{-k-1}D^{\alpha-k-1}(0)$, where $S\{y(t)\}$, is Sumudu transform.*

Proof. We can easily verify the above result using $\phi^\alpha(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$ in equation (4. 30)

$$S\Big\{\frac{d^n}{dt^n}\{{}_0I_t^{n-\alpha}y(t)\}\Big\} = \Big(\frac{1}{s}\Big)^\alpha S\{y(t)\} - \frac{1}{s}\Sigma_{k=0}^{n-1}\Big(\frac{1}{s}\Big)^k D^{\alpha-k-1}(0),$$
$$S\Big\{\frac{d^n}{dt^n}\{{}_0I_t^{n-\alpha}y(t)\}\Big\} = s^{-\alpha}S\{y(t)\} - \Sigma_{k=0}^{n-1}\Big(\frac{1}{s}\Big)^{k+1} D^{\alpha-k-1}(0).$$

Equivalently,

$$S\Big\{\frac{d^n}{dt^n}\{{}_0I_t^{n-\alpha}y(t)\}\Big\} = s^{-\alpha}S\{y(t)\} - \Sigma_{k=0}^{n-1}s^{-k-1}D^{\alpha-k-1}(0). \quad (4.\ 31)$$

Equation(4. 31) is equivalent to the results of Riemann-Liouville fractional derivative after applying Sumudu transform. □

4.9.1. ***Special cases of proposed transform via Riemann-Liouville Fractional derivative***. The Riemann-Liouville fractional derivative forms of existing integral transforms are generated by using the different values of $\phi^\alpha(s), \psi(s)$ and $\gamma(t) = 1$ in (4. 30). These cases are

(1) For $\phi^\alpha(s) = 1$ and $\psi(s) = s$, transform gives $s^\alpha L\{y(t)\} - \Sigma_{k=0}^{n-1}s^k D^{\alpha-k-1}(0)$, where $L\{y(t)\}$ is Laplace transform [39].
(2) For $\phi^\alpha(s) = \frac{1}{v}$ and $\psi(s) = v$, new transform gives $v^\alpha A\{y(t)\} - \Sigma_{k=0}^{n-1}v^{k-1}D^{\alpha-k-1}(0)$, where $A\{y(t)\}$ is Aboodh transform [2].
(3) For $\phi^\alpha(s) = p^2$ and $\psi(s) = \frac{1}{p}$, yields $p^{-\alpha}A\{y(t)\} - \Sigma_{k=0}^{n-1}p^{2-k}D^{\alpha-k-1}(0)$, where $A\{y(t)\}$ is Anuj transform [23].
(4) For $\phi^\alpha(s) = v$ and $\psi(s) = \frac{1}{v}$, produces $v^{-\alpha}E\{y(t)\} - \Sigma_{k=0}^{n-1}v^{1-k}D^{\alpha-k-1}(0)$, where $E\{y(t)\}$ is Elzaki transform [11].
(5) For $\phi^\alpha(s) = \frac{1}{\varphi(s)}$ and $\psi(s) = \varphi^2(s)$, it gives $\varphi^{2\alpha}(s)EF\{y(t)\} - \Sigma_{k=0}^{n-1}\varphi^{2k-1}(s)D^{\alpha-k-1}(0)$, where $EF\{y(t)\}$ is Emad-Falih transform [22].
(6) For $\phi^\alpha(s) = \frac{1}{u}$ and $\psi(s) = \frac{1}{u^2}$, transform gives $u^{-2\alpha}T\{y(t)\} - \Sigma_{k=0}^{n-1}u^{-2k-1}D^{\alpha-k-1}(0)$, where $T\{y(t)\}$ is Tarig transform [12].
(7) For $\phi^\alpha(s) = s^m$ and $\psi(s) = s^n$, transform corresponding to $s^{n\alpha}G\{y(t)\} - \Sigma_{k=0}^{n-1}s^{nk+m}D^{\alpha-k-1}(0)$, where $G\{y(t)\}$ is G-transform [4].
(8) For $\phi^\alpha(s) = \frac{s}{v}$ and $\psi(s) = \frac{s}{v}i$, transform results in the $\Big(\frac{si}{v}\Big)^\alpha HY\{y(t)\} - \Sigma_{k=0}^{n-1}\Big(\frac{s}{v}\Big)^{k+1}i^k D^{\alpha-k-1}(0)$, where $HY\{y(t)\}$ is HY transform [37].
(9) For $\phi^\alpha(s) = p(s)$ and $\psi(s) = q(s)$, transform results in the $q^\alpha(s)J\{y(t)\} - p(s)\Sigma_{k=0}^{n-1}q^k(s)D^{\alpha-k-1}(0)$, where $J\{y(t)\}$ is Jafari transform [15].
(10) For $\phi^\alpha(s) = 1$ and $\psi(s) = \frac{1}{v}$, transform yields the $v^{-\alpha}K\{y(t)\} - \Sigma_{k=0}^{n-1}v^{-k}D^{\alpha-k-1}(0)$, where $K\{y(t)\}$ is Kamal transform [43].
(11) For $\phi^\alpha(s) = 1$ and $\psi(s) = \frac{1}{v^2}$, new transform gives $v^{-2\alpha}KF\{y(t)\} - \Sigma_{k=0}^{n-1}v^{-2k}D^{\alpha-k-1}(0)$, where $KF\{y(t)\}$ is Kushuri-Fundo transform [19].

(12) For $\phi^{\alpha}(s) = s^3$ and $\psi(s) = \frac{1}{s^2}$, transform corresponding to $s^{-2\alpha}KT\{y(t)\} - \Sigma_{k=0}^{n-1} s^{-2k+3} D^{\alpha-k-1}(0)$, where $KT\{y(t)\}$ is Kharrat-Toma transform [21].

(13) For $\phi^{\alpha}(s) = p$ and $\psi(s) = \frac{1}{p}$, transform results in the $p^{-\alpha}ET\{y(t)\} - \Sigma_{k=0}^{n-1} p^{-k+1} D^{\alpha-k-1}(0)$, where $ET\{y(t)\}$ is Elzaki-Tarig transform [32].

(14) For $\phi^{\alpha}(s) = v$ and $\psi(s) = v^{\alpha}$, transform yields the $v^{2\alpha}KU\{y(t)\} - \Sigma_{k=0}^{n-1} v^{\alpha k+1} D^{\alpha-k-1}(0)$, where $KU\{y(t)\}$ gives Kushare transform [25].

(15) For $\phi^{\alpha}(s) = v$ and $\psi(s) = v$, transform results in the $v^{\alpha}MG\{y(t)\} - \Sigma_{k=0}^{n-1} v^{k+1} D^{\alpha-k-1}(0)$, where $MG\{y(t)\}$ is Mahgoub transform [27].

(16) For $\phi^{\alpha}(s) = v^2$ and $\psi(s) = v$, transform yields the $v^{\alpha}M\{y(t)\} - \Sigma_{k=0}^{n-1} v^{k+2} D^{\alpha-k-1}(0)$, where $M\{y(t)\}$ is Mohand transform [28].

(17) For $\phi^{\alpha}(s) = \frac{1}{u}$ and $\psi(s) = \frac{s}{u}$, transform corresponding to $\left(\frac{s}{u}\right)^{\alpha} RG\{y(t)\} - \Sigma_{k=0}^{n-1} \frac{s^k}{u^{k+1}} D^{\alpha-k-1}(0)$, where $RG\{y(t)\}$ is Ramdan Group transform [45].

(18) For $\phi^{\alpha}(s) = \frac{1}{u}$ and $\psi(s) = \frac{s}{u}$, transform results in $\left(\frac{s}{u}\right)^{\alpha} N\{y(t)\} - \Sigma_{k=0}^{n-1} \frac{s^k}{u^{k+1}} D^{\alpha-k-1}(0)$, where $N\{y(t)\}$ is N-transform [20].

(19) For $\phi^{\alpha}(s) = \frac{\sigma}{\varepsilon}$ and $\psi(s) = \frac{\sigma}{\varepsilon}$, our new transform gives $\left(\frac{\sigma}{\varepsilon}\right)^{\alpha} R\{y(t)\} - \Sigma_{k=0}^{n-1} \left(\frac{\sigma}{\varepsilon}\right)^{k+1} D^{\alpha-k-1}(0)$, where $R\{y(t)\}$ is Rishi transform [24].

(20) For $\phi^{\alpha}(s) = \frac{1}{v^{\beta}}$ and $\psi(s) = v^{\alpha}$, transform corresponding to $v^{2\alpha}S\{y(t)\} - \Sigma_{k=0}^{n-1} v^{\alpha k-\beta} D^{\alpha-k-1}(0)$, where $S\{y(t)\}$ is Sadik transform [44].

(21) For $\phi^{\alpha}(s) = \frac{1}{v^2}$ and $\psi(s) = \frac{1}{v}$, transform gives $v^{-\alpha}Sa\{y(t)\} - \Sigma_{k=0}^{n-1} v^{-k-2} D^{\alpha-k-1}(0)$, where $Sa\{y(t)\}$ is Sawi transform [29].

(22) For $\phi^{\alpha}(s) = 1$ and $\psi(s) = \frac{s}{u}$, transform corresponding to $\left(\frac{s}{u}\right)^{\alpha} Sh\{y(t)\} - \Sigma_{k=0}^{n-1} \left(\frac{s}{u}\right)^{k} D^{\alpha-k-1}(0)$, where $Sh\{y(t)\}$ is Shehu transform [30].

(23) For $\phi^{\alpha}(s) = \frac{1}{v}$ and $\psi(s) = v^{\alpha}$, transform yields $v^{2\alpha}So\{y(t)\} - \Sigma_{k=0}^{n-1} v^{\alpha k-1} D^{\alpha-k-1}(0)$, where $So\{y(t)\}$ is Soham transform [38].

(24) For $\phi^{\alpha}(s) = s$ and $\psi(s) = s$, our new transform gives $s^{\alpha}LC\{y(t)\} - \Sigma_{k=0}^{n-1} s^{k+1} D^{\alpha-k-1}(0)$, where $LC\{y(t)\}$ is Laplace-Carson transform [31].

(25) For $\phi^{\alpha}(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$, our new transform corresponding to $s^{-\alpha}S\{y(t)\} - \Sigma_{k=0}^{n-1} s^{-k-1} D^{\alpha-k-1}(0)$, where $S\{y(t)\}$ is Samudu transform [47].

(26) For $\phi^{\alpha}(s) = s$ and $\psi(s) = s^2$, our transform yields $s^{2\alpha}AJ\{y(t)\} - \Sigma_{k=0}^{n-1} s^{2k+1} D^{\alpha-k-1}(0)$, where $AJ\{y(t)\}$ is Pourreza transform [3].

(27) For $\phi^{\alpha}(s) = 1$ and $\psi(s) = s^{\frac{1}{\alpha}}$, transform corresponding to $sAL\{y(t)\} - \Sigma_{k=0}^{n-1} s^{\frac{k}{\alpha}} D^{\alpha-k-1}(0)$, where $AL\{y(t)\}$ is α-Laplace transform [35].

(28) For $\phi^{\alpha}(s) = \frac{s}{u}$ and $\psi(s) = \frac{s}{u}$, transform gives $\left(\frac{s}{u}\right)^{\alpha} Na\{y(t)\} - \Sigma_{k=0}^{n-1} \left(\frac{s}{u}\right)^{k+1} D^{\alpha-k-1}(0)$, where $Na\{y(t)\}$ is Natural-transform [20].

(29) For $\phi^\alpha(s) = \frac{1}{q^3}$ and $\psi(s) = q$, new transform corresponds to $q^\alpha G\{y(t)\} - \Sigma_{k=0}^{n-1} q^{k-3} D^{\alpha-k-1}(0)$, where $G\{y(t)\}$ is Gupta transform [13].
(30) For $\phi^\alpha(s) = p^5$ and $\psi(s) = p$, our transform produce the result $p^\alpha DV\{y(t)\} - \Sigma_{k=0}^{n-1} p^{k+5} D^{\alpha-k-1}(0)$, where $DV\{y(t)\}$ is Dinesh-Verma transform [46].
(31) For $\phi^\alpha(s) = s$ and $\psi(s) = s$, transform results in $s^\alpha Ra\{y(t)\} - \Sigma_{k=0}^{n-1} s^{k+1} D^{\alpha-k-1}(0)$, where $Ra\{y(t)\}$ is Raj transform [18].
(32) For $\phi^\alpha(s) = \frac{1}{u}$ and $\psi(s) = \frac{1}{us}$, new transform gives $(us)^{-\alpha} AS\{y(t)\} - \Sigma_{k=0}^{n-1} (s)^{-k} (u)^{-k-1} D^{\alpha-k-1}(0)$, where $AS\{y(t)\}$ is Abaoub-Skheam transform [1].
(33) For $\phi^\alpha(s) = u$ and $\psi(s) = u$, our new transform yields $u^\alpha JT\{y(t)\} - \Sigma_{k=0}^{n-1} u^{k+1} D^{\alpha-k-1}(0)$, where $JT\{y(t)\}$ is Jabber-Tawfiq transform [14].
(34) For $\phi^\alpha(s) = 1$ and $\psi(s) = \frac{1}{u}$, new transform results in $u^{-\alpha} Y\{y(t)\} - \Sigma_{k=0}^{n-1} u^{-k} D^{\alpha-k-1}(0)$, where $Y\{y(t)\}$ is Yang transform [9].

4.10. **Proposed Integral Transform on New Fractional Derivative (NFD).** In 2022, Jassim and Hussein [16] introduced a New Fractional Derivative (NFD). The NFD converges faster to the classical calculus. In this section, the general form after applying the proposed transform to NFD is presented. The NFD is defined as

$$ {}_a^M D_t^\alpha y(t) = M_\alpha \int_a^t y^{(n)}(\tau) e^{-M_\alpha (t-T)} dT, \qquad n-1 < \alpha \leq n, \tag{4. 32}$$

where M_α is the function of α, such that $\lim_{\alpha \longrightarrow n} M_\alpha = \infty$.

Theorem 4.11. *If ${}_0^M D_t^\alpha y(t)$ is the NFD, then the proposed transform gives* $\frac{M_\alpha}{\psi(s)+M_\alpha} \Big\{ \psi^n(s) S\{y(t)\} - \phi^\alpha(s) \Sigma_{k=0}^{n-1} \psi^k(s) y^{n-k-1}(0) \Big\}$.

Proof. After applying the proposed transform (2. 1) on NFD defined in (4. 32) and using properties from Table 1, we have

$$ S\{{}_0^M D_t^\alpha y(t)\} = M_\alpha \frac{1}{\phi^\alpha(s)} \Big\{ S\{y^{(n)}(t)\} S\{e^{-M_\alpha (t)}\} \Big\}. $$

This implies

$$ S\{{}_0^M D_t^\alpha y(t)\} = \frac{M_\alpha}{\phi^\alpha(s)} \Big\{ \psi^n(s) Y(s) - \phi^\alpha(s) \Sigma_{k=0}^{n-1} \psi^k(s) y^{n-k-1}(0) \Big\} \Big\{ \frac{\phi^\alpha(s)}{\psi(s) + M_\alpha} \Big\}. $$

After simplification, we get

$$ S\{{}_0^M D_t^\alpha y(t)\} = \frac{M_\alpha}{\psi(s) + M_\alpha} \Big\{ \psi^n(s) S\{y(t)\} - \phi^\alpha(s) \Sigma_{k=0}^{n-1} \psi^k(s) y^{n-k-1}(0) \Big\} \tag{4. 33}$$

Once the integral transform has been applied to the new fractional derivative, this is the general form. □

Proposition 4.12. *If $\phi^\alpha(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$ in equation (4. 33), then the proposed transform on the NFD converges to $\frac{M_\alpha}{s^{-1}+M_\alpha} \Big\{ s^{-n} S\{y(t)\} - \Sigma_{k=0}^{n-1} s^{-k-1} y^{n-k-1}(0) \Big\}$, where $S\{y(t)\}$, is Sumudu transform.*

Proof. We can easily verify the above result using $\phi^{\alpha}(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$ in equation (4. 33)

$$S\{{}_0^M D_t^{\alpha} y(t)\} = \frac{M_{\alpha}}{\left(\frac{1}{s}\right) + M_{\alpha}} \left\{ \left(\frac{1}{s}\right)^n S\{y(t)\} - \frac{1}{s} \Sigma_{k=0}^{n-1} \left(\frac{1}{s}\right)^k y^{n-k-1}(0) \right\},$$

$$S\{{}_0^M D_t^{\alpha} y(t)\} = \frac{M_{\alpha}}{s^{-1} + M_{\alpha}} \left\{ s^{-n} S\{y(t)\} - \Sigma_{k=0}^{n-1} \left(\frac{1}{s}\right)^{k+1} y^{n-k-1}(0) \right\}.$$

Or equivalently,

$$S\{{}_0^M D_t^{\alpha} y(t)\} = \frac{M_{\alpha}}{s^{-1} + M_{\alpha}} \left\{ s^{-n} S\{y(t)\} - \Sigma_{k=0}^{n-1} s^{-k-1} y^{n-k-1}(0) \right\}. \quad (4.\ 34)$$

Equation(4. 34) is equivalent to the results of NFD after applying Sumudu transform. □

4.12.1. ***Special cases of proposed transform via New Fractional Derivative***. The NFD forms of many existing integral transforms are generated after substituting the different values of $\phi^{\alpha}(s)$, $\psi(s)$ and $\gamma(t) = 1$ in (4. 33). These cases are

(1) For $\phi^{\alpha}(s) = 1$ and $\psi(s) = s$, transform gives $\frac{M_{\alpha}}{s+M_{\alpha}} \left\{ s^n L\{y(t)\} - \Sigma_{k=0}^{n-1} s^k y^{n-k-1}(0) \right\}$, where $L\{y(t)\}$ is Laplace transform [39].
(2) For $\phi^{\alpha}(s) = \frac{1}{v}$ and $\psi(s) = v$, new transform gives $\frac{M_{\alpha}}{v+M_{\alpha}} \left\{ v^n A\{y(t)\} \} - \Sigma_{k=0}^{n-1} v^{k-1} y^{n-k-1}(0) \right\}$, where $A\{y(t)\}$ is Aboodh transform [2].
(3) For $\phi^{\alpha}(s) = p^2$ and $\psi(s) = \frac{1}{p}$, yields $\frac{M_{\alpha}}{1+pM_{\alpha}} \left\{ p^{1-n} A\{y(t)\} \} - \Sigma_{k=0}^{n-1} p^{3-k} y^{n-k-1}(0) \right\}$, where $A\{y(t)\}$ is Anuj transform [23].
(4) For $\phi^{\alpha}(s) = v$ and $\psi(s) = \frac{1}{v}$, produces $\frac{M_{\alpha}}{1+vM_{\alpha}} \left\{ v^{1-n} E\{y(t)\} - \Sigma_{k=0}^{n-1} v^{2-k} y^{n-k-1}(0) \right\}$, where $E\{y(t)\}$ is Elzaki transform [11].
(5) For $\phi^{\alpha}(s) = \frac{1}{\varphi(s)}$ and $\psi(s) = \varphi^2(s)$, it gives $\frac{M_{\alpha}}{\varphi^2(s)+M_{\alpha}} \left\{ \varphi^{2n-1}(s) EF\{y(t)\} - \Sigma_{k=0}^{n-1} \varphi^{2k-1}(s) y^{n-k-1}(0) \right\}$, where $EF\{y(t)\}$ is Emad-Falih transform [22].
(6) For $\phi^{\alpha}(s) = \frac{1}{u}$ and $\psi(s) = \frac{1}{u^2}$, transform gives $\frac{M_{\alpha}}{1+u^2 M_{\alpha}} \left\{ u^{2-2n} T\{y(t)\} - \Sigma_{k=0}^{n-1} u^{1-2k} y^{n-k-1}(0) \right\}$, where $T\{y(t)\}$ is Tarig transform [12].
(7) For $\phi^{\alpha}(s) = s^m$ and $\psi(s) = s^n$, transform corresponding to $\frac{M_{\alpha}}{s^n+M_{\alpha}} \left\{ s^{2n} G\{y(t)\} - \Sigma_{k=0}^{n-1} s^{nk+m} y^{n-k-1}(0) \right\}$, where $G\{y(t)\}$ is G-transform [4].
(8) For $\phi^{\alpha}(s) = \frac{s}{v}$ and $\psi(s) = \frac{s}{v} i$, transform results in the $\Big(\frac{M_{\alpha}}{si+vM_{\alpha}} \left\{ \frac{(si)^n}{v^{n-1}} HY\{y(t)\} - \Sigma_{k=0}^{n-1} s^{k+1} (\frac{i}{v})^k y^{n-k-1}(0) \right\}$, where $HY\{y(t)\}$ is HY transform [37].
(9) For $\phi^{\alpha}(s) = p(s)$ and $\psi(s) = q(s)$, transform results in the $\frac{M_{\alpha}}{q(s)+M_{\alpha}} \left\{ q^n(s) J\{y(t)\} - p(s) \Sigma_{k=0}^{n-1} q^k(s) y^{n-k-1}(0) \right\}$, where $J\{y(t)\}$ is Jafari transform [15].

(10) For $\phi^{\alpha}(s) = 1$ and $\psi(s) = \frac{1}{v}$, transform yields the $\frac{M_\alpha}{v^{-1}+M_\alpha}\Big\{v^{-n}K\{y(t)\} - \Sigma_{k=0}^{n-1}v^{-k}y^{n-k-1}(0)\Big\}$, where $K\{y(t)\}$ is Kamal transform [43].

(11) For $\phi^{\alpha}(s) = 1$ and $\psi(s) = \frac{1}{v^2}$, new transform gives $\frac{M_\alpha}{v^{-2}+M_\alpha}\Big\{v^{-2n}KF\{y(t)\} - \Sigma_{k=0}^{n-1}v^{-2k}y^{n-k-1}(0)\Big\}$, where $KF\{y(t)\}$ is Kashuri-Fundo transform [19].

(12) For $\phi^{\alpha}(s) = s^3$ and $\psi(s) = \frac{1}{s^2}$, transform corresponding to $\frac{M_\alpha}{s^{-2}+M_\alpha}\Big\{s^{-2n}KT\{y(t)\} - \Sigma_{k=0}^{n-1}s^{3-2k}y^{n-k-1}(0)\Big\}$, where $KT\{y(t)\}$ is Kharrat-Toma transform [21].

(13) For $\phi^{\alpha}(s) = p$ and $\psi(s) = \frac{1}{p}$, transform results in the $\frac{M_\alpha}{p^{-1}+M_\alpha}\Big\{p^{-n}ET\{y(t)\} - \Sigma_{k=0}^{n-1}p^{1-k}y^{n-k-1}(0)\Big\}$, where $ET\{y(t)\}$ is Elzaki-Tarig transform [32].

(14) For $\phi^{\alpha}(s) = v$ and $\psi(s) = v^{\alpha}$, transform yields the $\frac{M_\alpha}{v^{\alpha}+M_\alpha}\Big\{v^{\alpha n}KU\{y(t)\} - \Sigma_{k=0}^{n-1}v^{1+\alpha k}y^{n-k-1}(0)\Big\}$, where $KU\{y(t)\}$ is Kushare transform [25].

(15) For $\phi^{\alpha}(s) = v$ and $\psi(s) = v$, transform results in the $\frac{M_\alpha}{v+M_\alpha}\Big\{v^{n}MG\{y(t)\} - \Sigma_{k=0}^{n-1}v^{1+k}y^{n-k-1}(0)\Big\}$, where $MG\{y(t)\}$ is Mahgoub transform [27].

(16) For $\phi^{\alpha}(s) = v^2$ and $\psi(s) = v$, transform yields the $\frac{M_\alpha}{v+M_\alpha}\Big\{v^{n}M\{y(t)\} - \Sigma_{k=0}^{n-1}v^{k+2}y^{n-k-1}(0)\Big\}$, where $M\{y(t)\}$ is Mohand transform [28].

(17) For $\phi^{\alpha}(s) = \frac{1}{u}$ and $\psi(s) = \frac{s}{u}$, transform corresponding to $\frac{M_\alpha}{s+uM_\alpha}\Big\{\frac{s^n}{u^{n-1}}RG\{y(t)\} - \Sigma_{k=0}^{n-1}(\frac{s}{u})^{k}y^{n-k-1}(0)\Big\}$, where $RG\{y(t)\}$ is Ramdan Group transform [45].

(18) For $\phi^{\alpha}(s) = \frac{1}{u}$ and $\psi(s) = \frac{s}{u}$, transform results in $\frac{M_\alpha}{s+uM_\alpha}\Big\{\frac{s^n}{u^{n-1}}N\{y(t)\} - \Sigma_{k=0}^{n-1}(\frac{s}{u})^{k}y^{n-k-1}(0)\Big\}$, where $N\{y(t)\}$ is N-transform [20].

(19) For $\phi^{\alpha}(s) = \frac{\sigma}{\varepsilon}$ and $\psi(s) = \frac{\sigma}{\varepsilon}$, our new transform gives $\frac{M_\alpha}{\sigma+\varepsilon M_\alpha}\Big\{\frac{\sigma^n}{\varepsilon^{n-1}}R\{y(t)\} - \Sigma_{k=0}^{n-1}\frac{\sigma^{1+k}}{\varepsilon^k}y^{n-k-1}(0)\Big\}$, where $R\{y(t)\}$ is Rishi transform [24].

(20) For $\phi^{\alpha}(s) = \frac{1}{v^{\beta}}$ and $\psi(s) = v^{\alpha}$, transform corresponding to $\frac{M_\alpha}{v^{\alpha}+M_\alpha}\Big\{v^{n\alpha}S\{y(t)\} - \Sigma_{k=0}^{n-1}v^{\alpha k-\beta}y^{n-k-1}(0)\Big\}$, where $S\{y(t)\}$ is Sadik transform [44].

(21) For $\phi^{\alpha}(s) = \frac{1}{v^2}$ and $\psi(s) = \frac{1}{v}$, transform gives $\frac{M_\alpha}{v^{-1}+M_\alpha}\Big\{v^{-n}Sa\{y(t)\} - \Sigma_{k=0}^{n-1}v^{-k-2}y^{n-k-1}(0)\Big\}$, where $Sa\{y(t)\}$ is Sawi transform [29].

(22) For $\phi^{\alpha}(s) = 1$ and $\psi(s) = \frac{s}{u}$, transform corresponding to $\frac{M_\alpha}{s+uM_\alpha}\Big\{\frac{s^n}{u^{n-1}}Sh\{y(t)\} - \Sigma_{k=0}^{n-1}\frac{s^k}{u^{k-1}}y^{n-k-1}(0)\Big\}$, where $Sh\{y(t)\}$ is Shehu-transform [30].

(23) For $\phi^{\alpha}(s)=\frac{1}{v}$ and $\psi(s)=v^{\alpha}$, transform yields $\frac{M_{\alpha}}{v^{\alpha}+M_{\alpha}}\left\{v^{\alpha n}So\{y(t)\}-\Sigma_{k=0}^{n-1}v^{\alpha k-1}y^{n-k-1}(0)\right\}$ where $So\{y(t)\}$, is Soham transform [38].

(24) For $\phi^{\alpha}(s)=s$ and $\psi(s)=s$, our new transform gives $\frac{M_{\alpha}}{s+M_{\alpha}}\left\{s^{n}LC\{y(t)\}-\Sigma_{k=0}^{n-1}s^{k+1}y^{n-k-1}(0)\right\}$, where $LC\{y(t)\}$ is Laplace-Carson transform [31].

(25) For $\phi^{\alpha}(s)=\frac{1}{s}$ and $\psi(s)=\frac{1}{s}$, our new transform corresponding to $\frac{M_{\alpha}}{s^{-1}+M_{\alpha}}\left\{s^{-n}S\{y(t)\}-\Sigma_{k=0}^{n-1}s^{-k-1}y^{n-k-1}(0)\right\}$, where $S\{y(t)\}$ is Sumudu transform [47].

(26) For $\phi^{\alpha}(s)=s$ and $\psi(s)=s^{2}$, our transform yields $\frac{M_{\alpha}}{s^{2}+M_{\alpha}}\left\{s^{2n}AJ\{y(t)\}-\Sigma_{k=0}^{n-1}s^{2k+1}y^{n-k-1}(0)\right\}$, where $AJ\{y(t)\}$ is Pourreza transform [3].

(27) For $\phi^{\alpha}(s)=1$ and $\psi(s)=s^{\frac{1}{\alpha}}$, transform corresponding to $\frac{M_{\alpha}}{s^{\frac{1}{\alpha}}+M_{\alpha}}\left\{s^{\frac{n}{\alpha}}AL\{y(t)\}-\Sigma_{k=0}^{n-1}s^{\frac{k}{\alpha}}y^{n-k-1}(0)\right\}$, where $AL\{y(t)\}$ is α-transform [35].

(28) For $\phi^{\alpha}(s)=\frac{s}{u}$ and $\psi(s)=\frac{s}{u}$, transform gives $\frac{M_{\alpha}}{s+uM_{\alpha}}\left\{\frac{s^{n}}{u^{n-1}}N\{y(t)\}-\Sigma_{k=0}^{n-1}\frac{s^{k+1}}{u^{k}}y^{n-k-1}(0)\right\}$, where $Na\{y(t)\}$ is Natural transform [20].

(29) For $\phi^{\alpha}(s)=\frac{1}{q^{3}}$ and $\psi(s)=q$, new transform corresponds to $\frac{M_{\alpha}}{q+M_{\alpha}}\left\{q^{n}G\{y(t)\}-\Sigma_{k=0}^{n-1}q^{k-3}y^{n-k-1}(0)\right\}$, where $G\{y(t)\}$ is Gupta transform [13].

(30) For $\phi^{\alpha}(s)=p^{5}$ and $\psi(s)=p$, our transform produce the result $\frac{M_{\alpha}}{p+M_{\alpha}}\left\{p^{n}DV\{y(t)\}-\Sigma_{k=0}^{n-1}p^{k+5}y^{n-k-1}(0)\right\}$, where $DV\{y(t)\}$ is Dinesh-Verma transform [46].

(31) For $\phi^{\alpha}(s)=s$ and $\psi(s)=s$, transform results in $\frac{M_{\alpha}}{s+M_{\alpha}}\left\{s^{n}Ra\{y(t)\}-\Sigma_{k=0}^{n-1}s^{k+1}y^{n-k-1}(0)\right\}$, where $Ra\{y(t)\}$ is Raj transform [18].

(32) For $\phi^{\alpha}(s)=\frac{1}{u}$ and $\psi(s)=\frac{1}{us}$, new transform gives $\frac{M_{\alpha}}{(us)^{-1}+M_{\alpha}}\left\{(us)^{-n}AS\{y(t)\}-\Sigma_{k=0}^{n-1}u^{-k-1}s^{-k}y^{n-k-1}(0)\right\}$, where $AS\{y(t)\}$ is Abaoub-Skheam transform [1].

(33) For $\phi^{\alpha}(s)=u$ and $\psi(s)=u$, our new transform yields $\frac{M_{\alpha}}{u+M_{\alpha}}\left\{u^{n}JT\{y(t)\}-\Sigma_{k=0}^{n-1}u^{k+1}y^{n-k-1}(0)\right\}$, where $JT\{y(t)\}$ is Jabber-Tawfiq transform [14].

(34) For $\phi^{\alpha}(s)=1$ and $\psi(s)=\frac{1}{u}$, new transform results in $\frac{M_{\alpha}}{u^{-1}+M_{\alpha}}\left\{u^{-n}Y\{y(t)\}-\Sigma_{k=0}^{n-1}u^{-k}y^{n-k-1}(0)\right\}$, where $Y\{y(t)\}$ is Yang transform [9].

4.13. **Proposed Integral Transform on Atangana-Baleanu Derivative in Caputo sense.** Let $f \in H^1(a,b),\ a < b,\ \alpha \in (0,1]$, then the Atangana-Baleanu [5] in Caputo sense is given as

$$ {}_{a}^{ABC}D_t^{\alpha} y(t) = \frac{L(\alpha)}{1-\alpha}\int_a^t y'(T)E_{\alpha}\Big(-\alpha\frac{(t-T)^{\alpha}}{1-\alpha}\Big)dT. \tag{4. 35}$$

where $E_{\alpha}\Big(-\alpha\frac{(t-T)^{\alpha}}{1-\alpha}\Big)dT = \sum_{m=o}^{\infty}\frac{\frac{-\alpha(t-T)^{m\alpha}}{1-\alpha}}{\Gamma(m\alpha+1)}$ and $L(\alpha)$ is a normalization function, such that $L(0) = L(1) = 1$.

Theorem 4.14. *If ${}_{0}^{ABC}D_t^{\alpha} y(t)$ is the Atangana-Baleanu derivative in the Caputo sense, then $S\{{}_{a}^{ABC}D_t^{\alpha} y(t)\}$ gives $\frac{L(\alpha)}{(1-\alpha)\psi^{\alpha}(s)+\alpha}\Big(\psi^{\alpha}(s)S\{y(t)\} - \phi^{\alpha}(s)\psi^{\alpha-1}y(0)\Big)$.*

Proof. After applying the proposed transform (2. 1) on the Atangana-Baleanu derivative defined in (4. 35), we have

$$S\{{}_{0}^{ABC}D_t^{\alpha} y(t)\} = \frac{L(\alpha)}{(1-\alpha)}\frac{1}{\phi^{\alpha}(s)}\Big\{\Big(\psi(s)Y(s) - \phi^{\alpha}(s)y(0)\Big)\phi^{\alpha}(s)\Big(\frac{\psi^{\alpha-1}(s)}{\psi^{\alpha}(s) - \Big(\frac{-\alpha}{1-\alpha}\Big)}\Big)\Big\}.$$

After simplification, we get

$$S\{{}_{0}^{ABC}D_t^{\alpha} y(t)\} = \frac{L(\alpha)}{(1-\alpha)}\Big\{\Big(\psi(s)Y(s) - \phi^{\alpha}(s)y(0)\Big)\Big(\frac{\psi^{\alpha-1}(s)}{\frac{(1-\alpha)\psi^{\alpha}(s)+\alpha}{(1-\alpha)}}\Big)\Big\}.$$

This implies

$$S\{{}_{0}^{ABC}D_t^{\alpha} y(t)\} = \frac{L(\alpha)}{(1-\alpha)\psi^{\alpha}(s)+\alpha}\Big(\psi^{\alpha}(s)S\{y(t)\} - \phi^{\alpha}(s)\psi^{\alpha-1}y(0)\Big). \tag{4. 36}$$

□

Proposition 4.15. *If $\phi^{\alpha}(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$ in equation (4. 36), then the proposed transform on the Atangana-Baleanu derivative in Caputo sense converges to $\frac{L(\alpha)}{(1-\alpha)+\alpha s^{\alpha}}\Big(S\{y(t)\} - y(0)\}\Big)$, where $S\{y(t)\}$, is Sumudu transform.*

Proof. We can easily verify the above result using $\phi^{\alpha}(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$ in equation (4. 36)

$$\begin{aligned} S\{{}_{0}^{ABC}D_t^{\alpha} y(t)\} &= \frac{L(\alpha)}{(1-\alpha)\Big(\frac{1}{s}\Big)^{\alpha}+\alpha}\Big(\Big(\frac{1}{s}\Big)^{\alpha}S\{y(t)\} - \Big(\frac{1}{s}\Big)\Big(\frac{1}{s}\Big)^{\alpha-1}y(0)\Big), \\ S\{{}_{0}^{ABC}D_t^{\alpha} y(t)\} &= \frac{s^{\alpha}L(\alpha)}{(1-\alpha)+\alpha s^{\alpha}}\Big(\frac{1}{s}\Big)^{\alpha}\Big(S\{y(t)\} - y(0)\Big). \end{aligned}$$

Or equivalently,

$$S\{{}_{0}^{ABC}D_t^{\alpha} y(t)\} = \frac{L(\alpha)}{(1-\alpha)+\alpha s^{\alpha}}\Big(S\{y(t)\} - y(0)\}\Big). \tag{4. 37}$$

Equation(4. 37) is equivalent to the results of Atangana-Baleanu derivative in Caputo sense after applying Sumudu transform. □

4.15.1. ***Special cases of proposed transform via Atangana-Baleanu derivative in Caputo sense.*** When different values of $\phi^{\alpha}(s)$, $\psi(s)$ and $\gamma(t) = 1$ are substituted in (4. 36), the Atangana-Baleanu derivative in Caputo forms of many existing integral transforms is produced. These unique instances are

(1) For $\phi^{\alpha}(s) = 1$ and $\psi(s) = s$, transform gives $\frac{L(\alpha)}{(1-\alpha)s^{\alpha}+\alpha}\Big(s^{\alpha}L\{y(t)\} - s^{\alpha-1}y(0)\Big)$, where $L\{y(t)\}$ is Laplace transform [39].
(2) For $\phi^{\alpha}(s) = \frac{1}{v}$ and $\psi(s) = v$, new transform gives $\frac{L(\alpha)}{(1-\alpha)v^{\alpha}+\alpha}\Big(v^{\alpha}A\{y(t)\} - v^{\alpha-2}y(0)\Big)$, where $A\{y(t)\}$ is Aboodh transform [2].
(3) For $\phi^{\alpha}(s) = p^2$ and $\psi(s) = \frac{1}{p}$, yields $\frac{L(\alpha)}{(1-\alpha)p^{-\alpha}+\alpha}\Big(p^{-\alpha}A\{y(t)\} - p^{-\alpha+3}y(0)\Big)$, where $A\{y(t)\}$ is Anuj transform [23].
(4) For $\phi^{\alpha}(s) = v$ and $\psi(s) = \frac{1}{v}$, produces $\frac{L(\alpha)}{(1-\alpha)v^{-\alpha}+\alpha}\Big(v^{-\alpha}E\{y(t)\} - v^{-\alpha+2}y(0)\Big)$, where $E\{y(t)\}$ is Elzaki transform [11].
(5) For $\phi^{\alpha}(s) = \frac{1}{\varphi(s)}$ and $\psi(s) = \varphi^2(s)$, it gives $\frac{L(\alpha)}{(1-\alpha)\varphi^{2\alpha}(s)+\alpha}\Big(\varphi^{2\alpha}(s)EF\{y(t)\} - \varphi^{2\alpha-3}(s)y(0)\Big)$, where $EF\{y(t)\}$ is Emad-Falih transform [22].
(6) For $\phi^{\alpha}(s) = \frac{1}{u}$ and $\psi(s) = \frac{1}{u^2}$, transform gives $\frac{L(\alpha)}{(1-\alpha)u^{-2\alpha}+\alpha}\Big(u^{-2\alpha}T\{y(t) - u^{-2\alpha+1}y(0)\Big)$, where $T\{y(t)\}$ is Tarig transform[12].
(7) For $\phi^{\alpha}(s) = s^m$ and $\psi(s) = s^n$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)s^{n\alpha}+\alpha}\Big(s^{n\alpha}G\{y(t)\} - s^{n\alpha+m-n}y(0)\Big)$, where $G\{y(t)\}$ is G-transform [4].
(8) For $\phi^{\alpha}(s) = \frac{s}{v}$ and $\psi(s) = \frac{s}{v}i$, transform results in the $\frac{L(\alpha)}{(1-\alpha)(si)^{\alpha}+v^{\alpha}\alpha}\Big((si)^{\alpha}HY\{y(t)\} - s^{\alpha}i^{\alpha-1}\Big)y(0)\Big)$, where $HY\{y(t)\}$ is HY transform [37].
(9) For $\phi^{\alpha}(s) = p(s)$ and $\psi(s) = q(s)$, transform results in the $\frac{L(\alpha)}{(1-\alpha)q^{\alpha}(s)+\alpha}\Big(q^{\alpha}(s)J\{y(t)\} - p(s)q^{\alpha-1}(s)y(0)\Big)$, where $J\{y(t)\}$ is Jafari transform [15].
(10) For $\phi^{\alpha}(s) = 1$ and $\psi(s) = \frac{1}{v}$, transform yields the $\frac{L(\alpha)}{(1-\alpha)v^{-\alpha}+\alpha}\Big(v^{-\alpha}K\{y(t)\} - v^{-\alpha+1}y(0)\Big)$, where $K\{y(t)\}$ is Kamal transform [43].
(11) For $\phi^{\alpha}(s) = 1$ and $\psi(s) = \frac{1}{v^2}$, new transform gives $\frac{L(\alpha)}{(1-\alpha)v^{-2\alpha}+\alpha}\Big(v^{-2\alpha}KF\{y(t)\} - v^{-2\alpha+2}y(0)\Big)$, where $KF\{y(t)\}$ is Kashuri-Fundo transform [19].
(12) For $\phi^{\alpha}(s) = s^3$ and $\psi(s) = \frac{1}{s^2}$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)s^{-2\alpha}+\alpha}\Big(s^{-2\alpha}KT\{y(t)\} - s^{-2\alpha+5}y(0)\Big)$, where $KT\{y(t)\}$ is Kharrat-Toma transform [21].
(13) For $\phi^{\alpha}(s) = p$ and $\psi(s) = \frac{1}{p}$, transform results in the $\frac{L(\alpha)}{(1-\alpha)p^{-\alpha}+\alpha}\Big(p^{-\alpha}ET\{y(t)\} - p^{-\alpha+2}y(0)\Big)$, where $ET\{y(t)\}$ is Elzaki-Tarig transform [32].

(14) For $\phi^\alpha(s) = v$ and $\psi(s) = v^\alpha$, transform yields the $\frac{L(\alpha)}{(1-\alpha)v^{2\alpha}+\alpha}\Big(v^{2\alpha}KU\{y(t)\} - v^{\alpha+1}y(0)\Big)$, where $KU\{y(t)\}$ is Kushare transform [25].

(15) For $\phi^\alpha(s) = v$ and $\psi(s) = v$, transform results in the $\frac{L(\alpha)}{(1-\alpha)v^{\alpha}+\alpha}\Big(v^{\alpha}MG\{y(t)\} - v^{\alpha}y(0)\Big)$, where $MG\{y(t)\}$ is Mahgoub transform [27].

(16) For $\phi^\alpha(s) = v^2$ and $\psi(s) = v$, transform yields the $\frac{L(\alpha)}{(1-\alpha)v^{\alpha}+\alpha}\Big(v^{\alpha}M\{y(t)\} - v^{\alpha+1}y(0)\Big)$, where $M\{y(t)\}$ is Mohand transform [28].

(17) For $\phi^\alpha(s) = \frac{1}{u}$ and $\psi(s) = \frac{s}{u}$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)s^{\alpha}+u^{\alpha}\alpha}\Big(s^{\alpha}RG\{y(t)\} - \frac{s^\alpha}{u^{\alpha+1}}y(0)\Big)$, where $RG\{y(t)\}$ is Ramdan Group transform [45].

(18) For $\phi^\alpha(s) = \frac{1}{u}$ and $\psi(s) = \frac{s}{u}$, transform results in $\frac{L(\alpha)}{(1-\alpha)s^{\alpha}+u^{\alpha}\alpha}\Big(s^{\alpha}N\{y(t)\} - \frac{s^\alpha}{u^{\alpha+1}}y(0)\Big)$, where $N\{y(t)\}$ is N-transform [20].

(19) For $\phi^\alpha(s) = \frac{\sigma}{\varepsilon}$ and $\psi(s) = \frac{\sigma}{\varepsilon}$, our new transform gives $\frac{L(\alpha)}{(1-\alpha)\sigma^{\alpha}+\varepsilon^{\alpha}\alpha}\Big(\sigma^{\alpha}R\{y(t)\} - \sigma^{\alpha}y(0)\Big)$, where $R\{y(t)\}$ is Rishi transform [24].

(20) For $\phi^\alpha(s) = \frac{1}{v^\alpha}$ and $\psi(s) = v^\alpha$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)v^{2\alpha}+\alpha}\Big(v^{2\alpha}S\{y(t)\} - v^{\alpha-\alpha}y(0)\Big)$, where $S\{y(t)\}$ is Sadik transform [44].

(21) For $\phi^\alpha(s) = \frac{1}{v^2}$ and $\psi(s) = \frac{1}{v}$, transform gives $\frac{L(\alpha)}{(1-\alpha)v^{-\alpha}+\alpha}\Big(v^{-\alpha}Sa\{y(t)\} - v^{-\alpha-1}y(0)\Big)$, where $Sa\{y(t)\}$ is Sawi transform [29].

(22) For $\phi^\alpha(s) = 1$ and $\psi(s) = \frac{s}{u}$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)s^{\alpha}+u^{\alpha}\alpha}\Big(s^{\alpha}Sh\{y(t)\} - \frac{u}{s^{1-\alpha}}y(0)\Big)$, where $Sh\{y(t)\}$ is Shehu transform [30].

(23) For $\phi^\alpha(s) = \frac{1}{v}$ and $\psi(s) = v^\alpha$, transform yields $\frac{L(\alpha)}{(1-\alpha)v^{2\alpha}+\alpha}\Big(v^{2\alpha}So\{y(t)\} - v^{\alpha-1}y(0)\Big)$, where $So\{y(t)\}$ is Soham transform [38].

(24) For $\phi^\alpha(s) = s$ and $\psi(s) = s$, our new transform gives $\frac{L(\alpha)}{(1-\alpha)s^{\alpha}+\alpha}\Big(s^{\alpha}LC\{y(t)\} - s^{\alpha}y(0)\}\Big)$, where $LC\{y(t)\}$ is Laplace-Carson transform [31].

(25) For $\phi^\alpha(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$, our new transform corresponding to $\frac{L(\alpha)}{(1-\alpha)+\alpha s^{\alpha}}\Big(S\{y(t)\} - y(0)\}\Big)$, where $S\{y(t)\}$ gives Sumudu transform [47].

(26) For $\phi^\alpha(s) = s$ and $\psi(s) = s^2$, our transform yields $\frac{L(\alpha)}{(1-\alpha)s^{2\alpha}+\alpha}\Big(s^{2\alpha}AJ\{y(t)\} - s^{2\alpha-1}y(0)\Big)$, where $AJ\{y(t)\}$ is Pourreza transform [3].

(27) For $\phi^\alpha(s) = 1$ and $\psi(s) = s^{\frac{1}{\alpha}}$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)s+\alpha}\Big(sAL\{y(t)\} - s^{1-\frac{1}{\alpha}}y(0)\Big)$, where $AL\{y(t)\}$ is α-transform [35].

(28) For $\phi^\alpha(s) = \frac{s}{u}$ and $\psi(s) = \frac{s}{u}$, transform gives $\frac{L(\alpha)}{(1-\alpha)s^{\alpha}+u^{\alpha}\alpha}\Big(s^{\alpha}N\{y(t)\} - s^{\alpha}y(0)\Big)$, where $Na\{y(t)\}$ is Natural transform [20].

(29) For $\phi^\alpha(s) = \frac{1}{q^3}$ and $\psi(s) = q$, new transform corresponds to $\frac{L(\alpha)}{(1-\alpha)q^\alpha+\alpha}\Big(q^\alpha G\{y(t)\} - q^{\alpha-4}y(0)\Big)$, where $G\{y(t)\}$ is Gupta transform [13].

(30) For $\phi^\alpha(s) = p^5$ and $\psi(s) = p$, our transform produce the result $\frac{L(\alpha)}{(1-\alpha)p^\alpha+\alpha}\Big(p^\alpha DV\{y(t)\} - p^{\alpha+4}y(0)\Big)$, where $DV\{y(t)\}$ is Dinesh-Verma transform [46].

(31) For $\phi^\alpha(s) = s$ and $\psi(s) = s$, transform results in $\frac{L(\alpha)}{(1-\alpha)s^\alpha+\alpha}\Big(s^\alpha Ra\{y(t)\} - s^\alpha y(0)\Big)$, where $Ra\{y(t)\}$ is Raj transform [18].

(32) For $\phi^\alpha(s) = \frac{1}{u}$ and $\psi(s) = \frac{1}{us}$, new transform gives $\frac{L(\alpha)}{(1-\alpha)(us)^{-\alpha}+\alpha}\Big((us)^{-\alpha} AS\{y(t)\} - \frac{s^{-\alpha+1}}{u^\alpha}y(0)\Big)$, where $AS\{y(t)\}$ is Abaoub-Skheam transform [1].

(33) For $\phi^\alpha(s) = u$ and $\psi(s) = u$, our new transform yields $\frac{L(\alpha)}{(1-\alpha)u^\alpha+\alpha}\Big(u^\alpha JT\{y(t)\} - u^\alpha y(0)\Big)$, where $JT\{y(t)\}$ is Jabber-Tawfiq transform [14].

(34) For $\phi^\alpha(s) = 1$ and $\psi(s) = \frac{1}{u}$, new transform results in $\frac{L(\alpha)}{(1-\alpha)u^{-\alpha}+\alpha}\Big(u^{-\alpha} Y\{y(t)\} - u^{-\alpha+1}y(0)\Big)$, where $Y\{y(t)\}$ is Yang transform [9].

4.16. **Proposed Integral Transform on Atangana-Baleanu derivative in Riemann Liouville sense.** Let $f \in H^1(a,b), a < b, \alpha \in (0,1]$, then the Atangana-Baleanu derivative [5] in Riemann-Liouville sense is given as

$$ {}_{a}^{ABR}D_t^\alpha y(t) = \frac{L(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t y(T)E_\alpha\Big(-\alpha\frac{(t-T)^\alpha}{1-\alpha}\Big)dT, \tag{4. 38}$$

where $E_\alpha\Big(-\alpha\frac{(t-T)^\alpha}{1-\alpha}(t-T)^\alpha\Big)dT = \sum_{m=o}^{\infty}\frac{-\alpha\frac{(t-T)^{m\alpha}}{1-\alpha}}{\Gamma(m\alpha+1)}$ and $L(\alpha)$ is a normalization function, such that $L(0) = L(1) = 1$.

Theorem 4.17. *If ${}_0^{ABR}D_t^\alpha y(t)$ is the Atangana-Baleanu derivative in the Riemann-Liouville sense, then $S\{{}_0^{ABR}D_t^\alpha y(t)\}$ gives $\frac{L(\alpha)}{(1-\alpha)\psi^\alpha(s)+\alpha}\Big(\psi^\alpha(s)S\{y(t)\}\Big)$.*

Proof. After applying the proposed transform (2. 1) on the Atangana-Baleanu derivative in Riemann-Liouville defined in (4. 38), we have

$$S\Big\{{}_0^{ABR}D_t^\alpha y(t)\Big\} = \frac{L(\alpha)}{1-\alpha}S\Big\{\frac{d}{dt}\Big[y(t)E_\alpha\Big(-\alpha\frac{t^\alpha}{1-\alpha}\Big)\Big]\Big\}.$$

This implies

$$S\Big\{{}_0^{ABR}D_t^\alpha y(t)\Big\} = \frac{L(\alpha)}{1-\alpha}\Big[\psi(s)S\Big\{y(t)E_\alpha\Big(-\alpha\frac{t^\alpha}{1-\alpha}\Big)\Big\} - \phi^\alpha(s)S\{y(0)E_\alpha(0)\}\Big].$$

By using the properties of Mittag Leffler [39], we have

$$S\Big\{{}_0^{ABR}D_t^\alpha y(t)\Big\} = \frac{L(\alpha)}{1-\alpha}\Big[\psi(s)\Big\{\frac{1}{\phi^\alpha(s)}Y(s)\Big(\frac{\psi^{\alpha-1}(s)}{\frac{(1-\alpha)\psi^\alpha(s)+\alpha}{1-\alpha}}\phi^\alpha(s)\Big)\Big\}\Big].$$

After simplification, we get

$$S\Big\{{}_{a}^{ABR}D_t^{\alpha}y(t)\Big\} = \frac{L(\alpha)}{(1-\alpha)\psi^{\alpha}(s)+\alpha}\Big(\psi^{\alpha}(s)Y(s)\Big).$$

This can be written as

$$S\Big\{{}_{a}^{ABR}D_t^{\alpha}y(t)\Big\} = \frac{L(\alpha)}{(1-\alpha)\psi^{\alpha}(s)+\alpha}\Big(\psi^{\alpha}(s)S\{y(t)\}\Big). \quad (4.\ 39)$$

□

Proposition 4.18. *If $\phi^{\alpha}(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$ in equation (4. 39), then the proposed transform on the Atangana-Baleanu derivative in Riemann Liouville sense converges to $\frac{L(\alpha)}{(1-\alpha)+\alpha s^{\alpha}}\Big(S\{y(t)\}\Big)$, where $S\{y(t)\}$, is Sumudu transform.*

Proof. We can easily verify the above result using $\phi^{\alpha}(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$ in equation (4. 39)

$$S\Big\{{}_{a}^{ABR}D_t^{\alpha}y(t)\Big\} = \frac{L(\alpha)}{(1-\alpha)\Big(\frac{1}{s}\Big)^{\alpha}+\alpha}\Big(\Big(\frac{1}{s}\Big)^{\alpha}S\{y(t)\}\Big),$$

$$S\{{}_{0}^{ABR}D_t^{\alpha}y(t)\} = \frac{s^{\alpha}L(\alpha)}{(1-\alpha)+\alpha s^{\alpha}}\Big(\frac{1}{s}\Big)^{\alpha}\Big(S\{y(t)\}\Big).$$

Or equivalently,

$$S\Big\{{}_{a}^{ABR}D_t^{\alpha}y(t)\Big\} = \frac{L(\alpha)}{(1-\alpha)+\alpha s^{\alpha}}\Big(S\{y(t)\}\Big). \quad (4.\ 40)$$

Equation(4. 40) is equivalent to the results of Atangana-Baleanu derivative in Riemann Liouville sense after applying Sumudu transform. □

4.18.1. ***Special cases of proposed transform via Atangana-Baleanu derivative in Riemann Liouville sense***. Many current integral transforms can be converted into their Atangana-Baleanu derivatives by substituting alternative values for $\phi^{\alpha}(s)$, $\psi(s)$ and $\gamma(t) = 1$ in (4. 39). These unique situations are

(1) For $\psi(s) = s$, transform gives $\frac{L(\alpha)}{(1-\alpha)s^{\alpha}+\alpha}\Big(s^{\alpha}L\{y(t)\}\Big)$, where $L\{y(t)\}$ is Laplace transform [39].
(2) For $\psi(s) = v$, new transform gives $\frac{L(\alpha)}{(1-\alpha)v^{\alpha}+\alpha}\Big(v^{\alpha}A\{y(t)\}\Big)$, where $A\{y(t)\}$ is Aboodh transform [2].
(3) For $\psi(s) = \frac{1}{p}$, yields $\frac{L(\alpha)}{(1-\alpha)p^{-\alpha}+\alpha}\Big(p^{-\alpha}A\{y(t)\}\Big)$, where $A\{y(t)\}$ is Anuj transform [23].
(4) For $\psi(s) = \frac{1}{v}$, produces $\frac{L(\alpha)}{(1-\alpha)v^{-\alpha}+\alpha}\Big(v^{-\alpha}E\{y(t)\}\Big)$, where $E\{y(t)\}$ is Elzaki transform [11].
(5) For $\psi(s) = \varphi^2(s)$, it gives $\frac{L(\alpha)}{(1-\alpha)\varphi^{2\alpha}(s)+\alpha}\Big(\varphi^{2\alpha}(s)EF\{y(t)\}\Big)$, where $EF\{y(t)\}$ is Emad-Falih transform [22].
(6) For $\psi(s) = \frac{1}{u^2}$, transform gives $\frac{L(\alpha)}{(1-\alpha)u^{-2\alpha}+\alpha}\Big(u^{-2\alpha}T\{y(t)\Big)$, where $T\{y(t)\}$ is Tarig transform [12].

(7) For $\psi(s) = s^n$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)s^{n\alpha}+\alpha}\Big(s^{n\alpha}G\{y(t)\}\Big)$, where $G\{y(t)\}$ is G-transform [4].
(8) For $\psi(s) = \frac{s}{v}i$, transform results in the $\frac{L(\alpha)}{(1-\alpha)si^{\alpha}+v^{\alpha}\alpha}\Big((si)^{\alpha}HY\{y(t)\}\Big)$, where $HY\{y(t)\}$ is HY transform [37].
(9) For $\psi(s) = q(s)$, transform results in the $\frac{L(\alpha)}{(1-\alpha)q^{\alpha}(s)+\alpha}\Big(q^{\alpha}(s)J\{y(t)\}\Big)$, where $J\{y(t)\}$ is Jafari transform [15].
(10) For $\psi(s) = \frac{1}{v}$, transform yields the $\frac{L(\alpha)}{(1-\alpha)v^{-\alpha}+\alpha}\Big(v^{-\alpha}K\{y(t)\}\Big)$, where $K\{y(t)\}$ is Kamal transform [43].
(11) For $\psi(s) = \frac{1}{v^2}$, new transform gives $\frac{L(\alpha)}{(1-\alpha)v^{-2\alpha}+\alpha}\Big(v^{-2\alpha}KF\{y(t)\}\Big)$, where $KF\{y(t)\}$ is Kashuri-Fundo transform [19].
(12) For $\psi(s) = \frac{1}{s^2}$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)s^{-2\alpha}+\alpha}\Big(s^{-2\alpha}KT\{y(t)\}\Big)$, where $KT\{y(t)\}$ is Kharrat-Toma transform [21].
(13) For $\psi(s) = \frac{1}{p}$, transform results in the $\frac{L(\alpha)}{(1-\alpha)p^{-\alpha}+\alpha}\Big(p^{-\alpha}ET\{y(t)\}\Big)$, where $ET\{y(t)\}$ is Elzaki-Tarig transform [32].
(14) For $\psi(s) = v^{\alpha}$, transform yields the $\frac{L(\alpha)}{(1-\alpha)v^{2\alpha}+\alpha}\Big(v^{2\alpha}KU\{y(t)\}\Big)$, where $KU\{y(t)\}$ is Kushare transform [25].
(15) For $\psi(s) = v$, transform results in the $\frac{L(\alpha)}{(1-\alpha)v^{\alpha}+\alpha}\Big(v^{\alpha}MG\{y(t)\}\Big)$, where $MG\{y(t)\}$ is Mahgoub transform [27].
(16) For $\psi(s) = v$, transform yields the $\frac{L(\alpha)}{(1-\alpha)v^{\alpha}+\alpha}\Big(v^{\alpha}M\{y(t)\}\Big)$, where $M\{y(t)\}$ is Mohand transform [28].
(17) For $\psi(s) = \frac{s}{u}$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)s^{\alpha}+u^{\alpha}\alpha}\Big(s^{\alpha}RG\{y(t)\}\}\Big)$, where $RG\{y(t)\}$ is Ramdan Group transform [45].
(18) For $\psi(s) = \frac{s}{u}$, transform results in $\frac{L(\alpha)}{(1-\alpha)s^{\alpha}+u^{\alpha}\alpha}\Big(s^{\alpha}N\{y(t)\}\}\Big)$, where $N\{y(t)\}$ is N-transform [20].
(19) For $\psi(s) = \frac{\sigma}{\varepsilon}$, our new transform gives $\frac{L(\alpha)}{(1-\alpha)\sigma^{\alpha}+\varepsilon^{\alpha}\alpha}\Big(\sigma^{\alpha}R\{y(t)\}\Big)$, where $R\{y(t)\}$ is Rishi transform [24].
(20) For $\psi(s) = v^{\alpha}$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)v^{2\alpha}+\alpha}\Big(v^{2\alpha}S\{y(t)\}\Big)$, where $S\{y(t)\}$ is Sadik transform [44].
(21) For $\psi(s) = \frac{1}{v}$, transform gives $\frac{L(\alpha)}{(1-\alpha)v^{-\alpha}+\alpha}\Big(v^{-\alpha}Sa\{y(t)\}\Big)$, where $Sa\{y(t)\}$ is Sawi transform [29].
(22) For $\psi(s) = \frac{s}{u}$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)s^{\alpha}+u^{\alpha}\alpha}\Big(s^{\alpha}Sh\{y(t)\}\Big)$, where $Sh\{y(t)\}$ is Shehu-transform [30].
(23) For $\psi(s) = v^{\alpha}$, transform yields $\frac{L(\alpha)}{(1-\alpha)v^{2\alpha}+\alpha}\Big(v^{2\alpha}So\{y(t)\}\Big)$, where $So\{y(t)\}$ is Soham transform [38].
(24) For $\psi(s) = s$, our new transform gives $\frac{L(\alpha)}{(1-\alpha)s^{\alpha}+\alpha}\Big(s^{\alpha}LC\{y(t)\}\Big)$, where $LC\{y(t)\}$ is Laplace-Carson transform [31].

(25) For $\psi(s) = \frac{1}{s}$, our new transform corresponding to $\frac{L(\alpha)}{(1-\alpha)+\alpha s^{\alpha}}\Big(S\{y(t)\}\Big)$, where $S\{y(t)\}$ is Sumudu transform [47].

(26) For $\psi(s) = s^2$, our transform yields $\frac{L(\alpha)}{(1-\alpha)s^{2\alpha}+\alpha}\Big(s^{2\alpha}AJ\{y(t)\}\Big)$ where $AJ\{y(t)\}$, is Pourreza transform [3].

(27) For $\psi(s) = s^{\frac{1}{\alpha}}$, transform corresponding to $\frac{L(\alpha)}{(1-\alpha)s+\alpha}\Big(sAL\{y(t)\}\Big)$, where $AL\{y(t)\}$ is α-transform [35].

(28) For $\psi(s) = \frac{s}{u}$, transform gives $\frac{L(\alpha)}{(1-\alpha)s^{\alpha}+u^{\alpha}\alpha}\Big(s^{\alpha}N\{y(t)\}\Big)$, where $Na\{y(t)\}$ is Natural transform [20].

(29) For $\psi(s) = q$, new transform corresponds to $\frac{L(\alpha)}{(1-\alpha)q^{\alpha}+\alpha}\Big(q^{\alpha}G\{y(t)\}\Big)$, where $G\{y(t)\}$ is Gupta transform [13].

(30) For $\psi(s) = p$, our transform produce the result $\frac{L(\alpha)}{(1-\alpha)p^{\alpha}+\alpha}\Big(p^{\alpha}DV\{y(t)\}\Big)$, where $DV\{y(t)\}$ is Dinesh-Verma transform [46].

(31) For $\psi(s) = s$, transform results in $\frac{L(\alpha)}{(1-\alpha)s^{\alpha}+\alpha}\Big(s^{\alpha}Ra\{y(t)\}\Big)$, where $Ra\{y(t)\}$ is Raj transform [18].

(32) For $\psi(s) = \frac{1}{us}$, new transform gives $\frac{L(\alpha)}{(1-\alpha)(us)^{-\alpha}+\alpha}\Big((us)^{-\alpha}AS\{y(t)\}\Big)$, where $AS\{y(t)\}$ is Abaoub-Skheam transform [1].

(33) For $\psi(s) = u$, our new transform yields $\frac{L(\alpha)}{(1-\alpha)u^{\alpha}+\alpha}\Big(u^{\alpha}JT\{y(t)\}\Big)$, where $JT\{y(t)\}$ is Jabber-Tawfiq transform [14].

(34) For $\psi(s) = \frac{1}{u}$, new transform results in $\frac{L(\alpha)}{(1-\alpha)u^{-\alpha}+\alpha}\Big(u^{-\alpha}Y\{y(t)\}\Big)$, where $Y\{y(t)\}$ is Yang transform [9].

5. Conclusions

In this paper, we introduced the fractional integral transform and applied it to One-Dimensional Fractional Viscous Burger's Equation. The nonlinear integer order PDE known as Burger's equation remains successful in gaining the attention of many researcher for many years due to its wide range of applications in mathematics, physics and engineering. But in our paper, we provide the exact solution of fractional Burger's equation using (proposed) transform decomposition method, and solution of this particular equation via all other Laplace family integral transforms can also be a special case of the solution via our proposed transform.

Our innovative approach encompasses over 200 distinct fractional differential transforms, all of which serve as specific instances derived from our foundational transforms. In our paper, we apply the proposed transform to a few fractional differential operators like Caputo, Caputo-Fabrizio, Riemann-Liouville,New Fractional Derivative, and Atangana-Baleanu in the Riemann-Liouville and Caputo senses. We also presented the other transforms as special cases of our proposed transform. It has been shown that for different values of $\phi^{\alpha}(s), \gamma(t)$ and $\psi(s)$ other integral transforms are special cases of the proposed transform.

Conflict of interests

The authors declare that there is no conflict of interest regarding the publication of this article and regarding the funding that they have received.

Acknowledgement

We thanks the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

Funding

This research is not funded by any government or private organization.

Authors Contribution

Sidra Younis: Conceptualization, Methodology, Writing Original Draft. Noreen Saba: Conceptualization, Formal Analysis, Writing Original Draft. Ghulam Mustafa: Supervision, Formal Analysis, Project Administration, Resources, Funding Acquisition, Writing-Review and Editing. Muhammad Asghar: Conceptualization, Methodology, Writing O-riginal Draft. Faheem Khan: Data Curation, Validation, Writing Review & Editing. All authors read and approved the final manuscript.

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