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On Solving the Kudryashov-Sinelshchikov Equation for Pressure Waves in Gas-Liquid Mixtures

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Abstract. In this work, our focus is on obtaining approximate analytic solutions for the Kudryashov-Sinelshchikov equation using the reduced differential transform method. This equation describes nonlinear waves in gas-liquid mixtures, with a specific emphasis on the impact of heat transfer and viscosity. The results present a theoretical basis of the numerical scheme and manifest its efficiency with different types of numerical examples, from solitary wave solutions to the soliton wave solutions as well as to the rational soliton solutions. This proposed scheme is backed by the calculated absolute error norms and illustrated results and is proven to be efficient in solving both linear and nonlinear problems of similar natures.

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1. Introduction

Solitary and periodic waves frequently appear in mixtures governed by nonlinear partial differential equations (PDEs), including the KdV-mKdV equation, the KdV-Burgers equation, the BBM-Burgers equation, etc. Nowadays, the Traveling wave solutions of such problems have attracted the attention of many researchers. Numerous efforts are underway to address a variety of nonlinear phenomena. The authors have also addressed similar model problems in their previous works [1, 2, 6, 7, 10, 11]. In the current work, we focus on Kudryashov-Sinelshchikov equation which is particularly used to model the behavior of pressure waves in liquid-gas mixtures, taking into account viscosity and thermal expansion. While studying the physical and biological aspects of the ocean, we encounter physical models described by nonlinear partial differential equations (PDEs) that support a broad range of solitary wave solutions. Among these nonlinear PDE models, the Kudryashov-Sinelshchikov equation stands out. It was initially proposed by Kudryashov and Sinelshchikov to describe pressure waves in a liquid mixture and gas bubbles [12].

Recently, Ali and Maneea presented a new technique focusing on the Kudryashov-Sinelshchikov (KS) equation and its variants. They assess different numerical methods, and showcases numerical results that affirm the method's precision. It highlights the efficiency of technique used for nonlinear fractional PDEs and its potential applications in biological systems and boundary value problems [5].

In order to find exact and approximate solutions, the Kudryashov-Sinelshchikov equation has recently been studied and solved using a variety of analytical, numerical and semi-analytical methods, including the residual power series method and the homotopy analysis method [4]. Using a radial basis function, Gupta and Ray numerically solved the fractional Kudryashov-Sinelshchikov problem. The approximate solution is in good agreement with the exact solution and achieves good accuracy [8]. Chen Yue et al. solve the fractional Kudryashov-Sinelshchikov equation using a modified Khater approach [18]. With the use of similarity reductions and the invariant subspace technique, Prakash employed Lie symmetry analysis to obtain many exact solutions [15].

In this work, authors analyzes nonlinear waves in a mixture of liquid and gas bubbles, focusing on the effects of viscosity and heat transfer on pressure wave propagation. The findings emphasize the importance of understanding wave dynamics in bubbly liquids, with implications for applications such as medical ultrasound imaging and insights into wave behavior influenced by thermal and viscous interactions.

In the present paper, we discuss the mathematical properties of the KS equation, which can be found in the literature [9, 13, 14, 16, 17]. Unlike exact solutions, there are very few numerical schemes documented. Therefore, our work focuses on developing a numerical scheme for the Kudryashov-Sinelshchikov equation. In Section 3, we discuss the application of the differential transform technique and the two-dimensional reduced differential transform method for solving the equation, respectively. The outcomes of the proposed schemes are presented in Section 4. The final section includes our concluding remarks, followed by a list of references used in this work.

2. Mathematical Modelling

Taking into account heat transfer and the viscosity of the liquid, a general partial differential equation describing pressure waves in a mixture of liquid and gas bubbles was introduced as

$$\frac{\partial u}{\partial t} + \gamma u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + \epsilon \left(u \frac{\partial^2 u}{\partial x^2} \right)_x - \kappa \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} - \nu \frac{\partial^2 u}{\partial x^2} - \delta \left(u \frac{\partial u}{\partial x} \right)_x = 0, \tag{2. 1}$$

where $\gamma, \epsilon, \kappa, \nu, \delta$ are constants playing a significant role with each term. u is a density modeling viscosity and heat transfer [14]. In particular if $\epsilon = \kappa = \nu = \delta = 0$, it will give rise to KdV equation. When $\epsilon = \kappa = \delta = 0$, it is called KdV-Burgers equation as discussed earlier. In the present work, we will consider Kudryashov-Sinelshchikov equation in the form

$$u_t + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} + \gamma \left(u \frac{\partial^2 u}{\partial x^2} \right)_x + \sigma u_x \frac{\partial^2 u}{\partial x^2} = 0, \tag{2. 2}$$

where α, β, γ and $\sigma = -3\gamma$ are arbitrary nonzero and real parameters.

3. Basic definitions

In this section, we introduce basic definitions related to differential transformation.

3.1. **1D-Differential transform method.** If $u(t) \in \mathbb{R}$ can be expressed as a Taylor series about the fixed point t_0, then $u(t)$ can be represented as the sum of its Taylor series expansion.

$$u(t) = \sum_{k=0}^{\infty} \frac{u^{(k)}(t_0)}{k!} (t - t_0)^k. \tag{3. 3}$$

If $u_n(t) = \sum_{k=0}^{n} \frac{u^{(k)}(t_0)}{k!} (t - t_0)^k$, is the n-partial sums of a Taylor series equation (3. 3), then

$$u_n(t) = \sum_{k=0}^{n} \frac{u^{(k)}(t_0)}{k!} (t - t_0)^k + R_n(t). \tag{3. 4}$$

Here, $u_n(t)$ represents the n-th polynomial for $u(t)$, and $R_n(t)$ stands for the remainder term. Now, if we define the differential transform function $U(k)$ as

$$U(k) = \frac{1}{k!} \left[\frac{d^k u(t)}{dt^k} \right]_{t=t_0}, \tag{3. 5}$$

where $k = 0, 1, 2, ...$ then Eq. (3. 3) is reduced to

$$u(t) = \sum_{k=0}^{\infty} U(k)(t - t_0)^k, \tag{3. 6}$$

and the n-partial sums of series expansion becomes

$$u_n(t) = \sum_{k=0}^{n} U(k)(t - t_0)^k + R_n(t). \tag{3. 7}$$

For simplicity, let's assume $t_0 = 0$. Then, Eq. (3. 7) is reduced to

$$u_n(t) = \sum_{k=0}^{n} U(k) t^k + R_n(t). \tag{3. 8}$$

Upon examining the definition above, it becomes evident that the roots of the differential transform method are embedded within the Taylor series expansion.

TABLE 1. The core operations within the one-dimensional DTM

Original Function	Transformed Function
$w(t) = u(t) \pm v(t)$	$W(k) = U(k) \pm V(k)$
$w(t) = \frac{d^m u(t)}{dt^m}$	$W(k) = \frac{(k+m)!}{k!} U(k+m)$
$w(t) = u(t)\, v(t)$	$W(k) = U(k) * V(k) = \sum_{r=0}^{k} U(r)\, V(k-r)$
$w(x) = x^m$	$W(k) = \delta(k-m) = \begin{cases} 1 & k=m \\ 0 & otherwise \end{cases}$
$w(t) = e^{\lambda t}$	$W(k) = \frac{\lambda^k}{k!}$
$w(t) = \sin(\alpha t + \beta)$	$W(k) = \frac{\alpha^k}{k!} \sin\left(\frac{k\pi}{2} + \beta\right)$
$w(t) = \cos(\alpha t + \beta)$	$W(k) = \frac{\alpha^k}{k!} \cos\left(\frac{k\pi}{2} + \beta\right)$

3.2. **2D-Differential transform method.** We denote $w(x,t)$ as $w(x,t) = f(x)g(t)$. Utilizing the properties of the 1D-DTM, it can be expressed as:

$$w(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} W(i,j) x^i t^j. \tag{3. 9}$$

Letting $W(i,j) = F(i)G(j)$ denote the spectrum of $w(x,t)$, the spectrum function for the analytical function $w(x,t)$ is given by:

$$W_k(x) = \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} w(x,t) \right]_{t=t_0}, \tag{3. 10}$$

as reduced transformed function whose inverse transform is defined as

$$w(x,t) = \sum_{k=0}^{\infty} W_k(x)(t-t_0)^k. \tag{3. 11}$$

From Eqs. (3. 10) and (3. 11), we get

$$w(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} w(x,t) \right]_{t=t_0} (t-t_0)^k. \tag{3. 12}$$

Thus, it becomes evident that the reduced differential transform technique is derived from the 2D differential transform technique.

TABLE 2. The core operations within the two-dimensional RDTM

Original Function	Transformed Function
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \frac{\partial}{\partial x} u(x,t)$	$W_k(x) = \frac{d}{dx} U_k(x)$
$w(x,t) = \frac{\partial}{\partial t} u(x,t)$	$W_k(x) = (k+1)\, U_{k+1}(x)$
$w(x,t) = \frac{\partial^{r+s}}{\partial x^r \partial t^s} u(x,t)$	$W_k(x) = \frac{(k+s)!}{k!} \frac{d^r}{dx^r} U_{k+s}(x)$
$w(x,t) = u(x,t)\, v(x,t)$	$W_k(x) = \sum_{r=0}^{k} U_r(x)\, V_{k-r}(x)$
$w(x,t) = u(x,t)\, v(x,t)\, z(x,t)$	$W_k(x) = \sum_{r=0}^{k} \sum_{s=0}^{k-r} U_r(x)\, V_s(x)\, Z_{k-r-s}(x)$
$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k-n) = \begin{cases} x^m & k=n \\ 0 & otherwise \end{cases}$

4. NUMERICAL APPLICATIONS

This section is dedicated to presenting test problems, aiming to illustrate the effectiveness of the technique. Absolute error norms are computed to validate the accuracy of the proposed method.

4.1. **Solitary wave solutions.** The solitary wave solution of Eq. (2. 2) is provided by [3]

$$u\left(x,t\right)=A\mathrm{sech}^{2}\left[B\left(x-ct\right)\right], \tag{4. 13}$$

where A is amplitude, B is inverse width, c is velocity and

$$A=\frac{3\kappa^{2}\mu^{2}\beta}{\alpha-\kappa^{2}\mu^{2}\gamma}\ ,\ B=\frac{\kappa\mu}{2}\ ,\ c=\kappa^{2}\mu^{2}\beta. \tag{4. 14}$$

Here, α, β, γ, κ and $\mu>0$ are arbitrary constants. We have Eq. (2. 2) with

$$u\left(x,0\right)=A\mathrm{sech}^{2}\left[Bx\right]. \tag{4. 15}$$

After applying reduced differential transform technique to Eq. (2. 2), we get

$$\begin{aligned}(k+1)\,U_{k+1}\left(x\right)+\alpha\sum_{r=0}^{k}U_{r}\left(x\right)\frac{d}{dx}U_{k-r}\left(x\right)&\\ +\left(\gamma+\sigma\right)\sum_{r=0}^{k}\frac{d}{dx}U_{r}\left(x\right)\frac{d^{2}}{dx^{2}}U_{k-r}\left(x\right)&\\ +\beta\frac{d^{3}}{dx^{3}}U_{k}\left(x\right)+\gamma\sum_{r=0}^{k}U_{r}\left(x\right)\frac{d^{3}}{dx^{3}}U_{k-r}\left(x\right)=0,&\end{aligned} \tag{4. 16}$$

and using the given condition (4. 15), we have

$$U_{0}\left(x\right)=A\mathrm{sech}^{2}\left[Bx\right], \tag{4. 17}$$

$$\begin{aligned}U_{1}\left(x\right)=&\,AB\left(-9B^{2}\beta+A\left(\alpha-4B^{2}(7\gamma+2\sigma)\right)\right.\\ &+\left(-8B^{2}\beta+A\left(\alpha+4B^{2}(2\gamma+\sigma)\right)\right)\cosh[2Bx]\\ &\left.+B^{2}\beta\cosh[4Bx]\right)\mathrm{sech}^{6}[Bx]\tanh[Bx],\end{aligned} \tag{4. 18}$$

$$\begin{aligned}U_{2}\left(x\right)=&\,AB^{2}\mathrm{sech}^{2}[Bx]\big(32B^{4}\beta^{2}\\ &+16B^{2}\beta\left(-63B^{2}\beta+5A\left(\alpha+4B^{2}(2\gamma+\sigma)\right)\right)\mathrm{sech}^{2}[Bx]\\ &+2\left(1680B^{4}\beta^{2}-16AB^{2}\beta\left(11\alpha+250B^{2}\gamma+98B^{2}\sigma\right)\right.\\ &\left.+3A^{2}\left(\alpha+4B^{2}(2\gamma+\sigma)\right)\left(\alpha+4B^{2}(5\gamma+2\sigma)\right)\right)\mathrm{sech}^{4}[Bx]\\ &-\left(2520B^{4}\beta^{2}-6AB^{2}\beta\left(49\alpha+20B^{2}\left(166\gamma+57\sigma\right)\right)\right.\\ &+A^{2}\left(7\alpha^{2}+4B^{2}\alpha\left(157\gamma+53\sigma\right)\right.\\ &\left.\left.+16B^{4}\left(574\gamma^{2}+403\gamma\sigma+70\sigma^{2}\right)\right)\right)\mathrm{sech}^{6}[Bx]\\ &+6AB^{2}\left(-36B^{2}\beta(60\gamma+19\sigma)+A\left(3\alpha(27\gamma+8\sigma)\right.\right.\\ &\left.\left.+4B^{2}\left(843\gamma^{2}+515\gamma\sigma+77\sigma^{2}\right)\right)\right)\mathrm{sech}^{8}[Bx]\\ &-132A^{2}B^{4}(3\gamma+\sigma)(31\gamma+7\sigma)\mathrm{sech}^{10}[Bx]\big),\end{aligned} \tag{4. 19}$$

Proceeding in a similar manner, we derive the subsequent terms. Substituting these into the inverse DTM, the Poisson series form yields the approximate solution for Eq. (2. 2):

$$U_2(x,t) = U_0(x) + U_1(x)\,t + U_2(x)\,t^2, \tag{4. 20}$$

$$\begin{aligned} U_2(x,t) = {} & A\text{sech}^2[Bx]\left(1 + 32B^6t^2\beta^2 + Bt\left(-Bt\left(2520B^4\beta^2\right.\right.\right. \\ & -6AB^2\beta\left(49\alpha + 20B^2(166\gamma + 57\sigma)\right) + A^2\left(7\alpha^2 + 4B^2\alpha(157\gamma + 53\sigma)\right. \\ & \left.\left.\left. +16B^4\left(574\gamma^2 + 403\gamma\sigma + 70\sigma^2\right)\right)\right)\text{sech}^6[Bx] \\ & +6AB^3t\left(-36B^2\beta(60\gamma + 19\sigma) + A\left(3\alpha(27\gamma + 8\sigma)\right.\right. \\ & \left.\left. +4B^2\left(843\gamma^2 + 515\gamma\sigma + 77\sigma^2\right)\right)\right)\text{sech}^8[Bx] \\ & -132A^2B^5t(3\gamma + \sigma)(31\gamma + 7\sigma)\text{sech}^{10}[Bx] + 8B^2\beta\tanh[Bx] \\ & +2B\text{sech}^4[Bx]\left(t\left(1680B^4\beta^2 - 16AB^2\beta\left(11\alpha + 250B^2\gamma + 98B^2\sigma\right)\right.\right. \\ & \left.\left. +3A^2\left(\alpha + 4B^2(2\gamma + \sigma)\right)\left(\alpha + 4B^2(5\gamma + 2\sigma)\right)\right) - 6AB(3\gamma + \sigma)\tanh[Bx]\right) \\ & +2\text{sech}^2[Bx]\left(8B^3t\beta\left(-63B^2\beta + 5A\left(\alpha + 4B^2(2\gamma + \sigma)\right)\right)\right. \\ & \left.\left.\left. + \left(-12B^2\beta + A\left(\alpha + 4B^2(2\gamma + \sigma)\right)\right)\tanh[Bx]\right)\right)\right), \end{aligned} \tag{4. 21}$$

which matches exactly with the is the first three terms of analytical result (4. 13).

The paramaters are chosen as $\alpha = 2$, $\beta = 1$, $\gamma = 0.1$, $\kappa = 1$, and $\mu = 0.5$. Subsequently, the algorithm continues to run until time $t = 5$. Table 3 presents the errors of the N-approximate solution obtained from RDTM and the analytical solution, denoted as $u - U_N$, at various N for specific points within the intervals $-15 \leq x \leq 15$ at selected time steps. In Fig. 1, $2-$ approximate solution are plotted as 3D and 2D, respectively. The absolute errors for $N-$approximate solitary wave solutions of Eq. (2. 2) are drawn as two dimensional in Fig. 2 at time $t = 1$. Then, the absolute errors for $N-$approximate solitary wave solutions of Eq. (2. 2) are depicted as three dimensional in Fig. 3 at $0 \leq t \leq 5$. Table 3, Fig. 2 and Fig. 3 verify the proposed technique as the number of terms are expanded in the series.

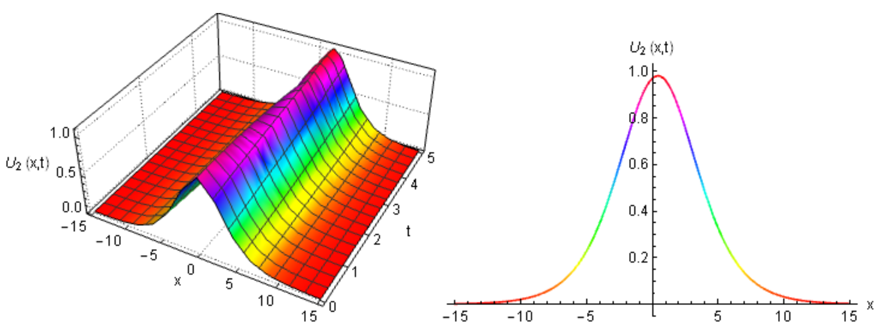

FIGURE 1. $2-$approximate solution for solitary wave solution of Eq. (2. 2) with $\alpha = 2$, $\beta = 1$, $\gamma = 0.1$, $\kappa = 1$, and $\mu = 0.5$.

TABLE 3. Calculating the errors for the N-approximate solitary wave solutions of Eq. (2. 2) under the conditions $\alpha = 2$, $\beta = 1$, $\gamma = 0.1$, $\kappa = 1$, and $\mu = 0.5$.

x	t	$\lvert u(x,t) - U_2(x,t)\rvert$	$\lvert u(x,t) - U_3(x,t)\rvert$	$\lvert u(x,t) - U_4(x,t)\rvert$
-15.00	0.10	3.421457×10^{-9}	1.803474×10^{-11}	7.533557×10^{-14}
-12.50	0.25	1.811225×10^{-7}	2.337145×10^{-9}	2.329880×10^{-11}
-10.00	0.50	4.623841×10^{-6}	1.098268×10^{-7}	1.778989×10^{-9}
-7.50	0.75	4.049229×10^{-5}	9.261687×10^{-7}	9.934641×10^{-9}
-5.00	1.00	7.006494×10^{-5}	1.083204×10^{-5}	1.163533×10^{-6}
-2.50	1.50	2.004236×10^{-3}	1.857150×10^{-4}	1.124277×10^{-5}
0.00	2.00	1.299918×10^{-3}	1.299918×10^{-3}	3.333081×10^{-5}
2.50	2.50	1.122572×10^{-2}	1.087055×10^{-3}	2.591817×10^{-4}
5.00	3.00	3.039990×10^{-4}	1.295289×10^{-3}	3.236481×10^{-4}
7.50	3.50	4.590416×10^{-3}	3.810729×10^{-4}	5.347095×10^{-5}
10.00	4.00	2.941583×10^{-3}	5.179452×10^{-4}	6.080790×10^{-5}
12.50	4.50	1.368788×10^{-3}	2.988510×10^{-4}	5.106108×10^{-5}
15.00	5.00	5.714126×10^{-4}	1.414762×10^{-4}	2.828821×10^{-5}

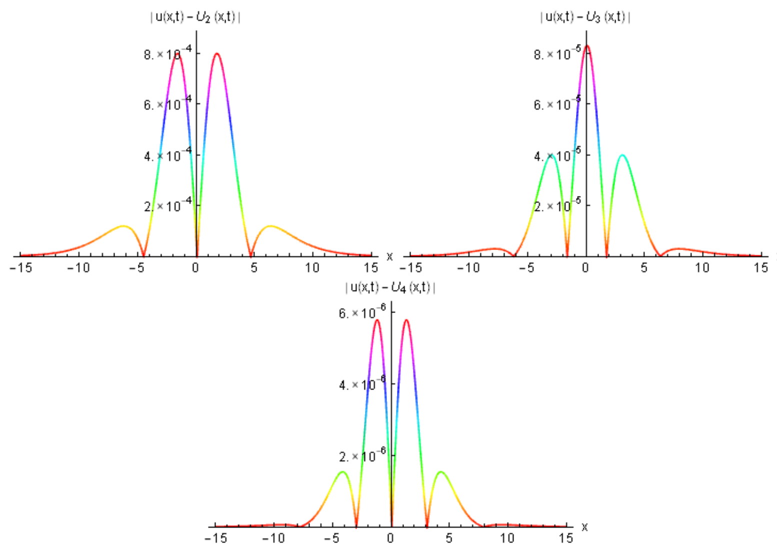

FIGURE 2. The absolute errors for $N-$approximate solitary wave solutions of Eq. (2. 2) with $\alpha = 2$, $\beta = 1$, $\gamma = 0.1$, $\kappa = 1$, and $\mu = 0.5$ at $t = 1$.

4.2. **Soliton wave solutions.** Analytical solution of Eq. (2. 2) is given by [3]

$$u(x,t) = -\frac{48 b_2 b_0^2 \kappa^2 \beta e^{2\sqrt{b_0}\kappa\left(x - 4b_0\kappa^2\beta t\right)}}{\left(\alpha - 4b_0\kappa^2\gamma\right)\left(-1 + b_0 b_2 e^{2\sqrt{b_0}\kappa\left(x - 4b_0\kappa^2\beta t\right)}\right)^2}, \quad (4.\ 22)$$

where $\sigma = -3\gamma$. We take Eq. (2. 2) subject to initial condition

$$u(x,0) = -\frac{48 b_2 b_0^2 \kappa^2 \beta e^{2\sqrt{b_0}\kappa x}}{\left(\alpha - 4b_0\kappa^2\gamma\right)\left(-1 + b_0 b_2 e^{2\sqrt{b_0}\kappa x}\right)^2}. \quad (4.\ 23)$$

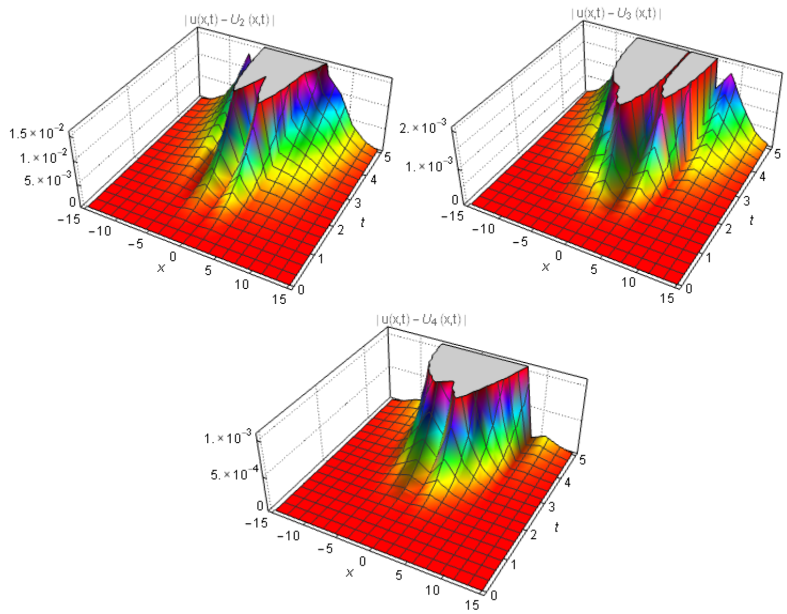

FIGURE 3. The absolute errors for $N-$approximate solitary wave solutions of Eq. (2. 2) with $\alpha = 2$, $\beta = 1$, $\gamma = 0.1$, $\kappa = 1$, and $\mu = 0.5$.

Thus, by applying the DTM on Eq. (2. 2), we get the recurrence relation as

$$\begin{aligned}(k+1)\,U_{k+1}(x) &+ \alpha\sum_{r=0}^{k}U_r(x)\,\frac{d}{dx}U_{k-r}(x)\\ &+(\gamma+\sigma)\sum_{r=0}^{k}\frac{d}{dx}U_r(x)\,\frac{d^2}{dx^2}U_{k-r}(x)\\ &+\beta\frac{d^3}{dx^3}U_k(x)+\gamma\sum_{r=0}^{k}U_r(x)\,\frac{d^3}{dx^3}U_{k-r}(x)=0,\end{aligned} \tag{4. 24}$$

with initial value

$$U_0(x) = -\frac{48b_2b_0^2\kappa^2\beta e^{2\sqrt{b_0}\kappa x}}{(\alpha-4b_0\kappa^2\gamma)\left(-1+b_0b_2e^{2\sqrt{b_0}\kappa x}\right)^2}, \tag{4. 25}$$

the first two terms from the recurrence relation (4. 24) are obtained as

$$\begin{aligned} U_1(x) = & \frac{384 e^{2x\kappa\sqrt{b_0}}\beta^2\kappa^5 b_0^{7/2} b_2}{(\alpha-4\gamma\kappa^2 b_0)^2\left(-1+e^{2x\kappa\sqrt{b_0}}b_0 b_2\right)^7}\left(-\alpha+b_0\left(4\gamma\kappa^2\right.\right. \\ & +e^{2x\kappa\sqrt{b_0}}b_2\left(3\alpha+b_0\left(12\kappa^2(11\gamma+4\sigma)+e^{2x\kappa\sqrt{b_0}}b_2\right.\right.\left(-2\alpha\right. \\ & +b_0\left(8\kappa^2(91\gamma+30\sigma)+e^{2x\kappa\sqrt{b_0}}b_2\left(-2\alpha+b_0\left(8\kappa^2(91\gamma+30\sigma)\right.\right.\right. \\ & +e^{2x\kappa\sqrt{b_0}}b_2\left(3\alpha+b_0\left(12\kappa^2(11\gamma+4\sigma)\right.\right. \\ & \left.\left.\left.\left.\left.\left.\left.\left.\left.\left.-e^{2x\kappa\sqrt{b_0}}\left(\alpha-4\gamma\kappa^2 b_0\right)b_2\right)\right)\right)\right)\right)\right)\right)\right)\right)\right), \end{aligned} \tag{4. 26}$$

$$\begin{aligned} U_2(x) = & \frac{1536 e^{2x\kappa\sqrt{b_0}}\beta^3\kappa^8 b_0^5 b_2}{(\alpha-4\gamma\kappa^2 b_0)^3\left(-1+e^{2x\kappa\sqrt{b_0}}b_0 b_2\right)^{12}}\left(-\left(\alpha-4\gamma\kappa^2 b_0\right)^2\right. \\ & -4e^{2x\kappa\sqrt{b_0}}b_0\left(-\alpha+4\gamma\kappa^2 b_0\right)\left(\alpha+4\kappa^2(89\gamma+30\sigma)b_0\right)b_2 \\ & -3e^{4x\kappa\sqrt{b_0}}b_0^2\big(-\alpha^2-8\alpha\kappa^2(1223\gamma+408\sigma)b_0 \\ & +16\kappa^4\left(5039\gamma^2+2544\gamma\sigma+288\sigma^2\right)b_0^2\big)b_2^2 \\ & -48e^{6x\kappa\sqrt{b_0}}b_0^3\big(\alpha^2-4\alpha\kappa^2(545\gamma+181\sigma)b_0 \\ & +16\kappa^4\left(4000\gamma^2+2269\gamma\sigma+312\sigma^2\right)b_0^2\big)b_2^3 \\ & -6e^{8x\kappa\sqrt{b_0}}b_0^4\left(-21\alpha^2+8\kappa^2 b_0\left(7\alpha(63\gamma+20\sigma)\right.\right. \\ & \left.\left.+2\kappa^2\left(142995\gamma^2+80360\gamma\sigma+10896\sigma^2\right)b_0\right)\right)b_2^4 \\ & -24e^{10x\kappa\sqrt{b_0}}b_0^5\left(7\alpha^2+8\kappa^2 b_0\left(7\alpha(170\gamma+57\sigma)\right.\right. \\ & \left.\left.+2\kappa^2\left(58525\gamma^2+32466\gamma\sigma+4320\sigma^2\right)b_0\right)\right)b_2^5 \\ & -6e^{12x\kappa\sqrt{b_0}}b_0^6\left(-21\alpha^2+8\kappa^2 b_0\left(7\alpha(63\gamma+20\sigma)\right.\right. \\ & \left.\left.+2\kappa^2\left(142995\gamma^2+80360\gamma\sigma+10896\sigma^2\right)b_0\right)\right)b_2^6 \\ & -48e^{14x\kappa\sqrt{b_0}}b_0^7\big(\alpha^2-4\alpha\kappa^2(545\gamma+181\sigma)b_0 \\ & +16\kappa^4\left(4000\gamma^2+2269\gamma\sigma+312\sigma^2\right)b_0^2\big)b_2^7 \\ & -3e^{16x\kappa\sqrt{b_0}}b_0^8\big(-\alpha^2-8\alpha\kappa^2(1223\gamma+408\sigma)b_0 \\ & +16\kappa^4\left(5039\gamma^2+2544\gamma\sigma+288\sigma^2\right)b_0^2\big)b_2^8 \\ & -4e^{18x\kappa\sqrt{b_0}}b_0^9\left(-\alpha+4\gamma\kappa^2 b_0\right)\left(\alpha+4\kappa^2(89\gamma+30\sigma)b_0\right)b_2^9 \\ & \left.-e^{20x\kappa\sqrt{b_0}}b_0^{10}\left(\alpha-4\gamma\kappa^2 b_0\right)^2 b_2^{10}\right). \end{aligned} \tag{4. 27}$$

Continuing in the same way, we get other terms using the (4. 24). Putting these in inverse DTM, the approximate solution of Eq. (2. 2) are:

$$U_2(x,t) = U_0(x) + U_1(x)\,t + U_2(x)\,t^2, \tag{4. 28}$$

$$\begin{aligned}
U_2(x,t) = {} & \frac{48e^{2x\kappa\sqrt{b_0}}\beta\kappa^2 b_0^2 b_2}{(\alpha-4\gamma\kappa^2 b_0)^3\left(-1+e^{2x\kappa\sqrt{b_0}}b_0b_2\right)^{12}}\Big(-\left(\alpha-4\gamma\kappa^2 b_0\right)^2 \\
& \left(-1+e^{2x\kappa\sqrt{b_0}}b_0b_2\right)^{10} + 32t^2\beta^2\kappa^6 b_0^3\Big(-\left(\alpha-4\gamma\kappa^2 b_0\right)^2 \\
& -4e^{2x\kappa\sqrt{b_0}}b_0\left(-\alpha+4\gamma\kappa^2 b_0\right)\left(\alpha+4\kappa^2(89\gamma+30\sigma)b_0\right)b_2 \\
& -3e^{4x\kappa\sqrt{b_0}}b_0^2\big(-\alpha^2-8\alpha\kappa^2(1223\gamma+408\sigma)b_0 \\
& +16\kappa^4\left(5039\gamma^2+2544\gamma\sigma+288\sigma^2\right)b_0^2\big)\,b_2^2 \\
& -48e^{6x\kappa\sqrt{b_0}}b_0^3\big(\alpha^2-4\alpha\kappa^2(545\gamma+181\sigma)b_0 \\
& +16\kappa^4\left(4000\gamma^2+2269\gamma\sigma+312\sigma^2\right)b_0^2\big)\,b_2^3 \\
& -6e^{8x\kappa\sqrt{b_0}}b_0^4\big(-21\alpha^2+8\kappa^2 b_0\big(7\alpha(63\gamma+20\sigma) \\
& +2\kappa^2\left(142995\gamma^2+80360\gamma\sigma+10896\sigma^2\right)b_0\big)\big)b_2^4 \\
& -24e^{10x\kappa\sqrt{b_0}}b_0^5\big(7\alpha^2+8\kappa^2 b_0\big(7\alpha(170\gamma+57\sigma) \\
& +2\kappa^2\left(58525\gamma^2+32466\gamma\sigma+4320\sigma^2\right)b_0\big)\big)b_2^5 \\
& -6e^{12x\kappa\sqrt{b_0}}b_0^6\big(-21\alpha^2+8\kappa^2 b_0\big(7\alpha(63\gamma+20\sigma) \\
& +2\kappa^2\left(142995\gamma^2+80360\gamma\sigma+10896\sigma^2\right)b_0\big)\big)b_2^6 \\
& -48e^{14x\kappa\sqrt{b_0}}b_0^7\big(\alpha^2-4\alpha\kappa^2(545\gamma+181\sigma)b_0 \\
& +16\kappa^4\left(4000\gamma^2+2269\gamma\sigma+312\sigma^2\right)b_0^2\big)b_2^7 \\
& -3e^{16x\kappa\sqrt{b_0}}b_0^8\big(-\alpha^2-8\alpha\kappa^2(1223\gamma+408\sigma)b_0 \\
& +16\kappa^4\left(5039\gamma^2+2544\gamma\sigma+288\sigma^2\right)b_0^2\big)b_2^8 \\
& -4e^{18x\kappa\sqrt{b_0}}b_0^9\left(-\alpha+4\gamma\kappa^2 b_0\right)\left(\alpha+4\kappa^2(89\gamma+30\sigma)b_0\right)b_2^9 \\
& -e^{20x\kappa\sqrt{b_0}}b_0^{10}\left(\alpha-4\gamma\kappa^2 b_0\right)^2 b_2^{10}\Big) \\
& +8t\beta\kappa^3 b_0^{3/2}\left(\alpha-4\gamma\kappa^2 b_0\right)\left(-1+e^{2x\kappa\sqrt{b_0}}b_0b_2\right)^5 \\
& \Big(-\alpha+b_0\Big(4\gamma\kappa^2+e^{2x\kappa\sqrt{b_0}}b_2\big(3\alpha+b_0\big(12\kappa^2(11\gamma+4\sigma) \\
& +e^{2x\kappa\sqrt{b_0}}b_2\big(-2\alpha+b_0\big(8\kappa^2(91\gamma+30\sigma) \\
& +e^{2x\kappa\sqrt{b_0}}b_2\big(-2\alpha+b_0\big(8\kappa^2(91\gamma+30\sigma) \\
& +e^{2x\kappa\sqrt{b_0}}b_2\big(3\alpha+b_0\big(12\kappa^2(11\gamma+4\sigma) \\
& -e^{2x\kappa\sqrt{b_0}}\left(\alpha-4\gamma\kappa^2 b_0\right)b_2\big)\big)\big)\big)\big)\big)\big)\big)\Big)\Big)\Big)\Big),
\end{aligned} \tag{4. 29}$$

which matches with the first three terms of analytical result (4. 22).

The computations run up to time $t = 5$ by taking the parameters as $\alpha = -0.3$, $\beta = -0.1$, $\gamma = -0.7$, $\kappa = 0.5$, $b_0 = 0.5$, and $b_2 = -0.1$. Table 4 displays the errors of the N-approximate solution derived from RDTM, represented as $u - U_N$. These errors are provided for different values of N at selected time steps, showcasing the comparison between the approximate solution and the exact solution within the intervals $-10 \leq x \leq 15$. In Fig. 4, $2-$approximate solutions are plotted as 3D and 2D, respectively. The absolute errors for $N-$approximate soliton wave solutions of Eq. (2. 2) are illustrated as two dimensional in Fig. 5 at time $t = 1$. Then, the absolute errors for $N-$approximate soliton wave solutions of Eq. (2. 2) are drawn as three dimensional in Fig. 6 at $0 \leq t \leq 5$. Observing Table 4, Fig. 5, and Fig. 6, it is evident that the error diminishes with an increase in the number of terms included in the series.

TABLE 4. Calculating the errors for the N-approximate soliton wave solutions of Eq. (2. 2) under the conditions $\alpha = -0.3$, $\beta = -0.1$, $\gamma = -0.7$, $\kappa = 0.5$, $b_0 = 0.5$, and $b_2 = -0.1$.

x	t	$\lvert u(x,t) - U_2(x,t)\rvert$	$\lvert u(x,t) - U_3(x,t)\rvert$	$\lvert u(x,t) - U_4(x,t)\rvert$
-10.00	0.10	3.754290×10^{-12}	3.316141×10^{-15}	2.493665×10^{-18}
-7.50	0.25	3.429394×10^{-10}	7.544369×10^{-13}	1.320125×10^{-15}
-5.00	0.50	1.579499×10^{-8}	6.809826×10^{-11}	2.290911×10^{-13}
-2.50	1.00	6.607382×10^{-7}	4.984779×10^{-9}	2.379720×10^{-11}
0.00	2.00	1.315241×10^{-5}	7.175303×10^{-8}	7.278341×10^{-9}
2.50	2.50	1.489075×10^{-4}	2.976294×10^{-6}	8.430426×10^{-8}
5.00	3.00	2.691813×10^{-4}	7.045310×10^{-6}	5.186909×10^{-7}
7.50	3.50	9.186402×10^{-6}	6.043706×10^{-6}	2.891235×10^{-7}
10.00	4.00	7.034321×10^{-5}	1.761287×10^{-6}	1.135005×10^{-8}
12.50	4.50	2.146755×10^{-5}	8.209124×10^{-7}	2.375126×10^{-8}
15.00	5.00	5.196413×10^{-6}	2.298583×10^{-7}	8.046319×10^{-9}

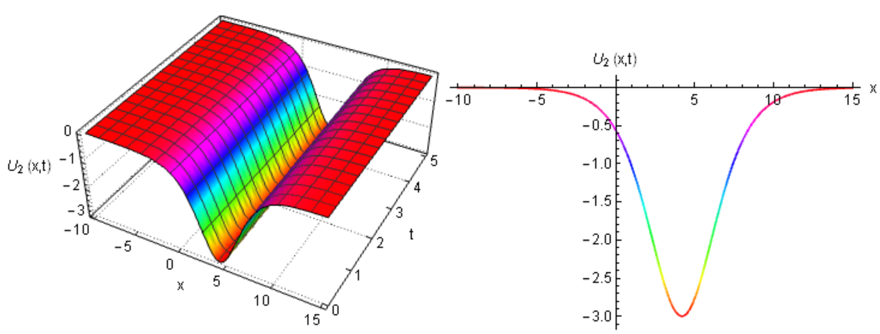

FIGURE 4. $2-$approximate solution for soliton wave solution of Eq. (2. 2) with $\alpha = -0.3$, $\beta = -0.1$, $\gamma = -0.7$, $\kappa = 0.5$, $b_0 = 0.5$, and $b_2 = -0.1$.

4.3. **Soliton rational solutions.** The exact solution of Eq. (2. 2) is given by [3]

$$u(x,t) = -\frac{2b_1 q_1 \beta e^{b_1 \kappa x}}{b_1 q_1 \gamma e^{b_1 \kappa x} - (b_0 q_1 - b_1 q_0)\, e^{b_1^3 \kappa^3 \beta t}}, \tag{4. 30}$$

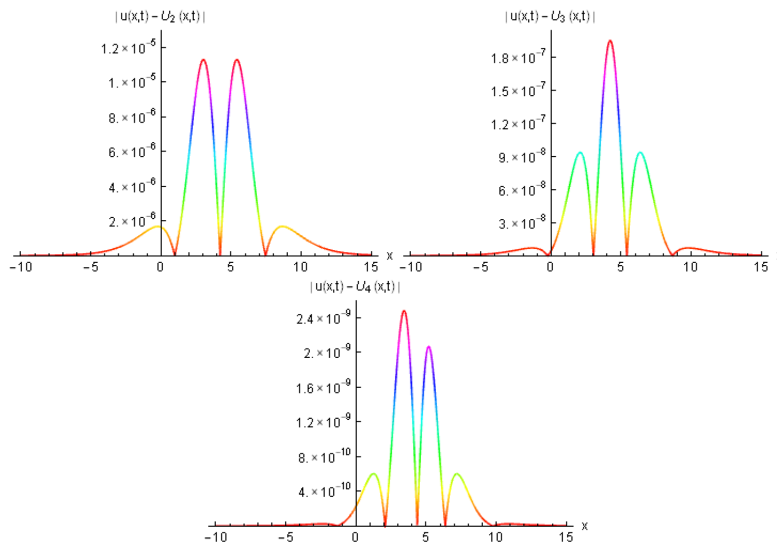

FIGURE 5. The absolute errors for $N-$approximate soliton wave solutions of Eq. (2. 2) with $\alpha = -0.3$, $\beta = -0.1$, $\gamma = -0.7$, $\kappa = 0.5$, $b_0 = 0.5$, and $b_2 = -0.1$ at $t = 1$.

where $\sigma = -3\gamma$. We have the Eq. (2. 2) with condition

$$u(x,0) = -\frac{2b_1 q_1 \beta e^{b_1 \kappa x}}{b_1 q_1 \gamma e^{b_1 \kappa x} - (b_0 q_1 - b_1 q_0)}. \tag{4. 31}$$

After implementing the reduced DTM on Eq. (2. 2), We get the recurrence relation equation

$$\begin{aligned}(k+1)U_{k+1}(x) &+ \alpha \sum_{r=0}^{k} U_r(x) \frac{d}{dx} U_{k-r}(x) \\ &+ (\gamma + \sigma) \sum_{r=0}^{k} \frac{d}{dx} U_r(x) \frac{d^2}{dx^2} U_{k-r}(x) \\ &+ \beta \frac{d^3}{dx^3} U_k(x) + \gamma \sum_{r=0}^{k} U_r(x) \frac{d^3}{dx^3} U_{k-r}(x) = 0,\end{aligned} \tag{4. 32}$$

and the corresponding initial value is

$$U_0(x) = -\frac{2b_1 q_1 \beta e^{b_1 \kappa x}}{b_1 q_1 \gamma e^{b_1 \kappa x} - (b_0 q_1 - b_1 q_0)}, \tag{4. 33}$$

Using the given initial guess, the first two terms of $U_{k+1}(x)$ are obtained as

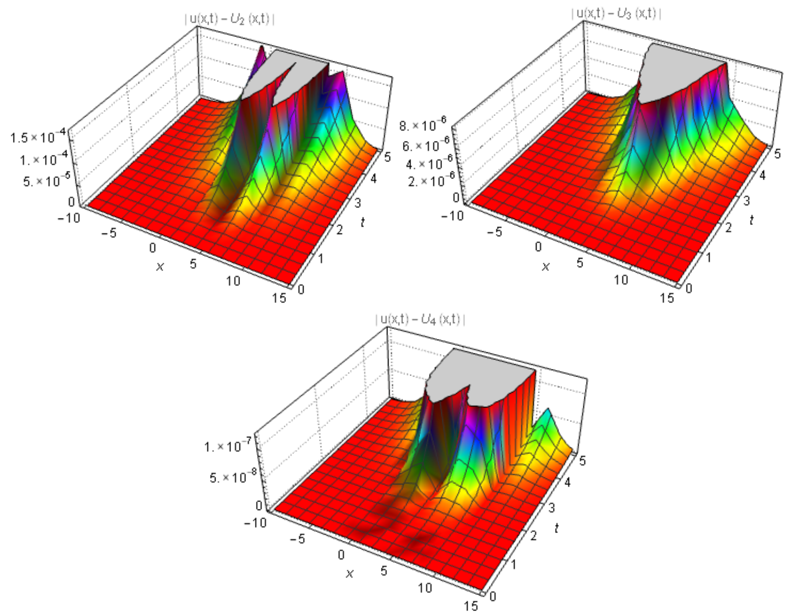

FIGURE 6. The absolute errors for $N-$approximate soliton wave solutions of Eq. (2. 2) with $\alpha = -0.3$, $\beta = -0.1$, $\gamma = -0.7$, $\kappa = 0.5$, $b_0 = 0.5$, and $b_2 = -0.1$.

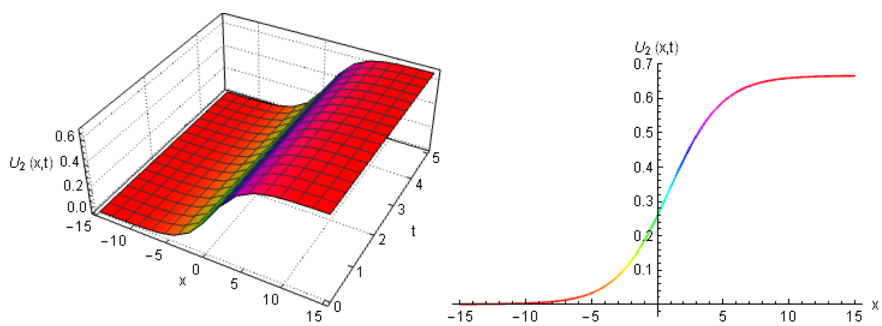

FIGURE 7. $2-$approximate solution for soliton rational solution of Eq. (2. 2) with $\alpha = -0.075$, $\beta = -0.1$, $\gamma = 0.3$, $\kappa = 1$, $b_0 = -1$, $b_1 = 0.5$, $q_0 = 0.4$ and $q_1 = -1$.

$$\begin{aligned}
U_1(x) = {} & \frac{2e^{x\kappa b_1}\beta^2\kappa b_1^3 q_1 (b_1 q_0 - b_0 q_1)}{\gamma^2 (-b_0 q_1 + b_1 (q_0 + e^{x\kappa b_1} q_1))^5} \left(-e^{3x\kappa b_1} b_1^2 \left(2\alpha + \gamma\kappa^2 b_1^2\right) q_1^3\right. \\
& + e^{2x\kappa b_1} b_1 \left(-4\alpha + \kappa^2 (7\gamma + 2\sigma) b_1^2\right) q_1^2 (b_1 q_0 - b_0 q_1) \\
& - e^{x\kappa b_1} \left(2\alpha + \kappa^2 (7\gamma + 2\sigma) b_1^2\right) q_1 (b_1 q_0 - b_0 q_1)^2 \\
& \left. + \gamma\kappa^2 b_1 (b_1 q_0 - b_0 q_1)^3\right),
\end{aligned} \tag{4. 34}$$

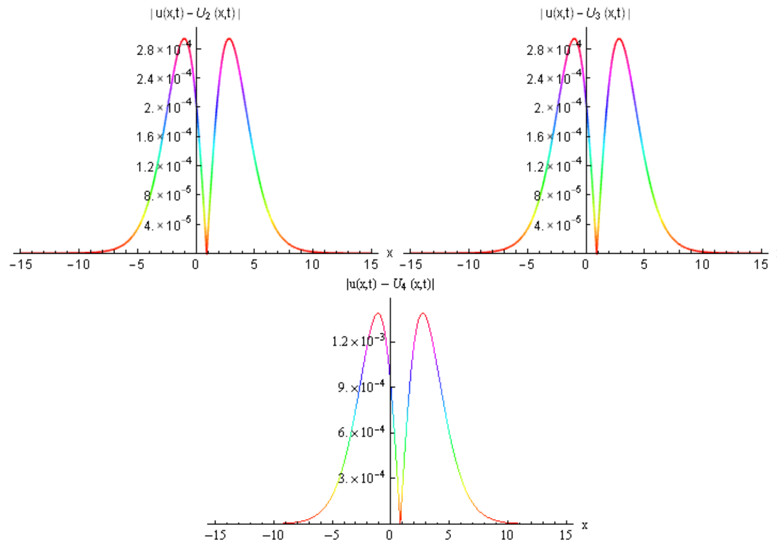

FIGURE 8. The absolute errors for $N-$approximate soliton wave solutions of Eq. (2. 2) with $\alpha = -0.075$, $\beta = -0.1$, $\gamma = 0.3$, $\kappa = 1$, $b_0 = -1$, $b_1 = 0.5$, $q_0 = 0.4$ and $q_1 = -1$ at $t = 1$.

$$\begin{aligned}
U_2(x) = &-\frac{e^{x\kappa b_1}\beta^3\kappa^2 b_1^5 q_1 (b_1 q_0 - b_0 q_1)}{\gamma^3(-b_0 q_1 + b_1(q_0 + e^{x\kappa b_1} q_1))^9} \Big(-12 e^{2x\kappa b_1}\alpha^2 b_0^5 q_1^7 + 4e^{x\kappa b_1}\alpha b_0^4 b_1 q_1^6 \\
&\Big(15 e^{x\kappa b_1}\alpha q_0 + \Big(11 e^{2x\kappa b_1}\alpha - 5\gamma\kappa^2 b_0^2\Big) q_1\Big) - b_0^3 b_1^2 q_1^5 \Big(120 e^{2x\kappa b_1}\alpha^2 q_0^2 \\
&+8e^{x\kappa b_1}\alpha \Big(22 e^{2x\kappa b_1}\alpha - 15\gamma\kappa^2 b_0^2\Big) q_0 q_1 + \Big(56 e^{4x\kappa b_1}\alpha^2 \\
&+12 e^{2x\kappa b_1}\alpha\kappa^2(10\gamma+3\sigma)b_0^2 + \gamma^2\kappa^4 b_0^4\Big) q_1^2\Big) + b_0^2 b_1^3 q_1^4 \Big(120 e^{2x\kappa b_1}\alpha^2 q_0^3 \\
&+12 e^{x\kappa b_1}\alpha\Big(22e^{2x\kappa b_1}\alpha - 25\gamma\kappa^2 b_0^2\Big) q_0^2 q_1 + \Big(168 e^{4x\kappa b_1}\alpha^2 \\
&+60 e^{2x\kappa b_1}\alpha\kappa^2(10\gamma+3\sigma)b_0^2 + 7\gamma^2\kappa^4 b_0^4\Big) q_0 q_1^2 \\
&+e^{x\kappa b_1}\Big(24 e^{4x\kappa b_1}\alpha^2 + 4e^{2x\kappa b_1}\alpha\kappa^2(11\gamma+\sigma)b_0^2 - 5\gamma\kappa^4(19\gamma+4\sigma)b_0^4\Big) q_1^3\Big) \\
&-b_0 b_1^4 q_1^3 \Big(60 e^{2x\kappa b_1}\alpha^2 q_0^4 + 16 e^{x\kappa b_1}\alpha\Big(11 e^{2x\kappa b_1}\alpha - 25\gamma\kappa^2 b_0^2\Big) q_0^3 q_1 \\
&+3\Big(56 e^{4x\kappa b_1}\alpha^2 + 40 e^{2x\kappa b_1}\alpha\kappa^2(10\gamma+3\sigma)b_0^2 + 7\gamma^2\kappa^4 b_0^4\Big) q_0^2 q_1^2 \\
&+2e^{x\kappa b_1}\Big(24 e^{4x\kappa b_1}\alpha^2 + 8e^{2x\kappa b_1}\alpha\kappa^2(11\gamma+\sigma)b_0^2 - 15\gamma\kappa^4(19\gamma+4\sigma)b_0^4\Big) q_0 q_1^3 \\
&+e^{2x\kappa b_1}\Big(-4e^{4x\kappa b_1}\alpha^2 - 12 e^{2x\kappa b_1}\alpha\kappa^2(22\gamma+7\sigma)b_0^2 + 3\kappa^4 \big(343\gamma^2 \\
&+120\gamma\sigma + 8\sigma^2\big) b_0^4\Big) q_1^4\Big) - \kappa^2 b_0 b_1^6 q_1^2 \Big(-120 e^{x\kappa b_1}\alpha\gamma q_0^5 + 5\big(12 e^{2x\kappa b_1}\alpha(10\gamma+3\sigma) \\
&+7\gamma^2\kappa^2 b_0^2\big) q_0^4 q_1 + 4e^{x\kappa b_1}\Big(4e^{2x\kappa b_1}\alpha(11\gamma+\sigma) - 25\gamma\kappa^2(19\gamma+4\sigma)b_0^2\Big) q_0^3 q_1^2 \\
&-6e^{2x\kappa b_1}\Big(6e^{2x\kappa b_1}\alpha(22\gamma+7\sigma) - 5\kappa^2\big(343\gamma^2+120\gamma\sigma+8\sigma^2\big) b_0^2\Big) q_0^2 q_1^3 \\
&-4e^{3x\kappa b_1}\Big(6e^{2x\kappa b_1}\alpha(7\gamma+3\sigma) + \kappa^2\big(2971\gamma^2+1284\gamma\sigma+136\sigma^2\big)b_0^2\Big) q_0 q_1^4 \\
&+e^{4x\kappa b_1}\Big(16e^{2x\kappa b_1}\alpha(5\gamma+\sigma)+\kappa^2\big(2971\gamma^2+1284\gamma\sigma+136\sigma^2\big)b_0^2\Big) q_1^5\Big) \\
&+b_1^5 q_1^2 \Big(12 e^{2x\kappa b_1}\alpha^2 q_0^5 + 4e^{x\kappa b_1}\alpha\Big(11e^{2x\kappa b_1}\alpha - 75\gamma\kappa^2 b_0^2\Big) q_0^4 q_1
\end{aligned}$$

$$
\begin{aligned}
&+\left(56e^{4x\kappa b_1}\alpha^2+120e^{2x\kappa b_1}\alpha\kappa^2(10\gamma+3\sigma)b_0^2+35\gamma^2\kappa^4 b_0^4\right)q_0^3q_1^2\\
&+3e^{x\kappa b_1}\left(8e^{4x\kappa b_1}\alpha^2+8e^{2x\kappa b_1}\alpha\kappa^2(11\gamma+\sigma)b_0^2-25\gamma\kappa^4(19\gamma+4\sigma)b_0^4\right)q_0^2q_1^3\\
&-e^{2x\kappa b_1}\left(4e^{4x\kappa b_1}\alpha^2+36e^{2x\kappa b_1}\alpha\kappa^2(22\gamma+7\sigma)b_0^2\right.\\
&\left.-15\kappa^4\left(343\gamma^2+120\gamma\sigma+8\sigma^2\right)b_0^4\right)q_0q_1^4-e^{3x\kappa b_1}\left(4e^{4x\kappa b_1}\alpha^2\right.\\
&\left.+12e^{2x\kappa b_1}\alpha\kappa^2(7\gamma+3\sigma)b_0^2+\kappa^4\left(2971\gamma^2+1284\gamma\sigma+136\sigma^2\right)b_0^4\right)q_1^5\\
&+\kappa^4b_1^9\left(q_0-e^{x\kappa b_1}q_1\right)\left(\gamma^2q_0^6-2e^{x\kappa b_1}\gamma(47\gamma+10\sigma)q_0^5q_1+e^{2x\kappa b_1}\left(935\gamma^2\right.\right.\\
&\left.+340\gamma\sigma+24\sigma^2\right)q_0^4q_1^2-4e^{3x\kappa b_1}\left(509\gamma^2+236\gamma\sigma+28\sigma^2\right)q_0^3q_1^3\\
&+e^{4x\kappa b_1}\left(935\gamma^2+340\gamma\sigma+24\sigma^2\right)q_0^2q_1^4-2e^{5x\kappa b_1}\gamma(47\gamma+10\sigma)q_0q_1^5\\
&\left.+e^{6x\kappa b_1}\gamma^2q_1^6\right)+\kappa^4b_0b_1^8q_1\big(-7\gamma^2q_0^6+30e^{x\kappa b_1}\gamma(19\gamma+4\sigma)q_0^5q_1\\
&-15e^{2x\kappa b_1}\left(343\gamma^2+120\gamma\sigma+8\sigma^2\right)q_0^4q_1^2\\
&+4e^{3x\kappa b_1}\left(2971\gamma^2+1284\gamma\sigma+136\sigma^2\right)q_0^3q_1^3\\
&-3e^{4x\kappa b_1}\left(2971\gamma^2+1284\gamma\sigma+136\sigma^2\right)q_0^2q_1^4\\
&+6e^{5x\kappa b_1}\left(343\gamma^2+120\gamma\sigma+8\sigma^2\right)q_0q_1^5-5e^{6x\kappa b_1}\gamma(19\gamma+4\sigma)q_1^6\big)\\
&+\kappa^2b_1^7q_1\big(-20e^{x\kappa b_1}\alpha\gamma q_0^6+3\left(4e^{2x\kappa b_1}\alpha(10\gamma+3\sigma)+7\gamma^2\kappa^2b_0^2\right)q_0^5q_1\\
&+e^{x\kappa b_1}\left(4e^{2x\kappa b_1}\alpha(11\gamma+\sigma)-75\gamma\kappa^2(19\gamma+4\sigma)b_0^2\right)q_0^4q_1^2\\
&-6e^{2x\kappa b_1}\big(2e^{2x\kappa b_1}\alpha(22\gamma+7\sigma)-5\kappa^2\left(343\gamma^2+120\gamma\sigma+8\sigma^2\right)b_0^2\big)q_0^3q_1^3\\
&-6e^{3x\kappa b_1}\big(2e^{2x\kappa b_1}\alpha(7\gamma+3\sigma)+\kappa^2\left(2971\gamma^2+1284\gamma\sigma+136\sigma^2\right)b_0^2\big)q_0^2q_1^4\\
&+e^{4x\kappa b_1}\big(16e^{2x\kappa b_1}\alpha(5\gamma+\sigma)+3\kappa^2\left(2971\gamma^2+1284\gamma\sigma+136\sigma^2\right)b_0^2\big)q_0q_1^5\\
&\left.-e^{5x\kappa b_1}\left(4e^{2x\kappa b_1}\alpha\gamma+3\kappa^2\left(343\gamma^2+120\gamma\sigma+8\sigma^2\right)b_0^2\right)q_1^6\big)\right)
\end{aligned}
\tag{4. 35}
$$

Continuing in this manner, we can deduce the remaining terms of the recurrence relation (4. 32). Substituting these terms into the inverse DTM, the approximate solution is obtained as

$$U_2\left(x,t\right)=U_0\left(x\right)+U_1\left(x\right)t+U_2\left(x\right)t^2, \tag{4. 36}$$

TABLE 5. Calculating the errors for the N-approximate soliton rational solutions of Eq. (2. 2) under the conditions $\alpha = -0.075$, $\beta = -0.1$, $\gamma = 0.3$, $\kappa = 1$, $b_0 = -1$, $b_1 = 0.5$, $q_0 = 0.4$ and $q_1 = -1$.

x	t	$\lvert u(x,t) - U_2(x,t)\rvert$	$\lvert u(x,t) - U_3(x,t)\rvert$	$\lvert u(x,t) - U_4(x,t)\rvert$
-15.00	0.10	2.000549×10^{-10}	1.999844×10^{-10}	1.999844×10^{-10}
-12.50	0.25	6.118709×10^{-9}	6.115434×10^{-9}	6.115435×10^{-9}
-10.00	0.50	1.485167×10^{-7}	1.484704×10^{-7}	1.484707×10^{-7}
-7.50	0.75	2.573567×10^{-6}	2.573727×10^{-6}	2.573706×10^{-6}
-5.00	1.00	3.353397×10^{-5}	3.351634×10^{-5}	3.351655×10^{-5}
-2.50	1.50	2.893326×10^{-4}	2.893442×10^{-4}	2.893217×10^{-4}
0.00	2.00	4.056308×10^{-4}	4.054906×10^{-4}	4.054886×10^{-4}
2.50	2.50	7.052504×10^{-4}	7.026420×10^{-4}	7.030204×10^{-4}
5.00	3.00	3.772943×10^{-4}	3.792805×10^{-4}	3.789540×10^{-4}
7.50	3.50	6.498536×10^{-5}	6.485422×10^{-5}	6.490473×10^{-5}
10.00	4.00	7.336962×10^{-6}	7.459426×10^{-6}	7.451839×10^{-6}
12.50	4.50	6.780834×10^{-7}	7.399809×10^{-7}	7.388150×10^{-7}
15.00	5.00	4.535004×10^{-8}	6.961987×10^{-8}	6.925285×10^{-8}

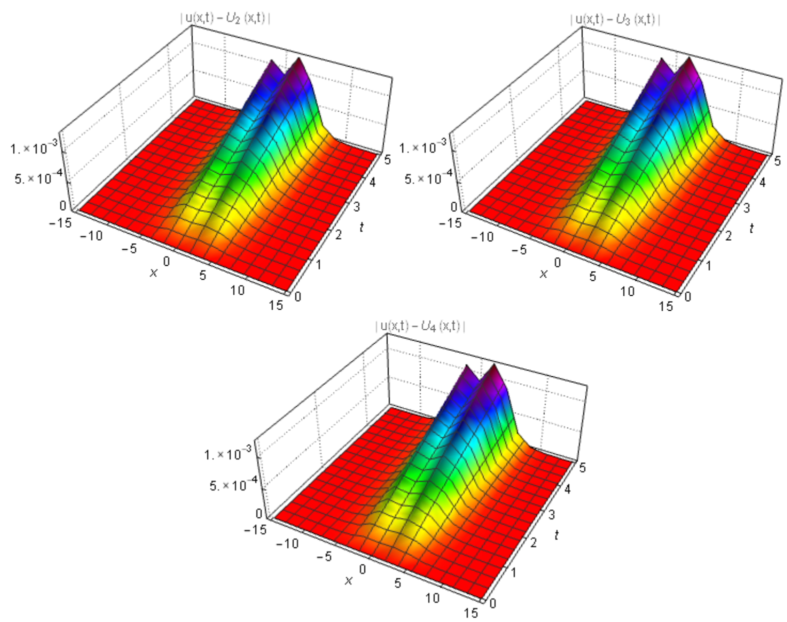

FIGURE 9. The absolute errors for $N-$approximate soliton wave solutions of Eq. (2. 2) with $\alpha = -0.075$, $\beta = -0.1$, $\gamma = 0.3$, $\kappa = 1$, $b_0 = -1$, $b_1 = 0.5$, $q_0 = 0.4$ and $q_1 = -1$.

$$
\begin{aligned}
U_2(x,t) = {} & \frac{e^{x\kappa b_1}\beta b_1 q_1}{\gamma^3\,(-b_0q_1 + b_1(q_0 + e^{x\kappa b_1}q_1))^9}\Big(- 2\gamma^2 \big(b_0q_1 - b_1 \big(q_0 + e^{x\kappa b_1}q_1\big)\big)^8 \\
& + 2t\beta\gamma\kappa b_1^2 (b_1q_0 - b_0q_1)\big(b_0q_1 - b_1\big(q_0 + e^{x\kappa b_1}q_1\big)\big)^4 \\
& \Big(-e^{3x\kappa b_1}b_1^2\,(2\alpha + \gamma\kappa^2 b_1^2)\,q_1^3 + e^{2x\kappa b_1}b_1\,(-4\alpha + \kappa^2(7\gamma + 2\sigma)b_1^2)\,q_1^2 \\
& (b_1q_0 - b_0q_1) - e^{x\kappa b_1}\,(2\alpha + \kappa^2(7\gamma + 2\sigma)b_1^2)\,q_1\,(b_1q_0 - b_0q_1)^2 \\
& + \gamma\kappa^2 b_1 (b_1q_0 - b_0q_1)^3\Big) - t^2\beta^2\kappa^2 b_1^4 (b_1q_0 - b_0q_1) \\
& \Big(- 12e^{2x\kappa b_1}\alpha^2 b_0^5 q_1^7 + 4e^{x\kappa b_1}\alpha b_0^4 b_1 q_1^6 \Big(15e^{x\kappa b_1}\alpha q_0 \\
& + \Big(11e^{2x\kappa b_1}\alpha - 5\gamma\kappa^2 b_0^2\Big) q_1\Big) - b_0^3 b_1^2 q_1^5 \Big(120e^{2x\kappa b_1}\alpha^2 q_0^2 \\
& + 8e^{x\kappa b_1}\alpha \Big(22e^{2x\kappa b_1}\alpha - 15\gamma\kappa^2 b_0^2\Big) q_0 q_1 + \Big(56e^{4x\kappa b_1}\alpha^2
\end{aligned}
$$

$$
\begin{aligned}
&+12e^{2x\kappa b_1}\alpha\kappa^2(10\gamma+3\sigma)b_0^2+\gamma^2\kappa^4 b_0^4\Big)\,q_1^2\big)+b_0^2b_1^3q_1^4\left(120e^{2x\kappa b_1}\alpha^2q_0^3\right.\\
&+12e^{x\kappa b_1}\alpha\left(22e^{2x\kappa b_1}\alpha-25\gamma\kappa^2b_0^2\right)q_0^2q_1+\Big(168e^{4x\kappa b_1}\alpha^2\\
&+60e^{2x\kappa b_1}\alpha\kappa^2(10\gamma+3\sigma)b_0^2+7\gamma^2\kappa^4b_0^4\Big)\,q_0q_1^2+e^{x\kappa b_1}\,\Big(24e^{4x\kappa b_1}\alpha^2\\
&+4e^{2x\kappa b_1}\alpha\kappa^2(11\gamma+\sigma)b_0^2-5\gamma\kappa^4(19\gamma+4\sigma)b_0^4\Big)\,q_1^3\big)-b_0b_1^4q_1^3\Big(60e^{2x\kappa b_1}\alpha^2q_0^4\\
&+16e^{x\kappa b_1}\alpha\left(11e^{2x\kappa b_1}\alpha-25\gamma\kappa^2b_0^2\right)q_0^3q_1+3\big(56e^{4x\kappa b_1}\alpha^2\\
&+40e^{2x\kappa b_1}\alpha\kappa^2(10\gamma+3\sigma)b_0^2+7\gamma^2\kappa^4b_0^4\big)q_0^2q_1^2+2e^{x\kappa b_1}\left(24e^{4x\kappa b_1}\alpha^2\right.\\
&+8e^{2x\kappa b_1}\alpha\kappa^2(11\gamma+\sigma)b_0^2-15\gamma\kappa^4(19\gamma+4\sigma)b_0^4\big)q_0q_1^3+e^{2x\kappa b_1}\big(-4e^{4x\kappa b_1}\alpha^2\\
&-12e^{2x\kappa b_1}\alpha\kappa^2(22\gamma+7\sigma)b_0^2+3\kappa^4\left(343\gamma^2+120\gamma\sigma+8\sigma^2\right)b_0^4\big)q_1^4\Big)\\
&-\kappa^2b_0b_1^6q_1^2\left(-120e^{x\kappa b_1}\alpha\gamma q_0^5+5\left(12e^{2x\kappa b_1}\alpha(10\gamma+3\sigma)+7\gamma^2\kappa^2b_0^2\right)q_0^4q_1\right.\\
&+4e^{x\kappa b_1}\big(4e^{2x\kappa b_1}\alpha(11\gamma+\sigma)-25\gamma\kappa^2(19\gamma+4\sigma)b_0^2\big)q_0^3q_1^2\\
&-6e^{2x\kappa b_1}\left(6e^{2x\kappa b_1}\alpha(22\gamma+7\sigma)-5\kappa^2\left(343\gamma^2+120\gamma\sigma+8\sigma^2\right)b_0^2\right)q_0^2q_1^3\\
&-4e^{3x\kappa b_1}\left(6e^{2x\kappa b_1}\alpha(7\gamma+3\sigma)+\kappa^2\left(2971\gamma^2+1284\gamma\sigma+136\sigma^2\right)b_0^2\right)q_0q_1^4\\
&+e^{4x\kappa b_1}\left(16e^{2x\kappa b_1}\alpha(5\gamma+\sigma)+\kappa^2\left(2971\gamma^2+1284\gamma\sigma+136\sigma^2\right)b_0^2\big)q_1^5\right)\\
&+b_1^5q_1^2\left(12e^{2x\kappa b_1}\alpha^2q_0^5+4e^{x\kappa b_1}\alpha\left(11e^{2x\kappa b_1}\alpha-75\gamma\kappa^2b_0^2\right)q_0^4q_1\right.\\
&+\left(56e^{4x\kappa b_1}\alpha^2+120e^{2x\kappa b_1}\alpha\kappa^2(10\gamma+3\sigma)b_0^2+35\gamma^2\kappa^4b_0^4\right)q_0^3q_1^2\\
&+3e^{x\kappa b_1}\left(8e^{4x\kappa b_1}\alpha^2+8e^{2x\kappa b_1}\alpha\kappa^2(11\gamma+\sigma)b_0^2-25\gamma\kappa^4(19\gamma+4\sigma)b_0^4\right)q_0^2q_1^3 \qquad (4.\ 37)\\
&-e^{2x\kappa b_1}\left(4e^{4x\kappa b_1}\alpha^2+36e^{2x\kappa b_1}\alpha\kappa^2(22\gamma+7\sigma)b_0^2-15\kappa^4\left(343\gamma^2+120\gamma\sigma\right.\right.\\
&\left.+8\sigma^2\right)b_0^4q_0q_1^4-e^{3x\kappa b_1}\left(4e^{4x\kappa b_1}\alpha^2+12e^{2x\kappa b_1}\alpha\kappa^2(7\gamma+3\sigma)b_0^2\right.\\
&+\kappa^4\left(2971\gamma^2+1284\gamma\sigma+136\sigma^2\right)b_0^4\big)\,q_1^5+\kappa^4b_1^9\left(q_0-e^{x\kappa b_1}q_1\right)\left(\gamma^2q_0^6\right.\\
&-2e^{x\kappa b_1}\gamma(47\gamma+10\sigma)q_0^5q_1+e^{2x\kappa b_1}\left(935\gamma^2+340\gamma\sigma+24\sigma^2\right)q_0^4q_1^2\\
&-4e^{3x\kappa b_1}\left(509\gamma^2+236\gamma\sigma+28\sigma^2\right)q_0^3q_1^3+e^{4x\kappa b_1}\left(935\gamma^2+340\gamma\sigma+24\sigma^2\right)q_0^2q_1^4\\
&-2e^{5x\kappa b_1}\gamma(47\gamma+10\sigma)q_0q_1^5+e^{6x\kappa b_1}\gamma^2q_1^6\Big)+\kappa^4b_0b_1^8q_1\Big(-7\gamma^2q_0^6\\
&+30e^{x\kappa b_1}\gamma(19\gamma+4\sigma)q_0^5q_1-15e^{2x\kappa b_1}\left(343\gamma^2+120\gamma\sigma+8\sigma^2\right)q_0^4q_1^2\\
&+4e^{3x\kappa b_1}\left(2971\gamma^2+1284\gamma\sigma+136\sigma^2\right)q_0^3q_1^3-3e^{4x\kappa b_1}\left(2971\gamma^2+1284\gamma\sigma\right.\\
&\left.+136\sigma^2\right)q_0^2q_1^4+6e^{5x\kappa b_1}\left(343\gamma^2+120\gamma\sigma+8\sigma^2\right)q_0q_1^5-5e^{6x\kappa b_1}\gamma(19\gamma+4\sigma)q_1^6\Big)\\
&+\kappa^2b_1^7q_1\Big(-20e^{x\kappa b_1}\alpha\gamma q_0^6+3\left(4e^{2x\kappa b_1}\alpha(10\gamma+3\sigma)+7\gamma^2\kappa^2b_0^2\right)q_0^5q_1\\
&+e^{x\kappa b_1}\left(4e^{2x\kappa b_1}\alpha(11\gamma+\sigma)-75\gamma\kappa^2(19\gamma+4\sigma)b_0^2\right)q_0^4q_1^2\\
&-6e^{2x\kappa b_1}\big(2e^{2x\kappa b_1}\alpha(22\gamma+7\sigma)-5\kappa^2\left(343\gamma^2+120\gamma\sigma+8\sigma^2\right)b_0^2\big)q_0^3q_1^3\\
&-6e^{3x\kappa b_1}\Big(2e^{2x\kappa b_1}\alpha(7\gamma+3\sigma)+\kappa^2\big(2971\gamma^2+1284\gamma\sigma+136\sigma^2\big)b_0^2\Big)q_0^2q_1^4\\
&+e^{4x\kappa b_1}\Big(16e^{2x\kappa b_1}\alpha(5\gamma+\sigma)+3\kappa^2\big(2971\gamma^2+1284\gamma\sigma+136\sigma^2\big)b_0^2\Big)q_0q_1^5\\
&-e^{5x\kappa b_1}\left(4e^{2x\kappa b_1}\alpha\gamma+3\kappa^2\left(343\gamma^2+120\gamma\sigma+8\sigma^2\right)b_0^2\right)q_1^6\Big)\Big)\Big)
\end{aligned}
$$

which matches with the first three terms of the analytical solution (4. 30).

The artificial interval is chosen as $[-15, 15]$ and the calculation is done up time $t = 5$ with $\alpha = -0.075$, $\beta = -0.1$, $\gamma = 0.3$, $\kappa = 1$, $b_0 = -1$, $b_1 = 0.5$, $q_0 = 0.4$ and $q_1 = -1$. The errors of the N-approximate solutions obtained from RDTM, denoted as $u - U_N$, for various values of N are presented in Table 5 at selected time steps, comparing the approximate solution with the exact solution. In Fig. 7, $2-$approximate solutions are showed as 3D and 2D, respectively. The absolute errors for $N-$approximate soliton rational solutions of Eq. (2. 2) are depicted as two dimensional in Fig. 8 at time $t = 1$. Then, the absolute errors for $N-$approximate soliton rational solutions of Eq. (2. 2) are illusrated as three dimensional in Fig. 3 at $0 \leq t \leq 5$. Table 5, Fig. 8, and Fig. 9 illustrate that the approximate solution remains nearly constant with an increase in the number of terms included in the series.

5. Conclusion

In this work, we developed a numerical scheme for the Kudryashov-Sinelshchikov equation using the reduced differential transform method. Such analysis provides a theoretical basis of the numerical scheme and can work practically with a number of the numerical examples, including those of rational soliton solution, solitary wave solution and soliton wave solution by the proposed approach. Moreover, this approach has been shown for the effectiveness in solving those linear and nonlinear problems related to the same types from the calculated absolute error norms and results. The accuracy and reliability of the proposed scheme are guaranteed by the results obtained.

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Conflict of Interests

No conflict of interest exists. We wish to confirm that there are no known conflicts of interest associated with this publication.

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