Punjab University Journal of Mathematics (2024), 56(7), 357- 368
https://doi.org/10.52280/pujm.2024.56(7)03

A research on space-like Bertrand curve pair in 3D lightlike cone

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Received: 05 December, 2023 / Accepted: 06 Febraury, 2025 / Published online:12 March, 2025

Abstract. In this study, the W-Bertrand curves lying on the $\mathbb{Q}^3$ are examined and the notion of $\gamma-$Bertrand curves(and $\alpha-$Bertrand curves, $\beta-$Bertrand curves, $y-$Bertrand curves, respectively). Also, the Bertrand pair $\{\gamma, \Gamma\}$ in terms of their curvature functions are obtained, and the necessary and sufficient conditions for the W-Bertrand curves are expressed using the asymptotic orthonormal frame in $\mathbb{Q}^3$. Furthermore, the helix curve is characterized in terms of curvature according to the condition of being W-Bertrand curve pair.

AMS (MOS) Subject Classification Codes: 53A35; 53A40
Key Words: The Bertrand curve pair, lightlike cone, asymptotic orthonormal frame.

1. Introduction

As it is well-know in differential geometry, space curves, associated curves, curves at one of its corresponding points where one of the Frenet vectors of one curve overlaps one of the other curve's Frenet vectors are known as Bertrand curves. That is to say that the curves will have the common principal normal, the Bertrand curve is a curve that shares the normal line with another curve. The curves play an important role in the theory of curves. On this occasion, to this day, a lot of mathematicians have worked on Bertrand curves, [3, 7, 8, 11, 15, 17, 19, 20]. In [16, 18], the authors studied the cylindrical spirals and Bertrand curves as curves on ruled surfaces. In [1, 10], the authors studied on different non-null curves in the null cone. In [2], the notion of the involute-evolute curves for the curves lying the surfaces in Minkowski 3-space E_1^3 was examined by the authors. In [6], Gluck investigated the Bertrand curves in E^n. In [5, 9, 14], the authors gave mathematical characterizations on some special curves, Smarandache curves, null slant helix, Legendre curves. In [12, 13], the cone curves and cone curvature function etc were studied by the authors. Also, they gave their representations and some examples of cone curves in Minkowski space. In [4, 21] many researchers examined Bertrand curves and their topological and geometrical properties and characterizations in Minkowski spacetime.

In this paper, by using asymptotic orthonormal frame $\{\gamma, \alpha, \beta, y\}$ of the curve γ expressing the notation of a W-Bertrand curve which the spacelike vector field W is given by

$$W(s) = c_1(s)\gamma(s) + c_2(s)\alpha(s) + c_3(s)\beta(s) + c_4(s)y(s), \tag{1.1}$$

where $c_1(s), c_2(s), c_3(s), c_4(s)$ are differentiable functions that satisfy the equality $c_2(s)^2 + 2c_1(s)c_4(s) + c_3(s)^2 = 1$ and one finds the necessary and sufficient conditions of these curves in lightlike cone 3-space to be W-Bertrand curves. Furthermore, giving some characterizations of $\gamma-$Bertrand curves, $\alpha-$Bertrand curves, $\beta-$Bertrand curves, $y-$Bertrand curves, respectively.

2. Notations and Preliminaries

The k-dimensional pseudo-Euclidean space E_q^k is given with the metric

$$d(A, B) = \sum_{i=1}^{k-q} a_i b_i - \sum_{j=m-q+1}^{m} a_j b_j,$$

where $A = (a_1, a_2, ..., a_k), B = (b_1, b_{2,}..., b_k) \in E_q^k$, E_q^k is a flat pseudo-Riemannian manifold of signature $(k-q, q)$, [13, 15].

Let M be a submanifold of E_q^k. If a pseudo-Riemannian metric m on M is induced the pseudo-Riemannian metric m of E_q^k then M is said to be timelike(respectively, spacelike, degenerate) submanifold of E_q^k.

The pseudo-Riemannian null cone is given by

$$\mathbb{Q}_q^n(\gamma_0, d) = \{\gamma \in E_q^{n+1} : m(\gamma - \gamma_0, \gamma - \gamma_0) = 0\},$$

where a fixed point in E_q^k is γ_0 and $d > 0$ is a constant, [13, 15]. $\mathbb{Q}_q^n(\gamma_0)$ is a degenerate hyper-surface in E_q^{n+1}. $\mathbb{Q}_q^n(\gamma_0)$ is said to be as pseudo-Riemannian space form. The point γ_0 is called the center of $\mathbb{Q}_q^n(\gamma_0, d)$. If $\gamma_0 = 0$ and $q = 1$, one expresses $\mathbb{Q}_1^n(0)$ by $\mathbb{Q}^n$ and it is said to be the lightlike cone, [13, 15].

Let E_1^{n+2} be the $(n+2)$-Minkowski space and $\mathbb{Q}^{n+1}$ be the null cone in E_1^{n+2}. A vector $w \neq 0$ in E_1^{n+2} is called spacelike, timelike or null, if $\langle w, w\rangle > 0$, $\langle w, w\rangle < 0$ or $\langle w, w\rangle = 0$, respectively. A frame field $\{e_1, e_2, ..., e_{n+1}, e_{n+2}\}$ on E_1^{n+2} is called as asymptotic orthonormal frame field, if

$$\begin{aligned} \langle e_{n+1}, e_{n+1}\rangle &= \langle e_{n+2}, e_{n+2}\rangle = 0, \langle e_{n+1}, e_{n+2}\rangle = 1, \\ \langle e_{n+1}, e_i\rangle &= \langle e_{n+2}, e_i\rangle = 0,\ \langle e_i, e_j\rangle = \delta_{ij},\ i, j = 1, 2, ..., n \end{aligned}$$

[15]. Let the curve $\gamma : I \longrightarrow \mathbb{Q}^{n+1} \subset E_1^{n+1}$, $v \longrightarrow \gamma(v) \in \mathbb{Q}^{n+1}$ be a regular curve in $\mathbb{Q}^{n+1}$ and $\gamma'(v) = \frac{d\gamma(v)}{dt}$, for $\forall v \in I \subset \mathbb{R}$, [13, 15].

Definition 2.1. *A curve $\gamma(v)$ in E_1^{n+2} is said to be a Frenet curve, for $\forall v \in I$, the vector fields $\gamma(v), \gamma'(v), \gamma''(v), ..., \gamma^{(n)}(v), \gamma^{(n+1)}(v)$ are linearly independent and the vector fields $\gamma(v), \gamma'(v), \gamma''(v), ..., \gamma^{(n)}(v), \gamma^{(n+1)}(v), \gamma^{(n+2)}(v)$ are linearly dependent, and the vector fields. Since $\langle \gamma, \gamma\rangle = 0$ and $\langle \gamma, d\gamma\rangle = 0$, $d\gamma(v)$ is spacelike. Then, according to arc length s of the curve $\gamma(v)$ can be written by*

$$ds^2 = \langle d\gamma(v), d\gamma(v)\rangle, [15].$$

If the arc length s of the curve $\gamma(t)=\gamma(v(s))$ is used, then $\gamma'(s)=\frac{d\gamma}{ds}$ is a spacelike unit tangent vector field of the curve $\gamma(s)$. Hence, we select the vector $y(s)$, the spacelike normal space of the curve $\gamma(s)$ is denoted as V^{n-1}. Then,

$$\begin{aligned}\langle\gamma(s),y(s)\rangle &= 1,\langle\gamma(s),\gamma(s)\rangle=\langle y(s),y(s)\rangle=\langle\gamma'(s),y(s)\rangle=0,\\ V^{n-1} &= \{span\{\gamma,y,\gamma'\}\}^{\perp}, span_R\{\gamma,y,\gamma',V^{n-1}\}=E_1^{n+2}\end{aligned}$$

are satisfied, [13, 15].

Remark 2.2. *For any asymptotic orthonormal frame $\{\gamma,\alpha,\beta,y\}$ of the curve $\gamma: I\longrightarrow \mathbb{Q}^3\subset E_1^4$ with*

$$\langle\gamma,\gamma\rangle=\langle y,y\rangle=\langle\gamma,\alpha\rangle=\langle\gamma,\beta\rangle=\langle y,\alpha\rangle=\langle y,\beta\rangle=\langle\alpha,\beta\rangle=0; \tag{2.1a}$$

$$\langle\gamma,y\rangle=\langle\alpha,\alpha\rangle=\langle\beta,\beta\rangle=1 \tag{2.1b}$$

$$\beta\in V^1=\{span\{\gamma,y,\alpha\}\}^{\perp}, span_R\{\gamma,y,\alpha,V^1\}=E_1^3$$

and hence, the Frenet equations are given by

$$\begin{aligned}\gamma'(s)&=\alpha(s)\\ \alpha'(s)&=\kappa(s)\gamma(s)-y(s)\\ \beta'(s)&=\tau(s)\gamma(s)\\ y'(s)&=-\kappa(s)\alpha(s)-\tau(s)\beta(s),\end{aligned} \tag{2.2}$$

where κ,τ are cone curvatures of the curve $\gamma(s)$ in $\mathbb{Q}^3\subset E_1^4$, [13].

3. The spacelike $W-$Bertrand pair curves in the lightlike Cone $\mathbb{Q}^3$

Let $\gamma(s)$ and $\Gamma(s^*)$ lying fully on $\mathbb{Q}^3$ be the arc length parameter curves. The asymptotic orthonormal frames of $\gamma(s)$ and $\Gamma(s^*)$ are given by $\{\gamma,\alpha,\beta,y\}$ and $\{\Gamma,\alpha^*,\beta^*,y^*\}$ with curvatures κ,τ and κ^*,τ^*, respectively. If there exists a corresponding relationship between the curves γ and Γ such that, at the corresponding points of the curves, the asymptotic orthonormal frame element β coincides with the asymptotic orthonormal frame element β^* of Γ, then γ is said to be as Bertrand curve, and Γ is a Bertrand partner curve of γ. Then, the pair $\{\gamma,\Gamma\}$ is said to be a Bertrand pair.

In this section, for the a unit speed curve $\gamma(s)=\int_0^s W(t)dt$ which is called the integral curve of $W(s)$, by thinking the asymptotic orthonormal frame $\{\gamma,\alpha,\beta,y\}$ and using the equation given by (1.1), defining the spacelike W-Bertrand curves and some characterizations of these curves in $\mathbb{Q}^3$ are given according to the vector fields γ,α,β,y respectively.

Theorem 3.1. *Let the spacelike vector field be $W(s)=c_1(s)\gamma(s)+c_2(s)\alpha(s)+c_3(s)\beta(s)+c_4(s)y(s)$; let $\gamma(s)=\int_0^s W(t)dt$ be the integral curve of $W(s)$ and $\Gamma: I\longrightarrow\mathbb{Q}^3\subset E_1^4$ be spacelike Bertrand curves with arc length parameter s and cone curvatures κ,τ and κ^*,τ^*, respectively. If $\{\gamma,\Gamma\}$ is a Bertrand pair and the curve γ is a W-Bertrand curve the following conditions are satisfied*

1) For $c_3(s)=0$ the distance function d is constant.

2) For $c_3(s)\neq 0$ the distance function is given as

$$d=|\lambda(s)|=\int c_3(s)\,ds.$$

3) The cone curvatures of the curve Γ *are given as*

$$\kappa^*(s^*) = \frac{2(\kappa\delta_2+\delta_4')(\delta_3'-\delta_2)+(\delta_4+\delta_2'-\kappa\delta_3)^2+\tau^2\delta_3^2}{-2c_4(s)(c_1(s)+\lambda(s)\tau(s))+c_2^2(s)}$$

$$\tau^*(s^*) = \frac{\sqrt{\begin{array}{c}2(\sigma_1'+\kappa\sigma_2+\tau\sigma_3)(\sigma_4'-\sigma_2)\\ +(\sigma_1+\sigma_2'-\kappa\sigma_4)^2+(\sigma_3'-\sigma_4\tau)^2-4\phi^2\kappa^{*2}\end{array}}}{\sqrt{2c_4(s)(c_1(s)+\lambda(s)\tau(s))+c_2^2(s)}}$$

where $c_1(s), c_2(s), c_3(s), c_4(s)$ *are differential functions and*

$$\phi = \sqrt{2c_4(s)(c_1(s)+\lambda(s)\tau(s))+c_2^2(s)}; c_i \in C^\infty;$$

$$\delta_1 = \frac{\sqrt{2c_4(c_1+\lambda\tau)}}{\phi}; \delta_2 = \frac{c_2}{\phi}; \delta_3 = \frac{c_4}{\phi}; \delta_4 = \frac{\delta_1}{2\delta_2};$$

$$\sigma_1 = \frac{\kappa\delta_2+\delta_4'}{\phi}; \sigma_2 = \frac{\delta_4+\delta_2'-\kappa\delta_3}{\phi}; \sigma_3 = \frac{-\tau\delta_3}{\phi}; \sigma_4 = \frac{\delta_3'-\delta_2}{\phi}.$$

Proof. Let the spacelike curves $\gamma, \Gamma : I \longrightarrow \mathbb{Q}^3 \subset E_1^4$ be Bertrand pair. Then, they can associate pseudo orthonormal frames $\{\gamma, \alpha, \beta, y\}$ and $\{\Gamma, \alpha^*, \beta^*, y^*\}$, respectively. From definition, for a spacelike Bertrand pair $\{\gamma, \Gamma\}$ in $\mathbb{Q}^3$, the position vector can be written as

$$\Gamma(s^*(s)) = \gamma(s) + \lambda(s)\beta(s), \tag{3.1a}$$

or

$$\Gamma(s^*(s)) = \int_0^s (c_1\gamma + c_2\alpha + c_3\beta + c_4 y)\,dt + \lambda(s)\beta(s), \tag{3.1b}$$

where $\gamma(s) = \int_0^s W(t)dt$ and $\lambda; c_i \in C^\infty$. Also, from (3.1a) the distance between the curves γ and Γ is obtained as

$$d(\gamma, \Gamma) = \|\lambda(s)\beta(s)\| = |\lambda(s)|.$$

By taking derivative of (3.1) with respect to s and applying frenet formulae (2.2), one obtains

$$\alpha^*\frac{ds^*}{ds} = (c_1 + \lambda(s)\tau(s))\gamma(s) + c_2\alpha(s) + (c_3 + \lambda'(s))\beta(s) + c_4 y(s). \tag{3.2}$$

Bertrand curves are a pair of curves that have a common principal normal vector at any point, since $\{\gamma, \Gamma\}$ is a Bertrand pair and by taking the inner product β to the both side of (3.2) and by using (2.2), one gets

$$c_3(s) + \lambda'(s) = 0 \Rightarrow \lambda(s) = -\int c_3(s)\,ds, \tag{3.3}$$

then if $c_3(s) = 0$ one has λ =constant. Here, by substituting the equation (3.3) in the equation (3.2), one gets

$$\alpha^*(s^*)\frac{ds^*}{ds} = (c_1 + \lambda(s)\tau(s))\gamma(s) + c_2\alpha(s) + c_4 y(s). \tag{3.4}$$

Also, if one takes the inner product of previous equation with itself, one obtains

$$\phi = \frac{ds^*}{ds} = \sqrt{2c_4(s)(c_1(s)+\lambda(s)\tau(s))+c_2^2(s)}. \tag{3.5}$$

Then, one assumes that

$$\delta_1 = \frac{\sqrt{2c_4\left(c_1+\lambda(s)\tau(s)\right)}}{\phi};\delta_2 = \frac{c_2}{\phi};\delta_3 = \frac{c_4}{\phi};\delta_4 = \frac{\delta_1}{2\delta_2} \tag{3.6}$$

and one can write

$$\alpha^*(s^*) = \delta_4\gamma(s) + \delta_2\alpha(s) + \delta_3 y(s). \tag{3.7}$$

By differentiating the equation (3.7) with respect to s and from (2.2) one gets

$$\left(\kappa^*\Gamma - y^*\right)\frac{ds^*}{ds} = \left(\kappa\delta_2+\delta_4'\right)\gamma + \left(\delta_4+\delta_2'-\kappa\delta_3\right)\alpha + \left(-\tau\delta_3\right)\beta + \left(\delta_3'-\delta_2\right)y. \tag{3.8}$$

If one takes the inner product of (3.8) with itself, one can write the following equation

$$-2\kappa^*\left(\frac{ds^*}{ds}\right)^2 = 2\left(\kappa\delta_2+\delta_4'\right)\left(\delta_3'-\delta_2\right) + \left(\delta_4+\delta_2'-\kappa\delta_3\right)^2 + \tau^2\delta_3^2 \tag{3.9}$$

and by substituting (3.5) in the equation (3.9) one writes the curvature as

$$\kappa^* = \frac{2\left(\kappa\delta_2+\delta_4'\right)\left(\delta_3'-\delta_2\right) + \left(\delta_4+\delta_2'-\kappa\delta_3\right)^2 + \tau^2\delta_3^2}{-2\left(2c_4\left(c_1+\lambda\tau\right)+c_2^2\right)}. \tag{3.10}$$

From (3.8) if one denotes

$$\sigma_1 = \frac{\kappa\delta_2+\delta_4'}{\phi};\sigma_2 = \frac{\delta_4+\delta_2'-\kappa\delta_3}{\phi};\sigma_3 = \frac{-\tau\delta_3}{\phi};\sigma_4 = \frac{\delta_3'-\delta_2}{\phi}, \tag{3.11}$$

one has

$$\kappa^*\Gamma - y^* = \sigma_1\gamma + \sigma_2\alpha + \sigma_3\beta + \sigma_4 y. \tag{3.12}$$

Also, by differentiating (3.12) with respect to s and using (2.2) one gets

$$\begin{aligned}\left(2\kappa^*\alpha^* + \kappa^{*\prime}\Gamma + \tau^*\beta^*\right)\frac{ds^*}{ds} \quad &= \quad \left(\sigma_1'+\kappa\sigma_2+\tau\sigma_3\right)\gamma + \left(\sigma_1+\sigma_2'-\kappa\sigma_4\right)\alpha \\ &+ \left(\sigma_3'-\sigma_4\tau\right)\beta + \left(-\sigma_2+\sigma_4'\right)y. \end{aligned} \tag{3.13}$$

If one denotes

$$\xi_1 = \frac{\sigma_1'+\kappa\sigma_2+\tau\sigma_3}{\phi};\xi_2 = \frac{\sigma_1+\sigma_2'-\kappa\sigma_4}{\phi}; \tag{3.14a}$$

$$\xi_3 = \frac{\sigma_3'-\sigma_4\tau}{\phi};\xi_4 = \frac{-\sigma_2+\sigma_4'}{\phi}, \tag{3.14b}$$

one can write

$$2\kappa^*\alpha^* + \kappa^{*\prime}\Gamma + \tau^*\beta^* = \xi_1\gamma + \xi_2\alpha + \xi_3\beta + \xi_4 y. \tag{3.15}$$

Furthermore, if one takes the inner product of (3.15) with itself, one has

$$4\kappa^{*2} + \tau^{*2} = 2\xi_1\xi_4 + \xi_2^2 + \xi_3^2,$$

then one gets

$$\tau^* = \sqrt{2\xi_1\xi_4 + \xi_2^2 + \xi_3^2 - 4\kappa^{*2}}. \tag{3.16}$$

□

Theorem 3.2. *Let $\{\gamma, \Gamma\}$ be a Bertrand pair and let $\gamma, \Gamma : I \longrightarrow \mathbb{Q}^3 \subset E_1^4$ be spacelike Bertrand curves with arc length parameter s and non-zero curvatures κ, τ and κ^*, τ^* respectively. Then, if the curve γ is a W-Bertrand curve the following conditions holds:*

1) If the curve γ is a $\gamma-$Bertrand curve, the following equations are satisfied

$$\lambda(s) = \textit{constant}; \tau(s) = -\frac{c_1(s)}{\lambda}.$$

2) If the curve γ is an $\alpha-$Bertrand curve, λ =constant and the curvatures of the curve Γ are given as

$$\begin{aligned} \kappa^*(s^*) &= \left(\lambda \frac{\tau(s)}{c_2^2}\right)^2 - \frac{2}{c_2^2(s)}\left(\kappa + \lambda \frac{d}{ds}\left(\frac{\tau(s)}{c_2(s)}\right)\right); \\ \tau^*(s^*) &= \pm\sqrt{\frac{1}{c_2^2}\begin{pmatrix} (A_2' + \kappa A_1)(-A_1 + A_3') \\ +(A_2 - \kappa A_3 + A_1')^2 + \tau^2 A_3^2 \end{pmatrix} - 4\kappa^{*2}}, \end{aligned}$$

where $A_2 = \left(\kappa + \lambda \frac{d}{ds}\left(\frac{\tau(s)}{c_2(s)}\right)\right)\frac{1}{c_2}$; $A_1 = \frac{\lambda\tau(s)}{c_2^2}$; $A_3 = \frac{-1}{c_2}$; $s^* = \int c_2(s)ds$.

3) If the curve γ is a $\beta-$Bertrand curve, the following equations are satisfied

$$\lambda(s) = -\int c_3(s)ds; \tau(s) = 0; c_3(s) \in C^\infty.$$

4) If the curve γ is a $y-$Bertrand curve, the following equations are satisfied

$$\lambda(s) = \textit{constant}; \tau(s) = 0,$$

where $c_1(s), c_2(s), c_3(s), c_4(s)$ are differential functions.

Proof. **Case 1**: Assume that $c_1 \neq 0$ and $c_2, c_3, c_4 = 0$. Then, one can write a $\gamma-$Bertrand curve as

$$\Gamma(s^*(s)) = \int_0^s c_1(t)\,\gamma(t)\,dt + \lambda(s)\beta(s), \tag{3.17}$$

where $\lambda; c_1 \in C^\infty$. By taking derivative of (3.17) with respect to s and applying (2.2), one obtains

$$\alpha^* \frac{ds^*}{ds} = (c_1 + \lambda(s)\tau(s))\gamma(s) + \lambda'(s)\beta(s), \tag{3.18}$$

since $\{\gamma, \Gamma\}$ is a Bertrand pair and by taking the inner product β to the both side of (3.18) and by using (2.2), one gets

$$\lambda'(s) = 0, \tag{3.19}$$

which λ =constant. Here, by substituting the equation (3.19) in the equation (3.18), one gets

$$\alpha^*(s^*)\frac{ds^*}{ds} = (c_1(s) + \lambda(s)\tau(s))\gamma(s). \tag{3.20}$$

Also, if one takes the inner product of previous equation with itself, since $\langle\gamma(s), \gamma(s)\rangle = 0$, one gets $\frac{ds^*}{ds} = 0$ and one can write

$$c_1(s) + \lambda\tau(s) = 0 \Rightarrow \tau(s) = -\frac{c_1(s)}{\lambda}. \tag{3.21}$$

Case 2: Assume that $c_2 \neq 0$ and $c_1, c_3, c_4 = 0$. Then, one can write an $\alpha-$Bertrand curve as

$$\Gamma(s^*(s)) = \int_0^s c_2(t)\,\alpha(t)\,dt + \lambda(s)\beta(s), \tag{3.22}$$

where $\lambda; c_2 \in C^\infty$. By taking derivative of (3.22) with respect to s and applying Frenet formulae (2.2), one obtains

$$\alpha^* \frac{ds^*}{ds} = c_2(s)\,\alpha(s) + \lambda'(s)\beta(s) + \lambda(s)\tau(s)\gamma(s), \tag{3.23}$$

since $\{\gamma, \Gamma\}$ is a Bertrand pair and by taking the inner product β to the both side of (3.23) and by applying (2.2), one gets $\lambda'(s) = 0$ which $\lambda =$constant. Here, by substituting λ in the equation (3.23), one gets

$$\alpha^*(s^*) \frac{ds^*}{ds} = c_2\alpha(s) + \lambda(s)\tau(s)\gamma(s). \tag{3.24}$$

Also, if one takes the inner product of previous equation with itself, since $\langle \alpha, \alpha \rangle = 1, \langle \gamma, \gamma \rangle = 0$, one gets

$$\frac{ds^*}{ds} = c_2(s) \Rightarrow s^* = \int c_2(s)ds$$

and then one can write

$$\alpha^*(s^*) = \alpha(s) + \frac{\lambda\tau(s)}{c_2(s)}\gamma(s). \tag{3.25}$$

By differentiating the equation (3.25) with respect to s and from (2.2) one has

$$\kappa^*\Gamma - y^* = A_1\alpha + A_2\gamma + A_3 y, \tag{3.26}$$

where $A_2 = \left(\kappa + \lambda\frac{d}{ds}\left(\frac{\tau(s)}{c_2(s)}\right)\right)\frac{1}{c_2}; A_1 = \frac{\lambda\tau(s)}{c_2^2}; A_3 = \frac{-1}{c_2}$. If one takes the inner product of (3.26) with itself, one gets

$$\kappa^*(s^*) = \left(\lambda\frac{\tau(s)}{c_2^2}\right)^2 - \frac{2}{c_2^2(s)}\left(\kappa + \lambda\frac{d}{ds}\left(\frac{\tau(s)}{c_2(s)}\right)\right). \tag{3.27}$$

By differentiating the equation (3.26) with respect to s and from (2.2) one gets

$$\kappa^{*\prime}\Gamma + 2\kappa^*\alpha^* + \tau^*\beta^* = C_1\gamma + C_2\alpha + C_3\beta + C_4 y, \tag{3.28}$$

where $C_1 = \frac{A_2' + \kappa A_1}{c_2}, C_2 = \frac{A_2 - \kappa A_3 + A_1'}{c_2}, C_3 = \frac{-\tau A_3}{c_2}, C_4 = \frac{-A_1 + A_3'}{c_2}$ and one takes the inner product of (3.28) with itself, one gets

$$4\kappa^{*2} + \tau^{*2} = \frac{1}{c_2^2}\left(\begin{array}{c} (A_2' + \kappa A_1)(-A_1 + A_3') \\ + (A_2 - \kappa A_3 + A_1')^2 + \tau^2 A_3^2 \end{array}\right)$$

or one has

$$\tau^*(s^*) = \pm\sqrt{\frac{1}{c_2^2}\left(\begin{array}{c} (A_2' + \kappa A_1)(-A_1 + A_3') \\ + (A_2 - \kappa A_3 + A_1')^2 + \tau^2 A_3^2 \end{array}\right) - 4\kappa^{*2}}. \tag{3.29}$$

Case 3: Assume that $c_3 \neq 0$ and $c_1, c_2, c_4 = 0$. Then, one can write a $\beta-$Bertrand curve as

$$\Gamma(s^*(s)) = \int_0^s c_3(t)\,\beta(t)\,dt + \lambda(s)\beta(s), \tag{3.30}$$

where $\lambda; c_3 \in C^{\infty}$. By taking derivative of (3.30) with respect to s and by using (2.2), one obtains

$$\alpha^* \frac{ds^*}{ds} = (c_3 + \lambda')\beta + \lambda\tau\gamma, \tag{3.31}$$

since $\{\gamma, \Gamma\}$ is a Bertrand pair and by taking the inner product β to the both side of (3.31) and from (2.2), one gets $c_3(s) + \lambda' = 0$ which $\lambda(s) = -\int c_3(s)ds$. Here, by substituting $\lambda(s)$ in the equation (3.31), one gets

$$\alpha^*(s^*) \frac{ds^*}{ds} = \lambda(s)\,\tau\gamma. \tag{3.32}$$

Also, by taking the inner product the equation (3.32) with itself, since $\langle \gamma, \gamma \rangle = 0$, one gets $\frac{ds^*}{ds} = 0$, and since $\lambda \neq 0$, one obtains $\tau(s) = 0$.

Case 4: Assume that $c_4 \neq 0$ and $c_1, c_2, c_3 = 0$. Then, one can write a $y-$Bertrand curve as

$$\Gamma(s^*(s)) = \int_0^s c_4(t)\, y(t)\, dt + \lambda(s)\beta(s), \tag{3.33}$$

where $\lambda, c_4 \in C^{\infty}$. By taking derivative of (3.33) with respect to s and applying (2.2), one obtains

$$\alpha^* \frac{ds^*}{ds} = c_4 y + \lambda'\beta + \lambda\tau\gamma, \tag{3.34}$$

since $\{\gamma, \Gamma\}$ is a Bertrand pair and by taking the inner product β to the both side of (3.34) and from (2.2), one gets $\lambda' = 0$ which $\lambda =$constant. Here, substituting λ in the equation (3.34), one gets

$$\alpha^*(s^*) \frac{ds^*}{ds} = c_4 y + \lambda\tau\gamma. \tag{3.35}$$

Also, if we take the inner product (3.15) with itself, since $\langle y, \gamma \rangle = 1$, one gets

$$\left(\frac{ds^*}{ds}\right)^2 = 2\lambda c_4(s)\,\tau(s) \Rightarrow s^* = \int \sqrt{2\lambda c_4(s)\,\tau(s)}ds, \tag{3.36}$$

by differentiating (3.35) with respect to s and from (2.2), one has

$$(\kappa^*\Gamma - y^*)\left(\frac{ds^*}{ds}\right)^2 + \alpha^* \frac{d^2 s^*}{ds^2} = \lambda\tau'\gamma + (-\tau c_4)\,\beta + (\lambda\tau - \kappa c_4)\,\alpha + c_4' y, \tag{3.37}$$

by taking the inner product of (3.37) with β, since $c_4(s) \neq 0$ one can write $\tau(s) = 0$. $\square$

Theorem 3.3. *Let $\{\gamma(s), \Gamma(s^*)\}$ be a unit speed spacelike Bertrand pair in $\mathbb{Q}^3$ and let θ be angle between γ and Γ for the cone curvatures $\kappa, \tau \neq 0$ and $\kappa^*, \tau^* \neq 0$, respectively. If Γ is a helix curve, then the following statements hold:*

i) *For the $\gamma-$Bertrand curve the following equations are satisfied*

$$\begin{aligned} \theta(s) &= 2\arctan(a_1 e^{a_2 s}) \text{ or } \theta(s) = -\int \kappa(s)\, ds; \\ \kappa(s) &= a_3 \frac{\cos\theta(s)}{\theta(s)}\left(\frac{d\theta(s)}{ds} + 1\right). \end{aligned}$$

ii) *For the $\alpha-$Bertrand curve the following equations are satisfied*

$$\begin{aligned}\theta(s) &= 2\arctan(a_1 e^{a_2 s});\ \kappa(s) = \frac{a_3}{\theta(s)}\left(\frac{d\theta(s)}{ds}+1\right)\cos\theta - \frac{c_2(s)}{b\theta(s)};\\ \tau(s) &= \frac{\widetilde{b}}{\lambda}\left(\frac{d\theta(s)}{ds}+\kappa(s)\right)\sin\theta,\end{aligned}$$

where $a_i, \widetilde{b}, b \in \mathbb{R}_0$ and $c_2(s)$ is differential function.

iii) *For the $\beta-$Bertrand curve there is no helix curve Γ.*

iv) *For the $y-$Bertrand curve there is no helix curve Γ.*

Proof. Suppose that the pair $\{\gamma, \Gamma\}$ is a spacelike $W-$Bertrand pair. Then, one says that $\Gamma \in Sp\{\gamma, \alpha, y\}$. Also, the position vector of Γ satisfies

$$\Gamma(s^*) = m\overrightarrow{\gamma(s)} + n\overrightarrow{\alpha(s)} + p\overrightarrow{y(s)};\ m, n, p \in \mathbb{R}_0^+.$$

Suppose that let Γ be a helix curve with s^* parameter. Hence, one can write the curve Γ as

$$\Gamma(s^*) = \widetilde{b}\cos\theta\overrightarrow{\gamma(s)} + \widetilde{b}\sin\theta\overrightarrow{\alpha(s)} + b\theta\overrightarrow{y(s)}, \tag{3.38}$$

where $\theta(s)$ is the angle between the vectors Γ and γ. By taking derivative of previous equation with respect to s and applying (2.2), one gets

$$\begin{aligned}\alpha^*\frac{ds^*}{ds} = &\left(\widetilde{b}\frac{d(\cos\theta)}{ds} + \widetilde{b}\kappa\sin\theta\right)\overrightarrow{\gamma} + \left(\widetilde{b}\frac{d(\sin\theta)}{ds} + \widetilde{b}\cos\theta - b\kappa\theta\right)\overrightarrow{\alpha}\\ &-(b\tau\theta)\overrightarrow{\beta} + \left(b\frac{d\theta}{ds} - \widetilde{b}\sin\theta\right)\overrightarrow{y},\end{aligned} \tag{3.39}$$

from (3.2) and (3.39), one writes the following equations

$$\widetilde{b}\frac{d(\cos\theta)}{ds} + \widetilde{b}\kappa\sin\theta = c_1(s) + \lambda(s)\tau(s); \tag{3.40a}$$

$$\widetilde{b}\frac{d(\sin\theta)}{ds} + \widetilde{b}\cos\theta - \kappa b\theta = c_2(s); \tag{3.40b}$$

$$-b\tau\theta(s) = c_3(s) + \lambda'(s); \tag{3.40c}$$

$$b\frac{d\theta}{ds} - \widetilde{b}\sin\theta = c_4(s). \tag{3.40d}$$

By using (3.40), one obtains the following cases;

1) If the curve γ is a $\gamma-$Bertrand curve, one can write $c_1(s) \neq 0, c_2, c_3, c_4 = 0$ and from theorem 2, by using the equations $\lambda(s)$ =constant, $\tau(s) = -\frac{c_1(s)}{\lambda}$. Then, from (3.40d) one gets

$$b\frac{d\theta}{ds} = \widetilde{b}\sin\theta \Rightarrow \theta(s) = 2\arctan(a_1 e^{a_2 s});\ a_i \in \mathbb{R}. \tag{3.41}$$

From (3.40b), since $c_2 = 0$ one has

$$\kappa(s) = \frac{a_3}{\theta(s)}\cos\theta\left(\frac{d\theta(s)}{ds}+1\right). \tag{3.42}$$

From (3.40c), since $\tau, \theta(s) \neq 0$ one get $b = 0$. Then, the helix curve Γ is written as

$$\Gamma = \widetilde{b}\cos\theta\overrightarrow{\gamma} + \widetilde{b}\sin\theta\overrightarrow{\alpha}. \tag{3.43}$$

Also, from (3.40a) one obtain the following equation

$$\theta\left(s\right)=-\int\kappa\left(s\right)ds. \tag{3.44}$$

2) If the curve γ is an $\alpha-$Bertrand curve, one can write $c_2\left(s\right)\neq 0, c_1, c_3, c_4=0$, $\lambda(s)$ =constant. Then, from (3.40d) one gets the equation (3.41). From (3.40b), since $c_2\neq 0$ one has

$$\kappa\left(s\right)=\frac{a_3}{\theta\left(s\right)}\left(\frac{d\theta\left(s\right)}{ds}+1\right)\cos\theta-\frac{c_2(s)}{b\theta(s)}. \tag{3.45}$$

From (3.40c), since $\tau,\theta(s)\neq 0$ one get $b=0$. Then, the helix curve Γ is obtained as (3.43). Also, from (3.40a) one obtains the following equation

$$\tau(s)=\frac{\widetilde{b}}{\lambda}\left(\frac{d\theta\left(s\right)}{ds}+\kappa\left(s\right)\right)\sin\theta. \tag{3.46}$$

3) If the curve γ is a $\beta-$Bertrand curve, one can write $c_3\left(s\right)\neq 0, c_1, c_2, c_4, \tau(s)=0, \lambda(s)=-\int c_3(s)ds$. Then, for the helix curve Γ one has $\tau(s)\neq 0$. Therefore, there is no helix curve Γ with a $\beta-$Bertrand curve for the pair $\{\gamma,\Gamma\}$.

4) If the curve γ is an $y-$Bertrand curve, one can write $c_4\left(s\right)\neq 0, c_1, c_2, c_3, \tau(s)=0, \lambda(s)$ =constant. Then, for the helix curve Γ one has $\tau(s)\neq 0$. Hence, there is no helix curve Γ with a $y-$Bertrand curve. □

Theorem 3.4. *Let $\gamma: I\longrightarrow\mathbb{Q}^3\subset E_1^4$ be a curve with arc length parameter s, for the cone curvature function τ,κ in the lightlike cone, then the following differential equations are satisfied*

1) *For the differential equation $\gamma''=\kappa\gamma-\gamma'''$, the following conditions hold:*
a) *If $\kappa=0$, $\gamma(s)=c_1+c_2s+c_3e^{-s}$.*
b) *If $\kappa\neq 0$,*

$$\gamma(s)=c_1e^{k_1^3s}+c_4e^{-\frac{k_1^3}{2}s}\cos\left((\sqrt[3]{\omega}+\frac{1}{9\sqrt[3]{\omega}}-\frac{1}{3})\frac{\sqrt{3}}{2}\right)s,$$

where $k_1^3=\sqrt[3]{\omega}+\frac{1}{9\sqrt[3]{\omega}}-\frac{1}{3};\omega=\frac{\sqrt{27\kappa^2-4\kappa}}{6\sqrt{3}}-\frac{2-27\kappa}{54}$.
2) *For the equation $\gamma'''=\tau\gamma$,* if $\tau\neq 0$, $\gamma(s)=(a_1+a_2s+a_3s^2)e^{\sqrt[3]{\tau}s}$.
3) *For the equation $\gamma''''=-\kappa\gamma'-\tau\gamma''$, the following conditions are satisfied*
a) *If $\frac{\kappa^2}{2^2}+\frac{\tau^3}{3^3}<0$,*

$$\gamma=c_0+c_1e^{t_1^1s}+c_5e^{-\frac{t_1^1}{2}s}\left(\cos(\frac{(\sqrt[3]{\omega}-\frac{\tau}{3\sqrt[3]{\omega}})\sqrt{3}}{2})s\right),$$

where $\omega=\frac{\sqrt{27\kappa^2+4\tau^3}}{6\sqrt{3}}-\frac{\kappa}{2}$.
b) *If $\kappa=\tau=0$, $\gamma(s)=\eta_0+\eta_1s+\eta_2s^2+\eta_3s^3$,* where $c_i,\eta_i\in R_0^+$.

Proof. From (2.2), one can write the following equations

$$\gamma''=\kappa\gamma-\gamma''' \tag{3.47}$$

$$\gamma'''=\tau\gamma \tag{3.48}$$

$$\gamma'''' = -\kappa\gamma' - \tau\gamma''. \tag{3.49}$$

By solving the equation (3.47), one obtains following cases;

$a)$ If $\kappa = 0$, one obtains

$$\gamma(s) = c_1 + c_2 s + c_3 e^{-s}.c_i \in \mathbb{R}.$$

$b)$ If $\kappa \neq 0$, by solving differential equation, one can write following equations,

$$k_1^3 = \sqrt[3]{\omega} + \frac{1}{9\sqrt[3]{\omega}} - \frac{1}{3}; k_{2,3}^3 = k_1^3(-\frac{1}{2} \mp i\frac{\sqrt{3}}{2}),$$

where $\omega = \frac{\sqrt{27\kappa^2 - 4\kappa}}{6\sqrt{3}} - \frac{2-27\kappa}{54}$ and from previous equations, one gets

$$\gamma(s) = c_1 e^{k_1^3 s} + c_4 e^{-\frac{k_1^3}{2}s} \cos\left((\sqrt[3]{\omega} + \frac{1}{9\sqrt[3]{\omega}} - \frac{1}{3})\frac{\sqrt{3}}{2}\right)s;\ c_i \in \mathbb{R}_0.$$

By solving equation (3.48), for $\tau \neq 0$ one gets

$$\gamma(s) = (a_1 + a_2 s + a_3 s^2)e^{\sqrt[3]{\tau(s)}s},$$

and by solving equation (3.49), the following cases can written;

$a)$ If $\frac{\kappa^2}{2^2} + \frac{\tau^3}{3^3} < 0$, one obtains

$$t_0^1 = 0; t_1^1 = \sqrt[3]{\omega} - \frac{\tau}{3\sqrt[3]{\omega}}; t_{2,3}^1 = \left(\sqrt[3]{\omega} - \frac{\tau}{3\sqrt[3]{\omega}}\right)(\frac{-1}{2} \pm \frac{i\sqrt{3}}{2})$$

from previous equations, one has

$$\gamma = c_0 + c_1 e^{t_1^1 s} + c_5 e^{-\frac{t_1^1}{2}s}\left(\cos((\sqrt[3]{\omega} - \frac{\tau}{3\sqrt[3]{\omega}})\frac{\sqrt{3}}{2})s\right).$$

where $\omega = \frac{\sqrt{27\kappa^2 + 4\tau^3}}{6\sqrt{3}} - \frac{\kappa}{2}$.

$b)$ If $\kappa = \tau = 0$, one writes

$$\gamma(s) = \eta_0 + \eta_1 s + \eta_2 s^2 + \eta_3 s^3; \eta_i \in \mathbb{R}^+.$$

□

4. CONCLUSION

In this paper, the $W-$Bertrand curves for curves lying on the $\mathbb{Q}^3$ are examined and some certain results of describing the $W-$Bertrand pair $\{\gamma, \Gamma\}$ due to differentiable functions are presented in detail. As a first instance, it is given that the conditions of being the $W-$Bertrand pair $\{\gamma, \Gamma\}$ according to asymptotic orthonormal frame in 3D lightlike cone. Also, an arbitrary helix curve in terms of their curvature functions are characterized satisfying condition the $W-$Bertrand curve. This study will accompany the scientists who will conduct new studies on similar subjects as a basic resource since it is one of the important studies on this subject. This study will be a resource for scientists who will work on new topics on similar subjects, as the work done on this subject in lightlike cone space is important and different. The fact that the work is in the four-dimension will allow us to think and interpret some physical concepts within this space.

5. Acknowledgments

The author wishes to express their thanks to the authors of literatures for the supplied scientific aspects and idea for this study.

6. Funding

Not applicable.

7. Conflicts of Interest

The author declares no conflict of interest.

References

[1] F. Almaz and M. Külahcı, *A note on special smarandache curves in the null cone*, Acta Universitatis Apulensis **56**, (2018) 111-124.

[2] F. Almaz and M. Külahcı, *Involute-evolute D-curves in Minkowski 3-space E_1^3*, Boletim da Sociedade Paranaense de Matematica **39**(1), (2021) 147-156.

[3] H. Balgetir, M. Bektaş and M. Ergüt, *Bertrand curves for non-null curves in 3-dimensional Lorentzian space*, Hadronic J. **27**, (2004) 299-236.

[4] J.H. Choi and H.K. Young, *Associated curves of a Frenet curve and their applications*, Appl. Math. Comput. **218**, (2012) 9116-9124.

[5] K. Eren, S. Şenyurt, *Some smarandache curves constructed from a spacelike salkowski curve with timelike principal normal*, Punjab Univ. J. Math. **53**(9), (2021) 679-690.

[6] H. Gluck, *Higher curvature of curves in Euclidean Space E^n*, Amer. Math. Monthly **73**, (1996) 699.

[7] E. Görgülü and E. Özdamar, *A generalizations of the Bertrand curves as general inclined curves in E^n*, Communications de la Fac. Sci. Uni. Ankara, Series A1, (1986) 53-60.

[8] H.H. Hacısalihoğlu, *Diferansiyel Geometri*, İnönü Üniversitesi Fen-Edebiyat Fakültesi Yayınları, 1983.

[9] H.B. Karadağ and M. Karadağ, *Null generalized slant helices in Lorentzian space*, Differential Geometry-Dynamical Systems **10**, (2008) 178-185.

[10] M. Külahcı and F. Almaz, *Assesment of smarandache curves in the null cone Q^2*, Punjab Univ. J. Math. **51**(3), (2019) 101-112.

[11] Y. Li, A. Uçum, K. İlarslan and Ç. Camcı, *A new class of Bertrand curves in Euclidean 4-space*, Symmetry **14**(6), (2022) 178-185.

[12] H. Liu and M.Q. Mong, *Representation formulas of curves in two-and-three-dimensional lightlike cone*, Results Math. **59**, (2011) 437-451.

[13] H. Liu, *Curves in the lightlike cone*, Beitrage zur Algebra and Geometrie Contributions to Algebra and Geometry **25**(1), (2004) 291-303.

[14] M. Mak and M. Demir, *Some special Legendre mates of spherical Legendre curves*, Punjab Univ. J. Math. **54**(5) (2022) 297-314.

[15] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

[16] G.Y. Sentürk and S. Yüce, *Bertrand offsets of ruled surfaces with Darboux frame*, Results Math. **72**, (2017) 1151-1159.

[17] M. Tosun and I. Özgür, *Bertrand curves in the $n-$dimensional Lorentz space L^n*, The Journal of arts and sciences **1**, Series A (1997).

[18] S. Uzumiya and N. Takeuchi, *Special curves and ruled surfaces*, Preprint Isaac Newton Institute for Mathematical Sciences, Cambridge Univ. (2000), preprint series, NI-00023-SGT.

[19] S. Uzumiya and N. Takeuchi, *Generic properties of helices and Bertrand curves*, Journal of Geometry **74**, (2002) 97-109.

[20] J.K. Whittemore, *Bertrand curves and helices*, Duke math. J. **6**(1), (1940) 235-245.

[21] C. Zhang and D. Pei *Generalized bertrand curves in minkowski 3-space*, Mathematics **8**, (2020) 2199.