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The Theory of T-Bipolar Soft Modules

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Abstract. Algebraic structures in mathematics and applications reveal an inherent duality or bipolarity that is seen in elements that have both positive and negative behaviors or properties. Integrating the already-known module theory concept with the capability of T-bipolar soft sets to handle such bipolar information can lead to discoveries and instruments that can be used to research and operate these algebraic structures. This integration of module theory and T-bipolar soft sets may result in new forms of analysis and methods of analysis. These new approaches could greatly improve the possibilities of studying, analyzing, and solving problems related to algebraic structures that are inherently bipolar. Thus, in this manuscript, we investigate the notion of a T-bipolar soft module. We also deduce the T-bipolar soft submodule, maximal T-bipolar soft submodule, and their associated results. Further, we investigate T-bipolar soft homomorphism, and T-bipolar soft isomorphism along with important results such as the First Isomorphism Theorem, Second Isomorphism Theorem, and Third Isomorphism Theorem in the structure of T-bipolar soft module. Additionally, we diagnose T-bipolar soft exactness and associated results.

Key Words: Soft set; Theory of modules; T-bipolar soft modules; Isomorphism.

1. Introduction

Soft sets (SS) [15] are a mathematical framework used to handle uncertainty and vagueness in decision-making and data analysis. They are particularly useful in situations where traditional crisp sets and classical set theory may not adequately represent the uncertainty or imprecision inherent in the problem. For instance, it may not always be possible to make a definitive diagnosis based on a patient's symptoms and test findings in the field of medicine, which makes SS an effective tool for simulating this a mbiguity. Decision-makers frequently communicate their preferences in general or imprecise words during decision-making (DM) processes, particularly when there are several competing factors at play. To make wise selections, SS can assist in collecting and assessing these preferences. Babitha and Sunil [7] investigated the relations and functions of SS. Ali et al. [5] devised a few novel operations for SS. Çağman and Enginoğlu [8] employed SS in DM. Yang and Yao [19] discussed three-way DM in the setting of SS and Voskoglou [18] fused the utilization of SS and grey numbers in DM. Georgiou and Megaritis [9] originated SS and topology. Alcantud et al. [3] provided a systematic review of the theory of SS. The notion of a soft group was deduced by Aktaş and Çağman [2]. Acar et al. [1] devised soft rings and Sun et al. [16] deduced the notion of soft modules. Türkmen and Pancar [17] investigated a few novel operations in the theory of soft modules and Atagün and Sezgin [6] devised substructures of rings, modules, and fields in the theory of SS.

There are various situations in real life where decision experts require both aspects of the element or object (i.e., positive and negative aspects) but the theory of SS can't model or portray such sort of information. Thus, Mahmood [11] devised the notion of T-bipolar SS (T-BSS) by expanding the notion of existing SS. The T-BSS is a parameterization technique to cope with uncertainty and ambiguous information that contain dual aspects or bipolarity. Siddique et al. [12] devised lattice ordered double framed soft rings. Ali et al. [4] analyzed the DM approach within bipolar soft expert sets. Hakim et al. [10] devised fuzzy bipolar soft quasi-ideals in ordered semigroups.

A module is an extension of the idea of a vector space, except that it functions across rings as opposed to scalar fields. As modules over the ring of integers perfectly correspond with abelian groups, this generalization also includes the idea of an abelian group. A module functions as an additive abelian group, much like a vector space, and scalar multiplication follows the distributive property of addition both inside the module and with regard to ring multiplication. The representation theory of groups and modules, which are important ideas in commutative algebra and homological algebra, are closely related. They have a wide range of applications in algebraic topology and geometry. The soft module devised by Sun et al. [16] extended the classical module. As time goes on, the information in real-world situations becomes more complicated, demanding a more sophisticated framework to handle it and up till now there has been no research on the module theory from the perspective of T-BSS. A significant research gap has emerged in this field: the absence of the module theory studied from the perspective of T-BSMs. This difference is even more conspicuous when one considers the fact that bipolarity is increasingly being realized in many mathematical as well as in real-life situations. Bipolarity is a condition in which one has positive and negative aspects in the same context and it is a concept that is often relevant in decision making and in the study of systems. The lack of research integrating T-BSSs with module theory presents several challenges: The following challenges arise when there is insufficient research on the integration of T-BSS and module theory.

- Incomplete theoretical framework: Therefore, the absence of the T-BSM theory leads to the absence of knowledge of how bipolar information influences modular structures.
- Missed opportunities for applications: Thus, if the T-BSM theory is not yet well-developed, it is impossible to determine the potential areas of application in data science, artificial intelligence, and other fields of complex systems analysis.
- Inadequate handling of uncertainty: Regarding the fact that soft sets can partly solve the problem of uncertainty, the introduction of bipolarity could enhance the ability to model and analyze the ambiguity or conflict in modular structures.

Thus, in this manuscript, we devise the theory of T-BSM. Additionally, we derive the maximal T-bipolar soft submodule (T-BSSM), the T-BSSM, and their related findings. We also look at T-bipolar soft homomorphism and T-bipolar soft exactness and provide a proper development of this new theory. The rest of the script is managed as follows: In section 2, we revise numerous theories such as SS, T-BSS, $\mathbb{R}$-module, submodule, maximal module, $\mathbb{R}$-homomorphism, etc. In section 3, we deduce the concept of T-BSM, T-BSSM, maximal T-BSSM, and related results. We also devise T-bipolar soft homomorphism, T-bipolar soft isomorphism, T-bipolar soft exactness, and their associated findings. The conclusion is devised in section 4.

2. Preliminaries

Here, we revise numerous theories such as SS, T-BSS, $\widehat{R}$-module, submodule, maximal module, $\widehat{R}$-homomorphism, etc. Further, $\mathcal{U}$ would be employed as a universal set, $\mathcal{P}(\mathcal{U})$ would be utilized as a power set of $\mathcal{U}$, $\mathcal{P}(\mathcal{X})$ and $\mathcal{P}(\mathcal{Y})$ would be identified as a power set of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Also noted that E would be treated as a set of parameters and $\Xi \subseteq E$.

Definition 1: [15] A set $(\underline{K}, \Xi)$ would be revealed as SS over $\mathcal{U}$. Noted that $\underline{K}; \Xi \to \mathcal{P}(\mathcal{U})$.

Definition 2: [11] Take $\mathcal{U}$ as a universal set, $\mathcal{X} \subset \mathcal{U}$, $\mathcal{Y} = \mathcal{U} - \mathcal{X}$, and $\Xi \subseteq E$. A triplet $(\underline{K}_P, \underline{K}_N, \Xi)$ would be demonstrated as T-BSS over $\mathcal{U}$, where $\underline{K}_P$ and $\underline{K}_N$ are maps that are $\underline{K}_P : \Xi \to \mathcal{P}(\mathcal{X})$ and

$\underline{K}_N : \Xi \to \mathcal{P}(\mathcal{Y})$. Mathematically,

$$(\underline{K}_P, \underline{K}_N, \Xi) = \{(\rho, \underline{K}_P(\rho), \underline{K}_N(\rho)) : \underline{K}_P(\rho) \in \mathcal{P}(\mathcal{X}), \underline{K}_N(\rho) \in \mathcal{P}(\mathcal{Y})\}$$

Definition 3: [11] Take two T-BSS// $(\underline{K}_{P-1}, \underline{K}_{N-1}, \Xi_1)$ and $(\underline{K}_{P-2}, \underline{K}_{N-2}, \Xi_2)$ over $\mathcal{U}$, then the intersection of $(\underline{K}_{P-1}, \underline{K}_{N-1}, \Xi_2)$ and $(\underline{K}_{P-2}, \underline{K}_{N-2}, \Xi_2)$ is a T-BSS (G_P, G_N, Ξ_3) and is revealed as underneath

$$(\underline{K}_{P-1}, \underline{K}_{N-1}, \Xi_2) \cap_{\text{T-BSS}} (\underline{K}_{P-2}, \underline{K}_{N-2}, \Xi_2) =$$

$(\mathrm{G}_P, G_N, \Xi_3) = \{(\rho, \underline{K}_{P-1}(\rho) \cap \underline{K}_{P-2}(\rho), \underline{K}_{N-1}(\rho) \cup \underline{K}_{N-2}(\rho)) : \forall \rho \in \Xi_3 = \Xi_1 \cap \Xi_2\}$ The union of $(\underline{K}_{P-1}, \underline{K}_{N-1}, \Xi_2)$ and $(\underline{K}_{P-2}, \underline{K}_{N-2}, \Xi_2)$ is a T-BSS $(H_P, H_N, \Xi_4 = \Xi_1 \cup \Xi_2)$ and is revealed as

$$(\underline{K}_{P-1}, \underline{K}_{N-1}, \Xi_2) \cup_{\text{T-BSS}} (\overline{K}_{P-2}, \underline{K}_{N-2}, \Xi_2) = (H_P, H_N, \Xi_4 = \Xi_1 \cup \Xi_2)$$

where

$$H_P(\rho) = \begin{cases} \underline{K}_{P-1}(\rho) & \text{if } \rho \in \Xi_1 - \Xi_2 \\ \underline{K}_{P-2}(\rho) & \text{if } \rho \in \Xi_2 - \Xi_1 \\ \underline{K}_{P-1}(\rho) \cup \underline{K}_{P-2}(\rho) & \text{if } \rho \in \Xi_1 \cap \Xi_2 \end{cases}$$

$$H_N(\rho) = \begin{cases} \underline{K}_{N-1}(\rho) & \text{if } \rho \in \Xi_1 - \Xi_2 \\ \underline{K}_{N-2}(\rho) & \text{if } \rho \in \Xi_2 - \Xi_1 \\ \underline{K}_{N-1}(\rho) \cap \underline{K}_{N-2}(\rho) & \text{if } \rho \in \Xi_1 \cap \Xi_2 \end{cases}$$

Definition 4: Take a ring $\widehat{R}$ with unity and $(\overline{M}, +)$ as an abelian group, then $\overline{M}$ will be devised as a left module over $\widehat{R}$ (left $\widehat{R}$-module) if an external law of composition $\widehat{R} \times \overline{M} \to \overline{M}$ exists and holds the underneath axioms.

$$(\widehat{r}_1 + \widehat{r}_2) \cdot \overline{m} = \widehat{r}_1 \cdot \overline{m} + \widehat{r}_2 \cdot \overline{m}, \quad \forall \widehat{r}_1, \widehat{r}_2 \in \widehat{R} \text{ and } \overline{m} \in \overline{M}$$

$$\widehat{r}(\overline{m}_1 + \overline{m}_2) = \widehat{r} \cdot \overline{m}_1 + \widehat{r} \cdot \overline{m}_2, \quad \forall \widehat{r} \in \widehat{R} \text{ and } \overline{m}_1, \overline{m}_2 \in \overline{M}$$

$$(\widehat{r}_1 \widehat{r}_2) \cdot \overline{m} = \widehat{r}_1(\widehat{r}_2 \cdot \overline{m}), \quad \forall \widehat{r}_1, \widehat{r}_2 \in \widehat{R} \text{ and } \overline{m} \in \overline{M}$$

$$1 \cdot \overline{m} = \overline{m}, \quad \forall \overline{m} \in \overline{M} \text{ and } 1 \in \widehat{R}$$

A subset $\overline{M}_S \neq \emptyset$ of $\overline{M}$ is identified as a submodule of an $\widehat{R}$-module $\overline{M}$ if $\overline{M}_S$ is an $\widehat{R}$-module with respect to the same addition and external law of composition.

Definition 5: A proper submodule $\overline{M}_L$ of $\overline{M}$ is identified as the maximal submodule of $\overline{M}$, if there exists a submodule $\overline{M}_S$ such that $\overline{M}_L \subseteq \overline{M}_S \subseteq \overline{M}$, then either $\overline{M}_L = \overline{M}_S$ or $\overline{M}_S = \overline{M}$.

Proposition 1 Take a gathering $\{M_{S-\zeta} \mid \zeta \in J\}$ of submodules of M, then $\bigcap_{\zeta \in J} M_{S-\zeta}$ and $\sum_{\zeta \in J} M_{S-\zeta}$ are submodules of M.

Definition 6 Take two R-modules M_1 and M_2, then a map $L : M_1 \to M_2$ would be termed as a homomorphism of R-modules if the following axioms hold:

$$L(y_1 + y_2) = L(y_1) + L(y_2) \quad \forall\, y_1, y_2 \in M,$$
$$L(ry) = rL(y) \quad \forall\, y \in M, r \in R.$$

Definition 7 The homomorphism sequence

$$\cdots \longrightarrow M_{n-1} \xrightarrow{I_{n-1}} M_n \xrightarrow{I_n} M_{n+1} \longrightarrow \cdots$$

would be called an exact sequence of modules if $\operatorname{Im} I_{n-1} = \ker I_n \,\forall\, n \in \mathbb{N}$.

3. T-Bipolar soft modules

In this section, we deduce the concept of T-BSM, T-BSSM, maximal T-BSSM, and related results. We also devise T-bipolar soft homomorphism, T-bipolar soft isomorphism, T-bipolar soft exactness, and their associated findings. The symbol $\preceq$ will indicate a submodule in the rest of the script.

Definition 8 A T-BSS $(\mathcal{K}_P, \mathcal{K}_N, \Xi)$ is revealed as a T-BSM over $\mathfrak{U}$ if and only if $\mathcal{K}_P(\varrho) \preceq \mathcal{M}_1$ and $\mathcal{K}_N(\varrho) \preceq \mathcal{M}_2$ for all $\varrho \in \Xi$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are two disjoint $\mathbb{R}$-modules such that $\mathfrak{U} = \mathcal{M}_1 \cup \mathcal{M}_2$, and $\mathcal{K}_P : \Xi \to \mathcal{P}(\mathcal{M}_1), \mathcal{K}_N : \Xi \to \mathcal{P}(\mathcal{M}_2)$.//

Remark Take $\Xi = \{\varrho_1, \varrho_2, \ldots, \varrho_o\} \subseteq E$, $\mathcal{M}_1 = \{\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_m\}$, $\mathcal{M}_2 = \{\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_n\}$ and a corresponding T-BSM $(\mathcal{K}_P, \mathcal{K}_N, \Xi)$. Then the tabular description of T-BSM is demonstrated in Table 1.

TABLE 1. The tabular interpretation of T-BSM.

$(\mathscr{F}_P, \mathscr{F}_N, \Xi)$	$(1,1)$	$(1,2)$	$\cdots$	$(1,n)$	$(2,1)$	$(2,2)$	$\cdots$	$(2,n)$	$\cdots$	$(m,1)$	(m,n)
ϱ_1	$\mathscr{P}_{111}$	$\mathscr{P}_{112}$	$\cdots$	$\mathscr{P}_{11n}$	$\mathscr{P}_{121}$	$\mathscr{P}_{122}$	$\cdots$	$\mathscr{P}_{12n}$	$\cdots$	$\mathscr{P}_{1m1}$	$\mathscr{P}_{1mn}$
ϱ_2	$\mathscr{P}_{211}$	$\mathscr{P}_{212}$	$\cdots$	$\mathscr{P}_{21n}$	$\mathscr{P}_{221}$	$\mathscr{P}_{222}$	$\cdots$	$\mathscr{P}_{22n}$	$\cdots$	$\mathscr{P}_{2m1}$	$\mathscr{P}_{2mn}$
ϱ_3	$\mathscr{P}_{311}$	$\mathscr{P}_{312}$	$\cdots$	$\mathscr{P}_{31n}$	$\mathscr{P}_{321}$	$\mathscr{P}_{322}$	$\cdots$	$\mathscr{P}_{32n}$	$\cdots$	$\mathscr{P}_{3m1}$	$\mathscr{P}_{3mn}$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
ϱ_o	$\mathscr{P}_{o11}$	$\mathscr{P}_{o12}$	$\cdots$	$\mathscr{P}_{o1n}$	$\mathscr{P}_{o21}$	$\mathscr{P}_{o22}$	$\cdots$	$\mathscr{P}_{o2n}$	$\cdots$	$\mathscr{P}_{om1}$	$\mathscr{P}_{omn}$

Note that:

$$\mathcal{P}_{\varepsilon\zeta\tau} = (\mathcal{P}^*_{\varepsilon\zeta\tau}, \mathcal{P}'_{\varepsilon\zeta\tau}) = \begin{cases} (0,0) & \text{if } \mathfrak{c}_\zeta \notin \underline{K}_P(\varrho_\varepsilon) \text{ and } \mathfrak{d}_\tau \notin \underline{K}_N(\varrho_\varepsilon) \\ (1,0) & \text{if } \mathfrak{c}_\zeta \in \underline{K}_P(\varrho_\varepsilon) \text{ and } \mathfrak{d}_\tau \notin \underline{K}_N(\varrho_\varepsilon) \\ (0,1) & \text{if } \mathfrak{c}_\zeta \notin \underline{K}_P(\varrho_\varepsilon) \text{ and } \mathfrak{d}_\tau \in \underline{K}_N(\varrho_\varepsilon) \\ (1,1) & \text{if } \mathfrak{c}_\zeta \in \underline{K}_P(\varrho_\varepsilon) \text{ and } \mathfrak{d}_\tau \in \underline{K}_N(\varrho_\varepsilon) \end{cases}$$

Definition 9 For two T-BSMs $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$, we demonstrate that $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ is a T-bipolar soft submodule (T-BSSM) of $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ if

- $\Xi_1 \subset \Xi_2$
- $\forall \varrho \in \Xi_1, \underline{K}_{(P-1)}(\varrho) \preceq \underline{K}_{(P-2)}(\varrho)$
- $\forall \varrho \in \Xi_1, \underline{K}_{(N-2)}(\varrho) \preceq \underline{K}_{(N-1)}(\varrho)$

It would be interpreted as $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1) \preceq (\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$.

Proposition 2: Assume two T-BSMs $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ over $\mathcal{U}$, then $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ is a T-BSSM of $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ if

$$\forall \varrho \in \Xi_2, \quad \underline{K}_{(P-1)}(\varrho) \subseteq \underline{K}_{(P-2)}(\varrho) \quad \text{and} \quad \underline{K}_{(N-2)}(\varrho) \subseteq \underline{K}_{(N-1)}(\varrho).$$

Proof: Omitted

Definition 10: Assume a set $\mathcal{I} = \{e\}$, where e is a unit element of Ξ. Then each T-BSM $(\underline{K}_P, \underline{K}_N, \Xi)$ has at least two T-BSSMs, that is $(\underline{K}_P, \underline{K}_N, \Xi)$ and $(\underline{K}_P, \underline{K}_N, \mathcal{I})$, which is identified as a trivial T-BSSM.

Proposition 3: Assume a T-BSM $(\underline{K}_P, \underline{K}_N, \Xi)$ and a non-empty collection of T-BSSMs $\{(\underline{K}_{(P-\zeta)}, \underline{K}_{(N-\zeta)}, \Xi_\zeta) : \zeta \in J\}$ of $(\underline{K}_P, \underline{K}_N, \Xi)$, then

- $\sum_{\zeta\in J}(\underline{K}_{(P-\zeta)}, \underline{K}_{(N-\zeta)}, \Xi_\zeta)$ is a T-BSSM of $(\underline{K}_P, \underline{K}_N, \Xi)$
- $\bigcap_{\zeta\in J}(\underline{K}_{(P-\zeta)}, \underline{K}_{(N-\zeta)}, \Xi_\zeta)$ is a T-BSSM of $(\underline{K}_P, \underline{K}_N, \Xi)$
- $\bigcup_{\zeta\in J}(\underline{K}_{(P-\zeta)}, \underline{K}_{(N-\zeta)}, \Xi_\zeta)$ is a T-BSSM of $(\underline{K}_P, \underline{K}_N, \Xi)$ if $\Xi_\zeta \cap \Xi_\tau = \emptyset \forall \zeta, \tau \in J$

Proof: Obvious.

Definition 11: Assume two T-BSMs $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ over $\mathcal{U}$, and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ as a T-BSSM of $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$, then we interpret that $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ is a maximal T-BSSM (MT-BSSM) of $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ if $\underline{K}_{(P-2)}(\varrho)$ and $\underline{K}_{(N-2)}(\varrho)$ are maximal submodules of $\underline{K}_{(P-1)}(\varrho)$ and $\underline{K}_{(N-1)}(\varrho)$ respectively.

Proposition 4: Assume a T-BSM $(\underline{K}_P, \underline{K}_N, \Xi)$ over $\mathcal{M}$, then

- If $\{(\underline{K}_{(P-\zeta)}, \underline{K}_{(N-\zeta)}, \Xi_\zeta) : \zeta \in J\}$ is a non-empty gathering of MT-BSSMs of $(\underline{K}_P, \underline{K}_N, \Xi)$, then $\bigcap_{\zeta\in J}(\underline{K}_{(P-\zeta)}, \underline{K}_{(N-\zeta)}, \Xi_\zeta)$ is an MT-BSSM of $(\underline{K}_P, \underline{K}_N, \Xi)$.
- If $\{(\underline{K}_{(P-\zeta)}, \underline{K}_{(N-\zeta)}, \Xi_\zeta) : \zeta \in J\}$ is a non-empty gathering of MT-BSSMs of $(\underline{K}_P, \underline{K}_N, \Xi)$, then $\sum_{\zeta\in J}(\underline{K}_{(P-\zeta)}, \underline{K}_{(N-\zeta)}, \Xi_\zeta)$ is an MT-BSSM of $(\underline{K}_P, \underline{K}_N, \Xi)$.

Proof: Obvious.

Definition 12: Assume a T-BSM $(\underline{K}_P, \underline{K}_N, \Xi)$ over $\mathcal{U} = \mathcal{M}_1 \cup \mathcal{M}_2$, then

- $(\underline{K}_P, \underline{K}_N, \Xi)$ would be demonstrated as a null T-BSM if $\underline{K}_P(\varrho) = 0$ and $\underline{K}_N(\varrho) = \mathcal{M}_2 \forall \varrho \in \Xi$
- $(\underline{K}_P, \underline{K}_N, \Xi)$ would be demonstrated as an absolute T-BSM over $\mathcal{M}$ if $\underline{K}_P(\varrho) = \mathcal{M}_1$ and $\underline{K}_N(\varrho) = 0 \forall \varrho \in \Xi$

Proposition 5: Assume two T-BSMs $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ over $\mathcal{U}$, then

- $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1) \cap_{(T-BSS)} (\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_1)$ is a T-BSM over $\mathcal{U} = \mathcal{M} \cup \mathcal{M}^*$.
- $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1) \cup_{(T-BSS)} (\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_1)$ is a T-BSM over $\mathcal{U} = \mathcal{M} \cup \mathcal{M}^*$ if $\Xi_1 \cap \Xi_2 = \emptyset$.

Proof:

(1) As from Def (3), we have that $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1) \cap_{(T-BSS)} (\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_1) = (G_P, G_N, \Xi_3 = \Xi_1 \cap \Xi_2)$ is a T-BSS over $\mathcal{U} = \mathcal{M} \cup \mathcal{M}^*$ and $G_P(\varrho) = \underline{K}_{(P-1)}(\varrho) \preceq \mathcal{M}$ or $G_P(\varrho) = \underline{K}_{(P-2)}(\varrho) \preceq \mathcal{M}$ and $G_N(\varrho) = \underline{K}_{(N-1)}(\varrho) \preceq \mathcal{M}^*$ or $G_N(\varrho) = \underline{K}_{(N-2)}(\varrho) \preceq \mathcal{M}^* \forall \varrho \in \Xi_3$. As $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ are T-BSMs over $\mathcal{U}$, hence, $(G_P, G_N, \Xi_3 = \Xi_1 \cap \Xi_2)$ is a T-BSM over $\mathcal{U} = \mathcal{M} \cup \mathcal{M}^*$.

(2) From Def (3), we have that $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_2) \cup_{(T-BSS)} (\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2) = (H_P, H_N, \Xi_4 = \Xi_1 \cup \Xi_2)$. If $\varrho \in \Xi_1 - \Xi_2$ or $\varrho \in \Xi_2 - \Xi_1$, then $\Xi_1 \cap \Xi_2 = \emptyset$ and $H_P(\varrho) = \underline{K}_{(P-1)}(\varrho) \preceq \mathcal{M}$ or $H_P(\varrho) = \underline{K}_{(P-2)}(\varrho) \preceq \mathcal{M}$ and $H_N(\varrho) = \underline{K}_{(N-1)}(\varrho) \preceq \mathcal{M}^*$ or $H_N(\varrho) = \underline{K}_{(N-2)}(\varrho) \preceq \mathcal{M}^*$. Thus, $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_2) \cup_{(T-BSS)} (\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2) = (H_P, H_N, \Xi_4 = \Xi_1 \cup \Xi_2)$ is a T-BSM over $\mathcal{U} = \mathcal{M} \cup \mathcal{M}^*$.

Definition 13: For two T-BSMs $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ over $\mathcal{U} = \mathcal{M} \cup \mathcal{M}^*$, we interpret $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1) + (\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ as underneath

$$(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1) + (\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2) = (J_P, J_N, \Xi_1 \times \Xi_2)$$

Where $\forall(\varrho_1, \varrho_2) \in \Xi_1 \times \Xi_2$

$$J_P(\varrho_1, \varrho_2) = \underline{K}_{(P-1)}(\varrho_1) + \underline{K}_{(P-2)}(\varrho_2) \quad \text{and} \quad J_N(\varrho_1, \varrho_2) = \underline{K}_{(N-1)}(\varrho_1) + \underline{K}_{(N-2)}(\varrho_2)$$

Proposition 6: Assume two T-BSMs $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ over $\mathcal{U} = \mathcal{M} \cup \mathcal{M}^*$, then $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1) + (\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ is a T-BSM.

Proof: From Proposition (1), it can be easily proved.

Definition 14: For two T-BSMs $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ over $\mathcal{U}_1 = \mathcal{M}_1 \cup \mathcal{M}_1^*$ and $\mathcal{U}_2 = \mathcal{M}_2 \cup \mathcal{M}_2^*$ respectively, we interpret $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1) \times (\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ as underneath

$$(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1) \times (\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2) = (T_P, T_N, \Xi_1 \times \Xi_2)$$

Where $\forall(\varrho_1, \varrho_2) \in \Xi_1 \times \Xi_2$

$$T_P(\varrho_1, \varrho_2) = \underline{K}_{(P-1)}(\varrho_1) \times \underline{K}_{(P-2)}(\varrho_2) \quad \text{and} \quad T_N(\varrho_1, \varrho_2) = \underline{K}_{(N-1)}(\varrho_1) \times \underline{K}_{(N-2)}(\varrho_2)$$

Proposition 7: Assume two T-BSMs $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ over $\mathcal{U}_1 = \mathcal{M}_1 \cup \mathcal{M}_1^*$ and $\mathcal{U}_2 = \mathcal{M}_2 \cup \mathcal{M}_2^*$, then $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1) \times (\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ is a T-BSM over $\mathcal{U}_1 \times \mathcal{U}_2$.

Proof: Omitted.

4. T-BIPOLAR SOFT HOMOMORPHISM

Here, in this sub-segment, we devise T-bipolar soft homomorphism and related findings.

Proposition 4.1. *Assume two T-BSMs $(\underline{K}_{P-1}, \underline{K}_{N-1}, \Xi_1)$ and $(\underline{K}_{P-2}, \underline{K}_{N-2}, \Xi_2)$ over $\mathbb{U}_1 = \mathbb{M}_1 \cup \mathbb{M}_1^*$ and $(\underline{K}_{P-2}, \underline{K}_{N-2}, \Xi_2)$ is a T-BSSM of $(\underline{K}_{P-1}, \underline{K}_{N-1}, \Xi_1)$. If $L_P : \mathbb{M}_1 \to \mathbb{M}_2$ and $L_N : \mathbb{M}_1^* \to \mathbb{M}_2^*$ are R-homomorphisms then $(L_P(\underline{K}_{P-1}), L_N(\underline{K}_{N-1}), \Xi_1)$ and $(L_P(\underline{K}_{P-2}), L_N(\underline{K}_{N-2}), \Xi_2)$ are T-BSMs over $\mathbb{U}_2 = \mathbb{M}_2 \cup \mathbb{M}_2^*$ and $(L_P(\underline{K}_{P-2}), L_N(\underline{K}_{N-2}), \Xi_2)$ is a T-BSSM of $(L_P(\underline{K}_{P-1}), L_N(\underline{K}_{N-1}), \Xi_1)$.*

Proof. As the homomorphic image of a submodule is a submodule, this result can be proven easily. □

Definition 15: Let us take two T-BSMs $(\underline{K}_{P-1}, \underline{K}_{N-1}, \Xi_1)$ and $(\underline{K}_{P-2}, \underline{K}_{N-2}, \Xi_2)$ over $\mathbb{U}_1 = \mathbb{M}_1 \cup \mathbb{M}_1^*$ and $\mathbb{U}_2 = \mathbb{M}_2 \cup \mathbb{M}_2^*$ respectively, and three functions $L_P : \mathbb{M}_1 \to \mathbb{M}_2$, $L_N : \mathbb{M}_1^* \to \mathbb{M}_2^*$, $G : \Xi_1 \to \Xi_2$. We interpret that (L_P, L_N, G) is a T-bipolar soft homomorphism or $(\underline{K}_{P-1}, \underline{K}_{N-1}, \Xi_1)$ is T-bipolar soft homomorphic to $(\underline{K}_{P-2}, \underline{K}_{N-2}, \Xi_2)$, denoted as $(\underline{K}_{P-1}, \underline{K}_{N-1}, \Xi_1) \sim (\underline{K}_{P-2}, \underline{K}_{N-2}, \Xi_2)$ if the following axioms are satisfied:

(1) $L_P : \mathbb{M}_1 \to \mathbb{M}_2$ and $L_N : \mathbb{M}_1^* \to \mathbb{M}_2^*$ are module homomorphisms
(2) $G : \Xi_1 \to \Xi_2$ is a mapping
(3) $L_P(\underline{K}_{P-1}(\rho)) = \underline{K}_{P-2}(G(\rho))$ and $L_N(\underline{K}_{N-1}(\rho)) = \underline{K}_{N-2}(G(\rho))$ for all $\rho \in \Xi$

Remark: If $L_P : \mathbb{M}_1 \to \mathbb{M}_2$ and $L_N : \mathbb{M}_1^* \to \mathbb{M}_2^*$ are isomorphisms and $G : \Xi_1 \to \Xi_2$ is a one-to-one map, then $(\underline{K}_{P-1}, \underline{K}_{N-1}, \Xi_1)$ is T-bipolar soft isomorphic to $(\underline{K}_{P-2}, \underline{K}_{N-2}, \Xi_2)$ and denoted as $(\underline{K}_{P-1}, \underline{K}_{N-1}, \Xi_1) \cong (\underline{K}_{P-2}, \underline{K}_{N-2}, \Xi_2)$.

Proposition 4.2. (1) *Assume a T-BSM $(\underline{K}_P, \underline{K}_N, \Xi)$ over $\mathbb{U}_1 = \mathbb{M}_1 \cup \mathbb{M}_1^*$ and $L_P : \mathbb{M}_1 \to \mathbb{M}_2$, $L_N : \mathbb{M}_1^* \to \mathbb{M}_2^*$ as homomorphisms. If for all $\rho \in \Xi$, $\underline{K}_P(\rho) = \ker L_P$ and $\underline{K}_N(\rho) = \ker L_N$, then $(L_P(\underline{K}_P), L_N(\underline{K}_P), \Xi)$ is the null T-BSM over $\mathbb{U}_2 = \mathbb{M}_2 \cup \mathbb{M}_2^*$.*

(2) *Assume an absolute T-BSM $(\underline{K}_P, \underline{K}_N, \Xi)$ over $\mathbb{U}_1 = \mathbb{M}_1 \cup \mathbb{M}_1^*$ and $L_P : \mathbb{M}_1 \to \mathbb{M}_2$, $L_N : \mathbb{M}_1^* \to \mathbb{M}_2^*$ as epimorphisms. Then $(L_P(\underline{K}_P), L_N(\underline{K}_P), \Xi)$ is the absolute T-BSM over $\mathbb{U}_2 = \mathbb{M}_2 \cup \mathbb{M}_2^*$.*

Proof. Obvious. □

Next, we devise a First Isomorphism Theorem for T-BSM.

Theorem 1: Let us take two T-BSMs $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ over $\mathcal{U}_1 = \mathcal{M}_1 \cup \mathcal{M}_1^*$ and $\mathcal{U}_2 = \mathcal{M}_2 \cup \mathcal{M}_2^*$ respectively. If (L_P, L_N, G) is a T-bipolar soft homomorphism from $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ to $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ and Ker$L_P \subset \underline{K}_{(P-1)}(\varrho)$ and Ker$L_N \subset \underline{K}_{(N-1)}(\varrho)$ for all $\varrho \in \Xi_1$, then

$$(Y_P, Y_N, \Xi_1) \cong (Z_P, Z_N, \Xi_1)$$

Note that $Y_P(\varrho) = \underline{K}_{(P-1)}(\varrho)/\text{Ker}L_P$, $Y_N(\varrho) = \underline{K}_{(N-1)}(\varrho)/\text{Ker}L_N$, $Z_P(\varrho) = L_P(\underline{K}_{(P-1)}(\varrho))$ and $Z_N(\varrho) = L_N(\underline{K}_{(N-1)}(\varrho))$ for all $\varrho \in \Xi_1$. If G is bijective then $(Y_P, Y_N, \Xi_1) \approx (Z_P, Z_N, \Xi_1)$.

Proof: 1. As we know that KerL_P and KerL_N are submodules of $\mathcal{M}_1$ and $\mathcal{M}_1^*$ respectively, thus, $\mathcal{M}_1/\text{Ker}L_P$ and $\mathcal{M}_1^*/\text{Ker}L_N$ are modules. Similarly, as KerL_P is a submodule of $\underline{K}_{(P-1)}(\varrho)$ and KerL_N is a submodule of $\underline{K}_{(N-1)}(\varrho)$, so $\underline{K}_{(P-1)}(\varrho)/\text{Ker}L_P$ and $\underline{K}_{(N-1)}(\varrho)/\text{Ker}L_N$ are modules for all $\varrho \in \Xi_1$. But $\underline{K}_{(P-1)}(\varrho)/\text{Ker}L_P$ and $\underline{K}_{(N-1)}(\varrho)/\text{Ker}L_N$ are always submodules of $\mathcal{M}_1/\text{Ker}L_P$ and $\mathcal{M}_1^*/\text{Ker}L_N$ respectively. This implies that (Y_P, Y_N, Ξ_1) is a T-BSM over $\mathcal{M}_1/\text{Ker}L_P \cup \mathcal{M}_1^*/\text{Ker}L_N$.

As we have that $Z_P(\varrho) = L_P(\underline{K}_{(P-1)}(\varrho))$ and $Z_N(\varrho) = L_N(\underline{K}_{(N-1)}(\varrho))$ for all $\varrho \in \Xi_1$, then by the definition of T-bipolar soft homomorphism, we have that $Z_P(\varrho) = L_P(\underline{K}_{(P-1)}(\varrho)) = \underline{K}_{(P-2)}(G(\varrho))$ and $Z_N(\varrho) = L_N(\underline{K}_{(N-1)}(\varrho)) = \underline{K}_{(N-2)}(G(\varrho))$ for all $\varrho \in \Xi_1$. This implies that $Z_P(\varrho)$ and $Z_N(\varrho)$ are submodules of $\mathcal{M}_2$ and $\mathcal{M}_2^*$ respectively. This implies that (Z_P, Z_N, Ξ_1) is a T-BSM over $\mathcal{M}_2 \cup \mathcal{M}_2^*$.

Next, define

$$\underline{L}_P : \mathcal{M}_1/\text{Ker}L_P \to \mathcal{M}_2 \text{ as } \underline{L}_P(r + \text{Ker}L_P) = L_P(r) \quad \forall r \in \mathcal{M}_1$$

and

$$\underline{L}_N : \mathcal{M}_1^*/\text{Ker}L_N \to \mathcal{M}_2^* \text{ as } \underline{L}_N(r^* + \text{Ker}L_N) = L_N(r^*) \quad \forall r^* \in \mathcal{M}_1^*$$

Then, $\underline{L}_P$ and $\underline{L}_N$ are isomorphisms. Now assume $\underline{G} : \Xi_1 \to \Xi_1$ interpreted as $\underline{G}(\varrho) = \varrho$, then $\underline{G}$ is a bijective function. Further, we have

$$\underline{L}_P(Y_P(\varrho)) = \underline{L}_P(\underline{K}_{(P-1)}(\varrho)/\text{Ker}L_P) = L_P(\underline{K}_{(P-1)}(\varrho)) = Z_P(\varrho) = Z_P(\underline{G}(\varrho))$$

and

$$\underline{L}_N(Y_N(\varrho)) = \underline{L}_N(\underline{K}_{(N-1)}(\varrho)/\text{Ker}L_N) = L_N(\underline{K}_{(N-1)}(\varrho)) = Z_N(\varrho) = Z_N(\underline{G}(\varrho))$$

This implies that $(\underline{L}_P, \underline{L}_N, \underline{G})$ is a T-bipolar soft homomorphism.

Theorem 2: Assume a T-BSM $(\underline{K}_P, \underline{K}_N, \Xi)$ over $\mathcal{U}_1 = \mathcal{M}_1 \cup \mathcal{M}_1^*$. If $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ are T-BSSMs of $(\underline{K}_P, \underline{K}_N, \Xi)$, then

$$(\underline{K}_{(P-3)}, \underline{K}_{(N-3)}, \Xi_1) \cong (\underline{K}_{(P-4)}, \underline{K}_{(N-4)}, \Xi_1)$$

$$(\underline{K}_{(P-5)}, \underline{K}_{(N-5)}, \Xi_2) \cong (\underline{K}_{(P-6)}, \underline{K}_{(N-6)}, \Xi_2)$$

where, $\underline{K}_{(P-3)}(\varrho) = \underline{K}_{(P-1)}(\varrho)/(V_P \cap V_P')$, $\underline{K}_{(N-3)}(\varrho) = \underline{K}_{(N-1)}(\varrho)/(V_N \cap V_N')$, $\underline{K}_{(P-4)}(\varrho) = (\underline{K}_{(P-1)}(\varrho) + V_P')/V_P'$, $\underline{K}_{(N-4)}(\varrho) = (\underline{K}_{(N-1)}(\varrho) + V_N')/V_N'$, $\underline{K}_{(P-5)}(\varrho) = \underline{K}_{(P-2)}(\varrho)/(V_P \cap V_P')$, $\underline{K}_{(N-5)}(\varrho) = \underline{K}_{(N-2)}(\varrho)/(V_N \cap V_N')$, $\underline{K}_{(P-6)}(\varrho) = (\underline{K}_{(P-2)}(\varrho) + V_P')/V_P'$, and $\underline{K}_{(N-6)}(\varrho) = (\underline{K}_{(N-2)}(\varrho) + V_N')/V_N'$.

Note that V_P, V_N are soft submodules of $(\underline{K}_P, \underline{K}_N, \Xi)$.

Theorem 3: Assume a T-BSM $(\underline{K}_P, \underline{K}_N, \Xi)$ over $\mathcal{U}_1 = \mathcal{M}_1 \cup \mathcal{M}_1^*$. If $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ are T-BSSMs of $(\underline{K}_P, \underline{K}_N, \Xi)$, such that $\underline{K}_{(P-1)}(\varrho) = V_P$ and $\underline{K}_{(N-1)}(\varrho) = V_N = \bigcap_{\varrho\in\Xi_1} \underline{K}_{(N-1)}(\varrho)$ for all $\varrho \in \Xi_1$, then

$$(\underline{K}_{(P-3)}, \underline{K}_{(N-3)}, \Xi_1) \cong (\underline{K}_{(P-4)}, \underline{K}_{(N-4)}, \Xi_1)$$

Note that $\underline{K}_{(P-3)}(\varrho) = \underline{K}_{(P-1)}(\varrho)/(V_P \cap V'_P)$, $\underline{K}_{(N-3)}(\varrho) = \underline{K}_{(N-1)}(\varrho)/(V_N \cap V'_N)$, $\underline{K}_{(P-4)}(\varrho) = (\underline{K}_{(P-1)}(\varrho) + V'_P)/V'_P$, $\underline{K}_{(N-4)}(\varrho) = (\underline{K}_{(N-1)}(\varrho) + V'_N)/V'_N$, $V_P = \bigcap_{\varrho\in\Xi_1} \underline{K}_{(P-1)}(\varrho)$, $V_N = \bigcap_{\varrho\in\Xi_1} \underline{K}_{(N-1)}(\varrho)$, $V'_P = \bigcap_{\varrho\in\Xi_2} \underline{K}_{(P-2)}(\varrho)$ and $V'_N = \bigcap_{\varrho\in\Xi_2} \underline{K}_{(N-2)}(\varrho)$.

Proof: Similar to Theorem 2.

Theorem 4: Assume a T-BSM $(\underline{K}_P, \underline{K}_N, \Xi)$ over $\mathcal{U}_1 = \mathcal{M}_1 \cup \mathcal{M}_1^*$. If $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ are T-BSSMs of $(\underline{K}_P, \underline{K}_N, \Xi)$, with $\Xi_1 \cap \Xi_2 \neq \emptyset$ and $\underline{K}_{(P-2)}(\varrho) \subset \underline{K}_{(P-1)}(\varrho)$, $\underline{K}_{(N-1)}(\varrho) \subset \underline{K}_{(N-2)}(\varrho)$ for all $\varrho \in \Xi_1 \cap \Xi_2$, then we have that

$$(\underline{K}_{(P-3)}, \underline{K}_{(N-3)}, \Xi_1 \cap \Xi_2) \cong (\underline{K}_{(P-4)}, \underline{K}_{(N-4)}, \Xi_1 \cap \Xi_2)$$

Note that $\underline{K}_{(P-3)}(\varrho) = (\underline{K}_P(\varrho)/V'_P)/(V_P/V'_P)$, $\underline{K}_{(N-3)}(\varrho) = (\underline{K}_N(\varrho)/V'_N)/(V_N/V'_N)$, $\underline{K}_{(P-4)}(\varrho) = \underline{K}_P(\varrho)/V_P$, $\underline{K}_{(N-4)}(\varrho) = \underline{K}_N(\varrho)/V_N$, $V_P = \bigcap_{\varrho\in\Xi_1\cap\Xi_2} \underline{K}_{(P-1)}(\varrho)$, $V_N = \bigcap_{\varrho\in\Xi_1\cap\Xi_2} \underline{K}_{(N-1)}(\varrho)$, $V'_P = \bigcap_{\varrho\in\Xi_1\cap\Xi_2} \underline{K}_{(P-2)}(\varrho)$ and $V'_N = \bigcap_{\varrho\in\Xi_1\cap\Xi_2} \underline{K}_{(N-2)}(\varrho)$.

Proof: One can easily show that $V_P = \bigcap_{\varrho\in\Xi_1\cap\Xi_2} \underline{K}_{(P-1)}(\varrho)$ is a sub-module of $\mathcal{M}_1$, $V_N = \bigcap_{\varrho\in\Xi_1\cap\Xi_2} \underline{K}_{(N-1)}(\varrho)$ is a sub-module of $\mathcal{M}_2$, $V'_P = \bigcap_{\varrho\in\Xi_1\cap\Xi_2} \underline{K}_{(P-2)}(\varrho)$ is a sub-module of V_P and $V'_N = \bigcap_{\varrho\in\Xi_1\cap\Xi_2} \underline{K}_{(N-2)}(\varrho)$ is a submodule of V_N. Next, it is easy to reveal that $(\underline{K}_{(P-3)}, \underline{K}_{(N-3)}, \Xi_1 \cap \Xi_2)$ is T-BSM over $(\mathcal{M}_1/V'_P)/(V_P/V'_P) \times (\mathcal{M}_1^*/V'_N)/(V_N/V'_N)$ and $(\underline{K}_{(P-4)}, \underline{K}_{(N-4)}, \Xi_1 \cap \Xi_2)$ is T-BSM over $\mathcal{M}_1/V_P \times \mathcal{M}_1^*/V_N$.

Now define two maps that are

$$L_P : (\mathcal{M}_1/V'_P)/(V_P/V'_P) \to \mathcal{M}_1/V_P$$

interpreted as

$$L_P((m + V'_P) + (V_P/V'_P)) = m + V_P$$

and

$$L_N : (\mathcal{M}_1^*/V'_N)/(V_N/V'_N) \to \mathcal{M}_1^*/V_N$$

interpreted as

$$L_N((m^* + V'_N) + (V_N/V'_N)) = m^* + V_N$$

Next, define

$$G : \Xi_1 \cap \Xi_2 \to \Xi_1 \cap \Xi_2$$

interpreted as

$$G(\varrho) = \varrho \quad \forall \varrho \in \Xi_1 \cap \Xi_2$$

Then, clearly L_P and L_N are isomorphisms, G is bijective and $L_P(\underline{K}_{(P-3)}(\varrho)) = L_P((\underline{K}_P(\varrho)/V'_P)/(V_P/V'_P)) = (\underline{K}_P(\varrho)/V_P) = \underline{K}_{(P-4)}(\varrho) = \underline{K}_{(P-4)}(G(\varrho))$, $L_N(\underline{K}_{(N-3)}(\varrho)) = L_N((\underline{K}_N(\varrho)/V'_N)/(V_N/V'_N)) = (\underline{K}_N(\varrho)/V_N) = \underline{K}_{(N-4)}(\varrho) = \underline{K}_{(N-4)}(G(\varrho))$. Consequently,

$$(\underline{K}_{(P-3)}, \underline{K}_{(N-3)}, \Xi_1 \cap \Xi_2) \cong (\underline{K}_{(P-4)}, \underline{K}_{(N-4)}, \Xi_1 \cap \Xi_2)$$

5. T-BIPOLAR SOFT EXACTNESS

In this subsection, we deduce T-bipolar soft exactness and associated results.

Definition 16: Assume a T-BSMs $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ over $\Upsilon_1 = M_1 \cup M_1^*$, $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ over $\Upsilon_2 = M_2 \cup M_2^*$ and $(\underline{K}_{(P-3)}, \underline{K}_{(N-3)}, \Xi_3)$ over $\Upsilon_3 = M_3 \cup M_3^*$. Then we interpret T-bipolar soft exactness at $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ if the following axioms hold:

- $M_1 \to^{\perp} (T_{(P-1)})M_2 \to^{\perp} (T_{(P-2)})M_3$ and $M_1^* \to^{\perp} (T_{(N-1)})M_2^* \to^{\perp} (T_{(N-2)})M_3^*$ are exact.
- $\Xi_1 \to^{\perp} (I_1)\Xi_2 \to^{\perp} (I_2)\Xi_3$ is exact.
- $\forall \rho \in \Xi_1, T_{(P-1)}(\underline{K}_{(P-1)}(\rho)) = \underline{K}_{(P-2)}(I_1(\rho))$ and $T_{(N-1)}(\underline{K}_{(N-1)}(\rho)) = \underline{K}_{(N-2)}(I_1(\rho))$
- $\forall \rho \in \Xi_2, T_{(P-2)}(\underline{K}_{(P-2)}(\rho)) = \underline{K}_{(P-3)}(I_2(\rho))$ and $T_{(N-2)}(\underline{K}_{(N-2)}(\rho)) = \underline{K}_{(N-3)}(I_2(\rho))$

and identified as
$(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1) \to^{\perp} (T_{(P-1)}, T_{(N-1)}, I_1)(\underline{K}_{(P-2)},$
$\underline{K}_{(N-2)}, \Xi_2) \to^{\perp} (T_{(P-2)}, T_{(N-2)}, I_2)(\underline{K}_{(P-3)}, \underline{K}_{(N-3)}, \Xi_3)$.

If $(\underline{K}_{(P-\zeta)}, \underline{K}_{(N-\zeta)}, \Xi_\zeta), \forall \zeta \in J$ is T-bipolar soft exact, then $(\underline{K}_{(P-\zeta)}, \underline{K}_{(N-\zeta)}, \Xi_\zeta)_{\zeta \in J}$ is identified as T-bipolar soft exact.

Theorem 5: Assume a T-BSMs $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ over $\Upsilon_1 = M_1 \cup M_1^*$, $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ over $\Upsilon_2 = M_2 \cup M_2^*$. If $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1) \to^{\perp} (T_P, T_N, I)(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2) \to 0$ is T-bipolar soft exact, then (T_P, T_N, I) is T-bipolar soft homomorphism. Further, $0 \to (\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1) \to^{\perp} (T_P, T_N, I)(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2) \to 0$ is T-bipolar soft exact, then (T_P, T_N, I) is a T-bipolar soft isomorphism.

Proof: As $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1) \to^{\perp} (T_P, T_N, I)(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2) \to 0$ is T-bipolar soft exact, thus $M_1 \to^{\perp} (T_P)M_2 \to 0$, $M_1^* \to^{\perp} (T_N)M_2^* \to 0$, and $\Xi_1 \to^{\perp} I\Xi_2 \to 0$ are exact. This implies that T_P, T_N, and I are epimorphisms, and hence (T_P, T_N, I) is a T-bipolar soft homomorphism.

Next, as $0 \to (\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1) \to^{\perp} (T_P, T_N, I)(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2) \to 0$ is T-bipolar soft exact, thus $0 \to M_1 \to^{\perp} (T_P)M_2 \to 0$, $0 \to M_1^* \to^{\perp} (T_N)M_2^* \to 0$, and $0 \to \Xi_1 \to^{\perp} I\Xi_2 \to 0$ are exact. This implies that T_P, T_N, and I are isomorphisms, and hence (T_P, T_N, I) is a T-bipolar soft isomorphism.

Proposition 10: Assume a null T-BSMs
$(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ over $\Upsilon_1 = M_1 \cup M_1^*$ and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ an absolute T-BSM over $\Upsilon_3 = M_3 \cup M_3^*$. If $0 \to M_1 \to^{\perp} (g_{(P-1)})M_2 \to^{\perp} (g_{(P-2)})M_3 \to 0$ and $0 \to M_1^* \to^{\perp} (g_{(N-1)})M_2^* \to^{\perp} (g_{(N-2)})M_3^* \to 0$ are short exact sequences, then $\forall \rho_1 \in \Xi_1, \rho_2 \in \Xi_2, 0 \to \underline{K}_{(P-1)}(\rho) \to^{\perp} (g'_{(P-1)})M_2 \to^{\perp} (g'_{(P-2)})\underline{K}_{(P-2)}(\rho_2) \to 0$ and $0 \to \underline{K}_{(N-1)}(\rho) \to^{\perp} (g'_{(N-1)})M_2^* \to^{\perp} (g'_{(N-2)})\underline{K}_{(N-2)}(\rho_2) \to 0$ are short exact sequences.

Proof: As $(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ is a null T-BSM, so $\underline{K}_{(P-1)}(\rho) = 0$ and $\underline{K}_{(N-1)}(\rho) = M_1^*$, this implies that $g'_{(P-1)}$ and $g'_{(N-1)}$ are monomorphisms. As we have that $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ is an absolute T-BSM, so $\underline{K}_{(P-2)}(\rho_2) = M_3$ and $\underline{K}_{(N-2)}(\rho_2) = 0$, this implies that $g'_{(P-2)}$ and $g'_{(N-2)}$ are epimorphisms as $0 \to M_1 \to^{\perp} (g_{(P-1)})M_2 \to^{\perp} (g_{(P-2)})M_3 \to 0$ and $0 \to M_1^* \to^{\perp} (g_{(N-1)})M_2^* \to^{\perp} (g_{(N-2)})M_3^* \to 0$ are short exact sequences.

Proposition 11: Assume a null T-BSMs
$(\underline{K}_{(P-1)}, \underline{K}_{(N-1)}, \Xi_1)$ over $\Upsilon_1 = M_1 \cup M_1^*$ and $(\underline{K}_{(P-2)}, \underline{K}_{(N-2)}, \Xi_2)$ an absolute T-BSM over $\Upsilon_3 = M_3 \cup M_3^*$. If $0 \to M_1 \to^{\perp} (T_{(P-1)})M_2 \to^{\perp} (T_{(P-2)})M_3 \to 0$ and $0 \to M_1^* \to^{\perp} (T_{(N-1)})M_2^* \to^{\perp} (T_{(N-2)})M_3^* \to 0$ are short exact sequences, then $\forall \rho_1 \in \Xi_1, \rho_2 \in \Xi_2, 0 \to \underline{K}_{(P-1)}(\rho) \to^{\perp} (T_{(P-1)})M_2 \to^{\perp}$

$(T_{(P-2)})\underline{K}_{(P-2)}(y) \rightarrow 0$ and $0 \rightarrow \underline{K}_{(N-1)}(\rho) \rightarrow^{\perp} (T_{(N-1)})M_2^* \rightarrow^{\perp} (T_{(N-1)})\underline{K}_{(N-2)}(y) \rightarrow 0$ are short exact sequences.

Proof: Obvious.

6. Conclusion

This paper contributed a new concept of T-BSMs to the field of abstract algebra and SS theory. This novel concept is a combination of the traditional module theory and the T-BSS in dealing with bipolar information. To the best of our knowledge, our study filled a significant void in the literature by proposing a conceptual framework for T-BSMs. We have presented and formally defined some concepts such as T-BSSMs and maximal T-BSSMs, their properties, and theorems. Thus, this paper offers a theoretical background that will help to advance the understanding and practice of T-BSMs in the future. The major contribution of this study lies in the application of basic algebraic concepts to the T-BSM framework. In this paper, we have comprehensively discussed T-bipolar soft homomorphisms and isomorphisms and have proved the First, Second, and Third Isomorphism Theorems for T-BSMs. These results show that T-BSMs have a very rich algebraic structure and that they coincide with the classical theory of modules. In addition, the investigation of T-bipolar soft exactness and the related theorems expands the possibility of studying the behavior of T-BSMs in more complicated algebraic environments. This part of the work is pretty significant as it relates the standard homological algebra to the comparatively new T-BSS theory. T-BSM theory is a new type of perception and modeling of algebraic structures that have duality or bipolarity in their characteristics. It can provide a new look at numerous aspects of mathematics and their uses in various spheres of life. Thus, the T-BSMs can be regarded as the useful tool for the bipolar information management in the context of the modular structures and can be applied for the solution of several problems in such fields as theartificial intelligence, the data analysis and the decision theory. In the future, we would like to discuss the theory of modules in the notion of the fuzzy set (FS) and its modifications such as bipolar complex fuzzy soft set, [13]. Generalized Fuzzy Filters in Quantales and Their Approximations [14]

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