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Existence and Uniqueness of Solutions for Atangana-Baleanu Fractional Differential Equations in Caputo-Sense

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Abstract. In this work, we investigate the existence and uniqueness of Atangana-Baleanu solutions in Caputo sense fractional differential equations. The existence results are based on Mönch's fixed point theorem. The idea of uniqueness is investigated using the Banach contraction principle. Furthermore, a practical example of the results of solutions that include various fractional orders is provided to validate the theoretical insights provided in this paper.

AMS (MOS) Subject Classification Codes: 334A08; 26A33; 34K20; 34A12; 34B27; 34D20
Key Words: ABC-fractional differential equations; Mönch's fixed point theorem; fixed point theorem.

1. Introduction

In recent decades, fractional differential equations (FDEs)have been used in many diverse domains, including chemistry, physics, engineering, control theory, aerodynamics, complex media electrodynamics, control of dynamical systems, and more. FDEs are consequently receiving a lot of attention and importance. We encourage readers to (see [4, 5, 6, 7, 13, 14, 15, 16, 17, 23, 28, 29]) and the references therein for more information. The main factor contributing to the popularity of fractional calculus, which considers the inherited characteristics of many materials and processes, is the nonlocal nature of the fractional order operators.

The study of existence and uniqueness for fractional order differential equations has gained significant attention due to their applications in various fields see [9, 10, 12, 33]. Unlike classical differential equations, fractional order models incorporate non-local behavior and memory effects, leading to richer dynamics and complex solutions. Establishing the existence and uniqueness of solutions for these equations is crucial, as it ensures that the models accurately describe real-world phenomena without ambiguity. Recent advancements in fractional calculus have facilitated the development of robust mathematical frameworks that address these challenges, paving the way for more reliable modeling approaches. As a result, understanding these properties not only enhances theoretical insights but also aids in practical applications where fractional order models are applied. For more information (see [3, 2, 22])

The ABC-fractional derivative (FD) with multiple conditions has received increased attention from numerous researchers in a variety of fields in recent years for more detail (see [9, 19, 26, 32]). The following nonsingularity, sometimes referred to as non-locality of the kernel, that gain the generalized Mittag- Leffler function, are familiar with the AB-FD. Atangana and Koca, two of the most recent researchers on ABC-deriative s, discover chaos in a basic nonlinear system with AB-FDs [3]. The AB-fractional neutral integro- differntial equations were thoroughly examined by Ravichandran et al. [23]. Sene specifically addressed the AB-deriative for Stokes' first problem for heated flat plates [32]. A dynamical system using the Atangana-Baleanu FD was modeled and simulated by Owolabi [27]. The application of fractional neutral deriative is the subject of extensive investigation. A coupled system of nonlinear neutral FDEs was examined by Liu et al. [22]. The fractional of neutral DEs with infinite delay was investigated by Zhou et al. [34].

The measure of noncompactness (MNC) together with fixed point theorems(FPT) for example Darbo [11], Mönch [25] and Sadovski [30] is an effective tool for studying differential or integral equations. MNCs are particularly important in nonlinear analysis. The researchers are frequently employed in differential and integral equations, operator theory, and Banach space geometry. Since 1970, there have been several research presented on the subject and its various uses. Kuratowski [21] pioneered the notion of MNC, whose performed an essential role in fixed point theory. Afterwards, Darbo [11] utilized Kuratowski's MNC to extend Schauder's FPT.

Some recent contributions on FDEs involving ABC-FDs can be found in the following articles For instance, in [1], The BVP of AB-Caputo FD, presented by Abdeljawad is also one of the recent problems through which the higher fractional orders are addressed:

$$\begin{cases} {}^{ABC}D^{\vartheta}_{a+}\varsigma(\tau) + q(\tau,\varsigma(\tau)), & \tau \in J = [a,T], \quad \vartheta \in (1,2], \\ \varsigma(a) = \varsigma(T) = 0. \end{cases}$$

AB-Caputo fractional IVP is one of the studied problems by Jarad et al. [18], and has the form:

$$\begin{cases} {}^{ABC}D^{\vartheta}_{a+}\varsigma(\tau) = f(\tau,\varsigma(\tau)), & \tau \in J = [a,T], \quad \vartheta \in (0,1], \\ \varsigma(a) = \varsigma_a. \end{cases}$$

Muhammad and Rafeeq investigate the existence and unique solution of the nonlinear differential equation to the Atangana-Baleanu fractional derivative in the sense of Caputo with the initial periodic condition in [26].

$${}^{ABC}D^{\alpha}_{0}x(t) = f(t,x(t)), \quad t \in J = [0,T], \quad \alpha \in (1,2],$$

with

$$\begin{cases} x(0) = x(T). \\ x'(0) = \int_0^T x(s)ds \end{cases}$$

Motivated and inspired by the above works, we discuss the existence and uniqueness of solutions to the ABC-FDEs which has the form:

$$\bar{\Im}'(\xi) + {}^{ABC}D_0^{\daleth}\bar{\Im}(\xi) = \bar{\varpi}(\xi, \bar{\Im}(\xi)), \quad \xi \in J = [0, \dot{\mho}], \quad \daleth \in (1, 2], \tag{1. 1}$$

with the boundary condition

$$\begin{cases} \bar{\Im}'(0) = 0. \\ \bar{\Im}(\dot{\mho}) = \int_0^{\dot{\mho}} \bar{\Im}(s)ds \end{cases} \tag{1. 2}$$

where ${}^{ABC}D_{0+}^{\daleth}$, is Atangana-Baleanu in Caputo sense FDEs of order $\daleth$, with $\bar{\varpi} : J \times \mathbb{R} \to \mathbb{R}$, is continuous where $\mathbb{R}$ is Banach space. integrating nonlocal and volatile situations. The Hausdorff MNC and the Mönch-FPT are used for establishing our findings. The structure of this paper is as follows: We provide some definitions, notations, and initial concepts in section 2. Our primary findings on the existence of solutions to aforementioned problem are described in section 3, and section 4 provides an example demonstrating the practical use of the enhanced conditions.

This study advances the understanding of fractional calculus by establishing existence and uniqueness results for the Atangana-Baleanu fractional differential equations (AB-FDEs) in the Caputo sense, utilizing Mönch's fixed point theorem and the Banach contraction principle. The paper highlights the unique properties of the Atangana-Baleanu derivative, which incorporates nonlocality, offering a fresh perspective compared to traditional fractional derivatives. By demonstrating solutions through practical examples, the research underscores the applicability of AB-FDEs in modeling real-world phenomena across various fields, including engineering and physics.

2. Notations and Preliminaries

Here we recollect some definitions and lemmas that are basic and needed at various places in this work.
Let f be a continuous function and $C(J, \mathbb{R})$ be a the Banach space with the supermom norm

$$\left\|\bar{\Im}\right\| = sup\{\left|\bar{\Im}(\xi)\right|; \xi \in J\}.$$

Let $L_1(J, \mathbb{R})$ be the Banach space of measurable functions $\bar{\Im} : J \to R$ which are Bochner integrable, equipped with the norm

$$||\bar{\Im}||_{L_1} = \int_J \bar{\Im}(\xi)d\xi.$$

$AC^1(J, \mathbb{R})$ refers to the set of functions $\bar{\Im} : J \to \mathbb{R}$ whose first deriative is absolutely continuous.In addition, for a provided set of functions $\bar{\Psi} : J \to \mathbb{R}$, let us indicate by

$$\bar{\Psi}(\xi) = \{\bar{\psi}(\xi) : \bar{\psi} \in \bar{\Psi}\}, \xi \in J,$$

and

$$\bar{\Psi}(J) = \{\bar{\psi}(\xi) : \bar{\psi} \in \bar{\Psi}, \xi \in J\}.$$

Definition 2.1. [8] *Let $\gimel \in (0,1], \bar{\varpi}' \in H'(a,b)$ where $a \leq b$, then AB-derivative in the Caputo sense is defined as*

$$({}_{a}^{ABC}D^{\gimel}\bar{\varpi})(\xi) = \frac{B(\gimel)}{1-\gimel}\int_{a}^{\xi}\bar{\varpi}'(s)E_{\gimel}\left[-\gimel\frac{(\dot{\mho}-s)^{\gimel}}{1-\gimel}\right]ds.$$

Where $B(\gimel)$ a normalizing positive function satisfying $B(0) = B(\gimel) = 1$ and $E_{\gimel}$ is the Mittag-Leffler function described by

$$E_{\gimel}(p) = \sum_{j=0}^{\infty}\frac{p^{j}}{\Gamma(j\gimel+1)} \qquad Re(\gimel) > 0, \quad p \in \mathbb{C}.$$

Definition 2.2. [8] *The fractional integral associated with the new FD with non-local kernel (AB fractional integral) is defined as:*

$$({}_{0}^{AB}I^{\gimel}\bar{\varpi})(\xi) = \frac{1-\gimel}{B(\gimel)}\bar{\varpi}(\xi) + \frac{\gimel}{B(\gimel)\Gamma(\gimel)}\int_{0}^{\xi}(\xi-s)^{\gimel-1}\bar{\varpi}(s)ds.$$

Theorem 2.3. [1] *Let $m \in N$ and Assume that $\bar{\varpi}(x)$ defined on $[a,b]$and $\gimel \in (m, m+1]$. Then we have*

(1) $({}^{ABR}D_{a}^{\gimel}\ {}^{AR}I_{a}^{\gimel})(\bar{\varpi}(\xi)) = \bar{\varpi}(\xi)$
(2) $({}^{AR}I_{a}^{\gimel}\ {}^{ABR}D_{a}^{\gimel})(\bar{\varpi}(\xi)) = \bar{\varpi}(\xi) - \sum_{n=0}^{m-1}\frac{f^{(n)}(\xi)}{n!}(\xi-a)^{n}$
(3) $({}^{AR}I_{a}^{\gimel}\ {}^{ABC}D_{a}^{\gimel})(\bar{\varpi}(\xi)) = \bar{\varpi}(\xi) - \sum_{n=0}^{m-1}\frac{f^{(n)}(\xi)}{n!}(\xi-a)^{n}$

Lemma 2.4. [1] *The solution of the below problem*

$$({}_{a}^{ABC}D^{\gimel}\bar{\varpi})(\xi) = w(\xi).$$

is given by

$$\bar{\varpi}(\xi) = \bar{\Im}(a)+\bar{\Im}'(a)(\xi-a)+\frac{2-\gimel}{B(\gimel-1)}\int_{a}^{\xi}w(s)ds+\frac{\gimel-1}{B(\gimel-1)\Gamma(\gimel)}\int_{a}^{\xi}(\xi-s)^{\gimel-1}w(s)ds.$$

Where $\gimel \in (1,2]$ and $w \in C(J,R)$ with $w(a) = 0$.

Theorem 2.5. (Banach contraction mapping).
Let H be a Banach space. If $Z : H \to H$ is a contraction, then Z has a unique fixed point in H.

Theorem 2.6. [25](Mönch's Fixed Point Theorem).
Let $\bar{\tau}$ be a non empty bounded closed and convex subset of a Banach space E such that $0 \in \bar{\tau}$ and let A be a continuous mapping of $\bar{\tau}$ into itself. If the implication

$$\bar{\Psi} = \overline{conv}A(\bar{\Psi}), \quad or \ \ \bar{\Psi} = A(\bar{\Psi}) \cup \{0\} \Rightarrow \mu(\bar{\Psi}) = 0,$$

holds for every subset $\bar{\Psi}$ of $\bar{\tau}$, then A has at least one fixed point.

Definition 2.7. *Let F be a Banach space and D_F, the bounded subset of F The Kuratowski MNC is defined by the mapping $\mu : D_F \to [0, \infty)$.*

$$\mu(w) = \inf\{\epsilon > 0 : w \subseteq \bigcup_{j=1}^{k} w_j \ and \ diam(w_j) < \epsilon\}; \ where \ w \in D_F,$$

Important properties satisfies the MNC

(1) $\mu(w) = 0 \Leftrightarrow \overline{w}$ is compact (w is relatively compact),
(2) $\mu(w) = \mu(\overline{w})$,
(3) $w \subset z \Rightarrow \mu(w) \leq \mu(z)$,
(4) $\mu(w + z) \leq \mu(w) + \mu(z)$,
(5) $\mu(cw) \leq |c|\mu(w), \ c \in \mathbb{R}$,
(6) $\mu(conv(w)) = \mu(w)$,

Where $convA$ and $\overline{w}$ are called the convex hull and the closure of the bounded set w, respectively.

Definition 2.8. *A map $\bar{\varpi} : J \times E \to E$ is called Caratheodory if*

(a) $\xi \to \bar{\varpi}(\xi, x)$ *is measurable for each* $x \in E$.
(b) $x \to \bar{\varpi}(\xi, x)$ *is continuous for almost* $\xi \in J$.

Lemma 2.9. *Let k be a bounded, closed and convex subset of the Banach space $C(Z^*, E^*)$. And if Q_h*

be a continuous function on $Z^* \times Z^*$ and a function $\bar{\varpi} : Z^* \times E^* \to E^*$, which meets the requirements of Caratheodory conditions, and assume that there $\exists \ z \in L^1(Z^*, \mathbb{R}_+)$ such that, for each $\xi \in Z^*$ and each bounded set $w \subset E^*$, if $\bar{\Psi}$ is an equicontinuous subset of K, then

$$\mu\left(\left\{\int_J Q_h(\xi, s)\bar{\varpi}(s, \bar{\Im}(s))ds, \bar{\Im} \in \bar{\Psi}\right\}\right) \leq \int_J ||Q_h(\xi, s)||z(\xi)\mu(\bar{\Psi})ds.$$

2.10. **Assumptions.** In this study, we make several critical assumptions regarding the function and the solutions to our boundary value problem:

1. Continuity: The function $\bar{\varpi}(\xi, \bar{\Im}(\xi))$ is assumed to be continuous over the domain $J \times \mathbb{R}$. This continuity is fundamental for applying fixed-point theorems and ensuring the existence of solutions.

2. Boundedness: We assume that the solutions $\bar{\Im}(\xi)$ are bounded within the interval J. This boundedness allows us to utilize Banach's fixed-point theorem effectively.

3.Regularity: The first derivative $\bar{\Im}(\xi)$ is assumed to be absolutely continuous over the interval J, which is necessary for defining the Atangana-Baleanu derivative in the Caputo sense.

4. Lipschitz Condition: We assume that the nonlinear term $\bar{\varpi}(\xi, \bar{\Im}(\xi))$ satisfies a Lipschitz condition with respect to the second variable. This condition is essential for proving the uniqueness of solutions via contraction mapping principles.

These assumptions are based on previous literature and are standard in the analysis of fractional differential equations, ensuring the robustness of our findings.

3. Main Results

This section focuses on to establishing formulae of solutions to the boundary value issue (1. 1)-(1. 2). We utilize Mönch's FPT establish the existence and Banach's FPT to demonstrate the uniqueness of our result.

To prove the main results, we require to the next assumptions:
($\mathbf{H1}$) There $\exists$ a constant $\Omega_2 > 0$, such that $\left|\bar{\varpi}(\xi, \bar{\Im}(\xi))\right| \leq \Omega_2$.
($\mathbf{H2}$) There $\exists$ constants $M, Q > 0$, such that

$$\left|\bar{\varpi}(\xi, \bar{\Im}_1(\xi)) - \bar{\varpi}(\xi, \bar{\Im}_2(\xi))\right| \leq M|\bar{\Im}_1 - \bar{\Im}_2|,$$

and $Q = \sup_{\xi \in J} ||\bar{\varpi}(\xi, 0)||$, for every $\xi \in J$.
(**H3**) Assume $\bar{\varpi} : J \times \mathbb{R} \to \mathbb{R}$ and $\bar{\Im} : J \to \mathbb{R}$ are satisfies the Caratheodory conditions,
(**H4**) For each $\xi \in J$, and $\bar{\Im} \in \mathbb{R}$, there exist $z \in L^1(J, \mathbb{R}_+)$, such that

$$||\bar{\varpi}(\xi, \bar{\Im}(\xi))|| \leq z(\xi)||\bar{\Im}||.$$

(**H5**) For each $\xi \in J$ and each bounded set $w \subset \mathbb{R}$, we have

$$\lim_{h \to 0^+} \mu(\bar{\varpi}(J_{\xi,h} \times w)) \leq z(\xi)\mu(w), \ where \ J_{\xi,h} = [\xi - h, \xi] \cap J,$$

where μ is the Kuratowski MNC and $J_{\xi,h} = [\xi - h, \xi]$.

Lemma 3.1. *For any $\bar{\Im}(\xi) \in C(J, \mathbb{R})$, then the BVP (1. 1)-(1. 2) has a solution*

$$\begin{aligned}
\bar{\Im}(\xi) &= \frac{\gimel - 2}{B(\gimel - 1)}\bar{\Im}(\xi) + \frac{(\gimel - 1)}{B(\gimel - 1)\Gamma(\gimel - 1)\zeta}\int_0^{\dot{\mho}} ((\dot{\mho} - s)^{\gimel - 2} - (\dot{\mho} - s)^{(\gimel - 1)})\bar{\Im}(s)ds \\
&+ \frac{2 - \gimel}{B(\gimel - 1)\zeta}\int_0^{\dot{\mho}} \bar{\varpi}(s, \bar{\Im}(s))(\dot{\mho} - s - 1)ds + \frac{(\gimel - 1)}{B(\gimel - 1)\Gamma(\gimel)\zeta}\int_0^{\dot{\mho}} \bar{\varpi}(s, \bar{\Im}(s))((\dot{\mho} - s)^{\gimel} - (\dot{\mho} - s)^{(\gimel - 1)})ds \\
&- \frac{(\gimel - 1)}{B(\gimel - 1)\Gamma(\gimel - 1)}\int_0^{\xi} (\xi - s)^{\gimel - 2}\bar{\Im}(s)ds \\
&+ \frac{2 - \gimel}{B(\gimel - 1)}\int_0^{\xi} \bar{\varpi}(s, \bar{\Im}(s))ds + \frac{(\gimel - 1)}{B(\gimel - 1)\Gamma(\gimel)}\int_0^{\xi} (\xi - s)^{(\gimel - 1)}\bar{\varpi}(s, \bar{\Im}(s))ds,
\end{aligned}$$

Proof. Applying the AR-fractional integral operator of order $\gimel$ from 0 to ξ on both sides of fractional differential equations (1. 1)

$${}^{AR}I_a^{\gimel} D\bar{\Im}(\xi) + {}^{AR}I_a^{\gimel} \, {}^{ABC}D_{0+}^{\gimel}\bar{\Im}(\xi) = {}^{AR}I_a^{\gimel}\bar{\varpi}(\xi, \bar{\Im}(\xi), I(\bar{\Im}(\xi))),$$

Applying theorem (**??**thm2.3), we get

$$\bar{\Im}(\xi) = \bar{\Im}(0) + \xi\bar{\Im}'(0) - {}^{AR}I_0^{\gimel - 1}\bar{\Im}(\xi) + {}^{AR}I_a^{\gimel}\bar{\varpi}(\xi, \bar{\Im}(\xi)),$$

$$\begin{aligned}
\bar{\Im}(\xi) =& \bar{\Im}(0) + \xi\bar{\Im}'(0) - \frac{2 - \gimel}{B(\gimel - 1)}\bar{\Im}(\xi) - \frac{(\gimel - 1)}{B(\gimel - 1)\Gamma(\gimel - 1)}\int_0^{\xi} (\xi - s)^{\gimel - 2}\bar{\Im}(s)ds \\
&+ \frac{2 - \gimel}{B(\gimel - 1)}\int_0^{\xi} \bar{\varpi}(s, \bar{\Im}(s))ds + \frac{(\gimel - 1)}{B(\gimel - 1)\Gamma(\gimel)}\int_0^{\xi} (\xi - s)^{(\gimel - 1)}\bar{\varpi}(s, \bar{\Im}(s))ds,
\end{aligned}$$

To find $\bar{\Im}(0)$ and $\bar{\Im}'(0)$, we apply the boundary condition (1. 2).
Since $\bar{\Im}'(0) = 0$. Hence

$$\bar{\Im}(\xi) = \bar{\Im}(0) + \frac{\daleth - 2}{B(\daleth - 1)}\bar{\Im}(\xi) - \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth - 1)} \int_0^{\xi} (\xi - s)^{\daleth - 2}\bar{\Im}(s)ds + \frac{2 - \daleth}{B(\daleth - 1)} \int_0^{\xi} \bar{\varpi}(s, \bar{\Im}(s))ds + \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth)} \int_0^{\xi} (\xi - s)^{(\daleth - 1)}\bar{\varpi}(s, \bar{\Im}(s))ds,$$

To find $\bar{\Im}(0)$, we have

$$\bar{\Im}(\dot{\mho}) = \bar{\Im}(0) + \frac{\daleth - 2}{B(\daleth - 1)}\bar{\Im}\dot{\mho} - \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth - 1)} \int_0^{\dot{\mho}} (\dot{\mho} - s)^{\daleth - 2}\bar{\Im}(s)ds + \frac{2 - \daleth}{B(\daleth - 1)} \int_0^{\dot{\mho}} \bar{\varpi}(s, \bar{\Im}(s))ds + \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth)} \int_0^{\dot{\mho}} (\dot{\mho} - s)^{(\daleth - 1)}\bar{\varpi}(s, \bar{\Im}(s))ds,$$

$$\bar{\Im}(\dot{\mho})(1 - \frac{\daleth - 2}{B(\daleth - 1)}) = \bar{\Im}(0) - \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth - 1)} \int_0^{\dot{\mho}} (\dot{\mho} - s)^{\daleth - 2}\bar{\Im}(s)ds + \frac{2 - \daleth}{B(\daleth - 1)} \int_0^{\dot{\mho}} \bar{\varpi}(s, \bar{\Im}(s))ds + \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth)} \int_0^{\dot{\mho}} (\dot{\mho} - s)^{(\daleth - 1)}\bar{\varpi}(s, \bar{\Im}(s))ds,$$

$$\int_0^{\dot{\mho}} \bar{\Im}(s)ds = \bar{\Im}(0)(\dot{\mho}) + \frac{\daleth - 2}{B(\daleth - 1)} \int_0^{\dot{\mho}} \bar{\Im}(s)ds - \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth - 1)} \int_0^{\dot{\mho}} (\dot{\mho} - s)^{(\daleth - 1)}\bar{\Im}(s)ds + \frac{2 - \daleth}{B(\daleth - 1)} \int_0^{\dot{\mho}} \bar{\varpi}(s, \bar{\Im}(s))(\dot{\mho} - s)ds + \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth)} \int_0^{\dot{\mho}} \bar{\varpi}(s, \bar{\Im}(s))(\dot{\mho} - s)^{\daleth}ds,$$

To isolate($\int_0^{\dot{\mho}} \bar{\Im}(s)ds$), we can rearrange:

$$\left(1 - \frac{\daleth - 2}{B(\daleth - 1)}\right) \int_0^{\dot{\mho}} \bar{\Im}(\xi)d\xi = \bar{\Im}(0)\dot{\mho} + \frac{2 - \daleth}{B(\daleth - 1)} \int_0^{\dot{\mho}} \bar{\varpi}(s, \bar{\Im}(s))(\dot{\mho} - s)ds - \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth - 1)} \int_0^{\dot{\mho}} (\dot{\mho} - s)^{(\daleth - 1)}\bar{\Im}ds + \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth)} \int_0^{\dot{\mho}} \bar{\varpi}(s, \bar{\Im}(s))(\dot{\mho} - s)^{\daleth}ds,$$

$$\bar{\Im}(0)\,(1 - T) = \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth - 1)} \int_0^{\dot{\mho}} ((\dot{\mho} - s)^{\daleth - 2} - (\dot{\mho} - s)^{(\daleth - 1)})\bar{\Im}(s)ds + \frac{2 - \daleth}{B(\daleth - 1)} \int_0^{\dot{\mho}} \bar{\varpi}(s, \bar{\Im}(s))(\dot{\mho} - s - 1)ds + \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth)} \int_0^{\dot{\mho}} \bar{\varpi}(s, \bar{\Im}(s))((\dot{\mho} - s)^{\daleth} - (\dot{\mho} - s)^{(\daleth - 1)})ds$$

Let $\zeta = (1 - \dot{\mho})$ then we obtain

$$\bar{\Im}(0) = \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth - 1)\zeta} \int_0^{\dot{\mho}} ((\dot{\mho} - s)^{\daleth - 2} - (\dot{\mho} - s)^{(\daleth - 1)})\bar{\Im}(s)ds + \frac{2 - \daleth}{B(\daleth - 1)\zeta} \int_0^{\dot{\mho}} \bar{\varpi}(s, \bar{\Im}(s))(\dot{\mho} - s - 1)ds + \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth)\zeta} \int_0^{\dot{\mho}} \bar{\varpi}(s, \bar{\Im}(s))((\dot{\mho} - s)^{\daleth} - (\dot{\mho} - s)^{(\daleth - 1)})ds,$$

$$\bar{\Im}(\xi) = \frac{\gimel-2}{B(\gimel-1)}\bar{\Im}(\xi) + \frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel-1)\zeta}\int_0^{\dot{\mho}}((\dot{\mho}-s)^{\gimel-2}-(\dot{\mho}-s)^{(\gimel-1)})\bar{\Im}(s)ds$$
$$+\frac{2-\gimel}{B(\gimel-1)\zeta}\int_0^{\dot{\mho}}\bar{\varpi}(s,\bar{\Im}(s))(\dot{\mho}-s-1)ds + \frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel)\zeta}\int_0^{\dot{\mho}}\bar{\varpi}(s,\bar{\Im}(s))((\dot{\mho}-s)^{\gimel}-(\dot{\mho}-s)^{(\gimel-1)})ds$$
$$-\frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel-1)}\int_0^{\xi}(\xi-s)^{\gimel-2}\bar{\Im}(s)ds + \frac{2-\gimel}{B(\gimel-1)}\int_0^{\xi}\bar{\varpi}(s,\bar{\Im}(s))(s)$$
$$+\frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel)}\int_0^{\xi}(\xi-s)^{(\gimel-1)}\bar{\varpi}(s,\bar{\Im}(s))ds,$$

□

In the first result, the uniqueness theorem is proved using the Banach contraction principle, and the existence of solutions for the BVP (1. 1)-(1. 2) is proven using the Mönch FPT.

For the sake of convenience, we set the notation:

$$\vartheta_0 = \left[\frac{\gimel-2}{\dot{\mho}^{(\gimel-1)}} + \frac{1}{\Gamma(\gimel-1)\zeta} + \frac{\dot{\mho}(\gimel-1)}{(\gimel)\Gamma(\gimel-1)\zeta} + \frac{1}{\Gamma(\gimel-1)}\right]\frac{\dot{\mho}^{(\gimel-1)}}{B(\gimel-1)},$$

$$\vartheta_1 = \left[\frac{2-\gimel(2\dot{\mho}^2-1)}{2\zeta\dot{\mho}^{\gimel-1}} + \frac{\gimel(\gimel-1)\dot{\mho}+(\gimel+1)(\gimel-1)}{\Gamma(\gimel+2)\zeta} + \frac{2-\gimel}{\dot{\mho}^{\gimel-1}} + \frac{(\gimel-1)}{\Gamma(\gimel+1)}\right]\frac{\dot{\mho}^{\gimel}}{B(\gimel-1)}.$$

Theorem 3.2. *Assume that $\bar{\varpi}: J\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies (**H1**)-(**H2**). If*

$$\vartheta_1 M < 1.$$

*Then the boundary value problem (**??**eqa1.1)-(**??**eqa1.2) has a unique solution.*

Proof. Define the operator $\Upsilon: C(J,\mathbb{R})\to C(J,\mathbb{R})$ as the following

$$\Upsilon(\bar{\Im})(\xi) = \frac{\gimel-2}{B(\gimel-1)}\bar{\Im}(\xi) + \frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel-1)\zeta}\int_0^{\dot{\mho}}((\dot{\mho}-s)^{\gimel-2}-(\dot{\mho}-s)^{(\gimel-1)})\bar{\Im}(s)ds$$
$$+\frac{2-\gimel}{B(\gimel-1)\zeta}\int_0^{\dot{\mho}}\bar{\varpi}(s,\bar{\Im}(s))(\dot{\mho}-s-1)ds$$
$$+\frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel)\zeta}\int_0^{\dot{\mho}}\bar{\varpi}(s,\bar{\Im}(s))((\dot{\mho}-s)^{\gimel}-(\dot{\mho}-s)^{(\gimel-1)})ds$$
$$-\frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel-1)}\int_0^{\xi}(\xi-s)^{\gimel-2}\bar{\Im}(s)ds + \frac{2-\gimel}{B(\gimel-1)}\int_0^{\xi}\bar{\varpi}(s,\bar{\Im}(s))(s)$$
$$+\frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel)}\int_0^{\xi}(\xi-s)^{(\gimel-1)}\bar{\varpi}(s,\bar{\Im}(s))ds,$$

We must demonstrate that Υ is a solution of the boundary value problem (**??**eqa1.1)-(**??**eqa1.2) with a fixed point on Θ_{d_0}. First, we demonstrate that $\Upsilon\Theta_{d_0}\subset\Theta_{d_0}$, where, $\Theta_{d_0} = \{\bar{\Im}\in C(J,\mathbb{R}) : ||\bar{\Im}|| < d_0\}$. For $\bar{\Im}\in\Theta_{d_0}$ The operator Υ is bounded set in

$C(J,\mathbb{R})$. For any $d_0>0$, then for each $\xi\in J$, we have

$$
\begin{aligned}
|(\Upsilon\bar{\Im})(\xi)| = \Bigg| &\frac{\gimel-2}{B(\gimel-1)}\bar{\Im}(\xi) + \frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel-1)\zeta}\int_0^{\dot{\mho}}((\dot{\mho}-s)^{\gimel-2}-(\dot{\mho}-s)^{(\gimel-1)})\bar{\Im}(s)ds \\
&+ \frac{2-\gimel}{B(\gimel-1)\zeta}\int_0^{\dot{\mho}}\bar{\varpi}(s,\bar{\Im}(s))(\dot{\mho}-s-1)ds \\
&+ \frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel)\zeta}\int_0^{\dot{\mho}}\bar{\varpi}(s,\bar{\Im}(s))((\dot{\mho}-s)^{\gimel}-(\dot{\mho}-s)^{(\gimel-1)})ds \\
&- \frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel-1)}\int_0^{\xi}(\xi-s)^{\gimel-2}\bar{\Im}(s)ds + \frac{2-\gimel}{B(\gimel-1)}\int_0^{\xi}\bar{\varpi}(s,\bar{\Im}(s))ds \\
&+ \frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel)}\int_0^{\xi}(\xi-s)^{(\gimel-1)}\bar{\varpi}(s,\bar{\Im}(s))ds\Bigg|,
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\gimel-2}{B(\gimel-1)}||\bar{\Im}|| + \frac{(\gimel-1)||\bar{\Im}||}{B(\gimel-1)\Gamma(\gimel-1)\zeta}\int_0^{\dot{\mho}}\left|(\dot{\mho}-s)^{\gimel-2}-(\dot{\mho}-s)^{(\gimel-1)}\right|ds \\
&+ \frac{2-\gimel}{B(\gimel-1)\zeta}\int_0^{\dot{\mho}}(\dot{\mho}-s-1)|\bar{\varpi}(s,\bar{\Im}(s))-\bar{\varpi}(s,0)|ds + \frac{2-\gimel}{B(\gimel-1)\zeta}\int_0^{\dot{\mho}}(\dot{\mho}-s-1)|\bar{\varpi}(s,0)|ds \\
&+ \frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel)\zeta}\int_0^{\dot{\mho}}\left((\dot{\mho}-s)^{\gimel}-(\dot{\mho}-s)^{(\gimel-1)}\right)|\bar{\varpi}(s,\bar{\Im}(s))-\bar{\varpi}(s,0)|ds \\
&+ \frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel)\zeta}\int_0^{\dot{\mho}}((\dot{\mho}-s)^{\gimel}-(\dot{\mho}-s)^{(\gimel-1)})|\bar{\varpi}(s,0)|ds \\
&+ \frac{(\gimel-1)||\bar{\Im}||}{B(\gimel-1)\Gamma(\gimel-1)}\left|\int_0^{\xi}(\xi-s)^{\gimel-2}ds\right| + \frac{2-\gimel}{B(\gimel-1)}\int_0^{\xi}|\bar{\varpi}(s,0)|ds \\
&+ \frac{2-\gimel}{B(\gimel-1)}\int_0^{\xi}|\bar{\varpi}(s,\bar{\Im}(s))-\bar{\varpi}(s,0)|ds + \frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel)}\int_0^{\xi}(\xi-s)^{(\gimel-1)}|\bar{\varpi}(s,\bar{\Im}(s))-\bar{\varpi}(s,0)|ds \\
&+ \frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel)}\int_0^{\xi}(\xi-s)^{(\gimel-1)}|\bar{\varpi}(s,0)|ds,
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\gimel-2}{B(\gimel-1)}||\bar{\Im}|| + \frac{(\gimel-1)||\bar{\Im}||}{B(\gimel-1)\Gamma(\gimel-1)\zeta}\left(\frac{\dot{\mho}^{(\gimel-1)}}{(\gimel-1)}+\frac{\dot{\mho}^{\gimel}}{\gimel}\right) + \frac{(2-\gimel)M||\bar{\Im}||}{B(\gimel-1)\zeta}\left(\frac{\dot{\mho}^2}{2}-\dot{\mho}\right) \\
&+ \frac{(2-\gimel)Q}{B(\gimel-1)\zeta}\left(\frac{\dot{\mho}^2}{2}-\dot{\mho}\right) + \frac{(\gimel-1)M||\bar{\Im}||}{B(\gimel-1)\Gamma(\gimel)\zeta}\left(\frac{\dot{\mho}^{\gimel+1}}{\gimel+1}+\frac{\dot{\mho}^{\gimel}}{\gimel}\right) \\
&+ \frac{(\gimel-1)M||\bar{\Im}||}{B(\gimel-1)\Gamma(\gimel)\zeta}\left(\frac{\dot{\mho}^{\gimel+1}}{\gimel+1}+\frac{\dot{\mho}^{\gimel}}{\gimel}\right) + \frac{(\gimel-1)||\bar{\Im}||}{B(\gimel-1)\Gamma(\gimel-1)}\left(\frac{\dot{\mho}^{(\gimel-1)}}{(\gimel-1)}\right) + \frac{(2-\gimel)Q\dot{\mho}}{B(\gimel-1)} \\
&+ \frac{(2-\gimel)M||\bar{\Im}||\dot{\mho}}{B(\gimel-1)} + \frac{(\gimel-1)M||\bar{\Im}||}{B(\gimel-1)\Gamma(\gimel)}\left(\frac{\dot{\mho}^{\gimel}}{\gimel}\right) + \frac{(\gimel-1)Q}{B(\gimel-1)\Gamma(\gimel)}\left(\frac{\dot{\mho}^{\gimel}}{\gimel}\right),
\end{aligned}
$$

$$\leq \left[\frac{\gimel-2}{\dot{\mho}^{(\gimel-1)}} + \frac{1}{\Gamma(\gimel-1)\zeta} + \frac{\dot{\mho}(\gimel-1)}{(\gimel)\Gamma(\gimel-1)\zeta} + \frac{1}{\Gamma(\gimel-1)}\right] \frac{\dot{\mho}^{(\gimel-1)}}{B(\gimel-1)}||\bar{\Im}||$$

$$+ \left[\frac{(2-\gimel)(2\dot{\mho}^2-T)}{2\zeta} + \left(\frac{(\gimel-1)\dot{\mho}^{\gimel+1}}{(\gimel+1)\Gamma(\gimel)\zeta} + \frac{(\gimel-1)\dot{\mho}^{\gimel}}{\gimel\Gamma(\gimel)\zeta}\right) + (\dot{\mho}-\dot{\mho}(\gimel-1)) + \frac{(\gimel-1)\dot{\mho}^{\gimel}}{\Gamma(\gimel+1)}\right] \frac{1}{B(\gimel-1)}(Q+M||\bar{\Im}||),$$

$$\leq \left[\frac{\gimel-2}{\dot{\mho}^{(\gimel-1)}} + \frac{1}{\Gamma(\gimel-1)\zeta} + \frac{\dot{\mho}(\gimel-1)}{(\gimel)\Gamma(\gimel-1)\zeta} + \frac{1}{\Gamma(\gimel-1)}\right] \frac{\dot{\mho}^{(\gimel-1)}}{B(\gimel-1)}d_0$$

$$+ \left[\frac{2-\gimel(2\dot{\mho}^2-1)}{2\zeta\dot{\mho}^{\gimel-1}} + \frac{\gimel(\gimel-1)\dot{\mho}+(\gimel^2-1)}{\Gamma(\gimel+2)\zeta} + \frac{2-\gimel}{\dot{\mho}^{\gimel-1}} + \frac{(\gimel-1)}{\Gamma(\gimel+1)}\right] \frac{\dot{\mho}^{\gimel}}{B(\gimel-1)}(Q+Md_0),$$

$$\leq \vartheta_0 d_0 + \vartheta_1 M d_0 + Q_2),$$

$$||(\Upsilon\bar{\Im}))(\xi)|| \leq \vartheta_0 d_0 + \vartheta_1 M d_0 + \vartheta_1 Q_2), \leq d_0,$$

$$d_0 \geq \frac{\vartheta_1 Q_2}{1-\vartheta_1 M - \vartheta_0}. \tag{3. 3}$$

Therefore, $\Upsilon\Theta_{d_0} \subset \Theta_{d_0}$. Now to show that Υ is a contraction mapping, let $\bar{\Im}_1, \bar{\Im}_2 \in \Theta_{d_0}$, and for all $\xi \in J$.

$$|(\Upsilon\bar{\Im}_1)(\xi) - (\Upsilon\bar{\Im}_2)(\xi)| \leq \frac{\gimel-2}{B(\gimel-1)}||\bar{\Im}_1(\xi) - \bar{\Im}_2(\xi)||$$

$$+ \frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel-1)\zeta}\int_0^{\dot{\mho}} ((\dot{\mho}-s)^{\gimel-2} - (\dot{\mho}-s)^{(\gimel-1)})|\bar{\Im}_1(s) - \bar{\Im}_2(\xi)|ds$$

$$+ \frac{2-\gimel}{B(\gimel-1)\zeta}\int_0^{\dot{\mho}} (\dot{\mho}-s-1)|\bar{\varpi}(s,\bar{\Im}_1(s)) - \bar{\varpi}(s,\bar{\Im}_2(s))|ds - \frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel-1)}\int_0^{\xi}(\xi-s)^{\gimel-2}\bar{\Im}_1(s)ds$$

$$+ \frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel)\zeta}\int_0^{\dot{\mho}} ((\dot{\mho}-s)^{\gimel} - (\dot{\mho}-s)^{(\gimel-1)})|\bar{\varpi}(s,\bar{\Im}_1(s)) - \bar{\varpi}(s,\bar{\Im}_2(s))|ds$$

$$+ \frac{2-\gimel}{B(\gimel-1)}\int_0^{\xi} |\bar{\varpi}(s,\bar{\Im}_1(s)) - \bar{\varpi}(s,\bar{\Im}_2(s))|ds + \frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel)}\int_0^{\xi}(\xi-s)^{(\gimel-1)}|\bar{\varpi}(s,\bar{\Im}_1(s)) - \bar{\varpi}(s,\bar{\Im}_2(s))|ds,$$

$$\leq \frac{\gimel-2}{B(\gimel-1)}||\bar{\Im}_1 - \bar{\Im}_2|| + \frac{(\gimel-1)||\bar{\Im}_1 - \bar{\Im}_2||}{B(\gimel-1)\Gamma(\gimel-1)\zeta}\left(\frac{\dot{\mho}^{(\gimel-1)}}{(\gimel-1)} + \frac{\dot{\mho}^{\gimel}}{\gimel}\right) + \frac{(2-\gimel)M||\bar{\Im}_1 - \bar{\Im}_2||}{B(\gimel-1)\zeta}\left(\frac{\dot{\mho}^2}{2} - \dot{\mho}\right)$$

$$+ \frac{(\gimel-1)||\bar{\Im}_1 - \bar{\Im}_2||}{B(\gimel-1)\Gamma(\gimel-1)}\left(\frac{\dot{\mho}^{(\gimel-1)}}{(\gimel-1)}\right) + \frac{(\gimel-1)M||\bar{\Im}_1 - \bar{\Im}_2||}{B(\gimel-1)\Gamma(\gimel)\zeta}\left(\frac{\dot{\mho}^{\gimel+1}}{\gimel+1} + \frac{\dot{\mho}^{\gimel}}{\gimel}\right)$$

$$+ \frac{(\gimel-1)M||\bar{\Im}_1 - \bar{\Im}_2||}{B(\gimel-1)\Gamma(\gimel)\zeta}\left(\frac{\dot{\mho}^{\gimel+1}}{\gimel+1} + \frac{(\xi)^{\gimel}}{\gimel}\right) + \frac{(2-\gimel)M||\bar{\Im}_1 - \bar{\Im}_2||\dot{\mho}}{B(\gimel-1)} + \frac{(\gimel-1)M||\bar{\Im}_1 - \bar{\Im}_2||}{B(\gimel-1)\Gamma(\gimel)}\left(\frac{\dot{\mho}^{\gimel}}{\gimel}\right),$$

$$\leq \left[\frac{\gimel-2}{\dot{\mho}^{(\gimel-1)}} + \frac{1}{\Gamma(\gimel-1)\zeta} + \frac{\dot{\mho}(\gimel-1)}{(\gimel)\Gamma(\gimel-1)\zeta} + \frac{1}{\Gamma(\gimel-1)}\right] \frac{\dot{\mho}^{(\gimel-1)}}{B(\gimel-1)}||\bar{\Im}_1 - \bar{\Im}_2||$$

$$+ \left[\frac{(2-\gimel)(2\dot{\mho}^2-\dot{\mho})}{2\zeta} + \left(\frac{(\gimel-1)\dot{\mho}^{\gimel+1}}{(\gimel+1)\Gamma(\gimel)\zeta} + \frac{(\gimel-1)\dot{\mho}^{\gimel}}{\gimel\Gamma(\gimel)\zeta}\right) + (\dot{\mho}-\dot{\mho}(\gimel-1)) + \frac{(\gimel-1)\dot{\mho}^{\gimel}}{\Gamma(\gimel+1)}\right] \frac{1}{B(\gimel-1)}M||\bar{\Im}_1 - \bar{\Im}_2||,$$

$$\leq \left[\frac{\gimel-2}{\dot{\mho}^{(\gimel-1)}}+\frac{1}{\Gamma(\gimel-1)\zeta}+\frac{\dot{\mho}(\gimel-1)}{(\gimel)\Gamma(\gimel-1)\zeta}+\frac{1}{\Gamma(\gimel-1)}\right]\frac{\dot{\mho}^{(\gimel-1)}}{B(\gimel-1)}d_0$$
$$+\left[\frac{(2-\gimel)(2\dot{\mho}^2-1)}{2\zeta\dot{\mho}^{\gimel-1}}+\frac{\gimel(\gimel-1)\dot{\mho}+(\gimel^2-1)}{\Gamma(\gimel+2)\zeta}+\frac{2-\gimel}{\dot{\mho}^{\gimel-1}}+\frac{(\gimel-1)}{\Gamma(\gimel+1)}\right]\frac{\dot{\mho}^{\gimel}}{B(\gimel-1)}Md_0,$$

$$\leq \vartheta_0||\bar{\Im}_1-\bar{\Im}_2||+\vartheta_1 M||\bar{\Im}_1-\bar{\Im}_2||,$$

$$||(\Upsilon\bar{\Im}_1)(\xi)-(\Upsilon\bar{\Im}_2)(\xi)||\leq(\vartheta_0+\vartheta_1 M)||\bar{\Im}_1-\bar{\Im}_2||$$

If $(\vartheta_0+\vartheta_1 M)<1$. Then Υ is a contraction according to Banach's contraction theorem and so it has only one fixed point, which is which is a unique solution of the boundary value problem (**??**eqa1.1)-(**??**eqa1.2). □

Theorem 3.3. Assume that the conditions (**H3**), (**H4**) and (**H5**) are satisfied. If

$$(\vartheta_0+\vartheta_1\Omega_2)<1.$$

Then the exist at least one solution for the boundary value problem (**??**eqa1.1)-(**??**eqa1.2) on J.

Proof. Consider the operator $\Upsilon : C(J,\mathbb{R})\to C(J,\mathbb{R})$ defined by

$$\Upsilon(\bar{\Im})(\xi)=\frac{\gimel-2}{B(\gimel-1)}\bar{\Im}(\xi)+\frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel-1)\zeta}\int_0^{\dot{\mho}}((\dot{\mho}-s)^{\gimel-2}-(\dot{\mho}-s)^{(\gimel-1)})\bar{\Im}(s)ds$$
$$+\frac{2-\gimel}{B(\gimel-1)\zeta}\int_0^{\dot{\mho}}\bar{\varpi}(s,\bar{\Im}(s))(\dot{\mho}-s-1)ds+\frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel)\zeta}\int_0^{\dot{\mho}}\bar{\varpi}(s,\bar{\Im}(s))((\dot{\mho}-s)^{\gimel}-(\dot{\mho}-s)^{(\gimel-1)})ds$$
$$-\frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel-1)}\int_0^{\xi}(\xi-s)^{\gimel-2}\bar{\Im}(s)ds+\frac{2-\gimel}{B(\gimel-1)}\int_0^{\xi}\bar{\varpi}(s,\bar{\Im}(s))ds$$
$$+\frac{(\gimel-1)}{B(\gimel-1)\Gamma(\gimel)}\int_0^{\xi}(\xi-s)^{(\gimel-1)}\bar{\varpi}(s,\bar{\Im}(s))ds,$$

Let $\Theta_{r_0}=\{\bar{\Im}\in C(J,\mathbb{R}):||\bar{\Im}||\leq d_0,\ d_0>0\}$, be a closed bounded convex ball in E, with $\Omega_2=sup_{\xi\in J}z(\xi)$. We shall show that T satisfies the assumptions of Mönch's FPT. The proof will be given in three steps.
Step1. We show that Υ is continuous.

Let $\bar{\Im}_n$ be a sequence such that $\bar{\Im}_n \to y$, in $C(J,\mathbb{R})$ for each $\xi \in J$.

$$
\begin{aligned}
||(\Upsilon\bar{\Im}_n)(\xi) - (\Upsilon\bar{\Im})(\xi)|| \leq{}& \frac{\gimel - 2}{B(\gimel - 1)}||\bar{\Im}_n(\xi) - \bar{\Im}(\xi)|| \\
&+ \frac{(\gimel - 1)}{B(\gimel - 1)\Gamma(\gimel - 1)\zeta}\int_0^{\dot{\mho}}((\dot{\mho} - s)^{\gimel-2} - (\dot{\mho} - s)^{(\gimel-1)})||\bar{\Im}_n(s) - \bar{\Im}(s)||ds \\
&+ \frac{2 - \gimel}{B(\gimel - 1)\zeta}\int_0^{\dot{\mho}}(\dot{\mho} - s - 1)||\bar{\varpi}(s, \bar{\Im}_n(s)) - \bar{\varpi}(s, \bar{\Im}(s))||ds \\
&+ \frac{(\gimel - 1)}{B(\gimel - 1)\Gamma(\gimel)\zeta}\int_0^{\dot{\mho}}((\dot{\mho} - s)^{\gimel} - (\dot{\mho} - s)^{(\gimel-1)})||\bar{\varpi}(s, \bar{\Im}_n(s)) - \bar{\varpi}(s, \bar{\Im}(s))||ds \\
&- \frac{(\gimel - 1)}{B(\gimel - 1)\Gamma(\gimel - 1)}\int_0^{\xi}(\xi - s)^{\gimel-2}||\bar{\Im}_n(s) - \bar{\Im}(s)||ds \\
&+ \frac{2 - \gimel}{B(\gimel - 1)}\int_0^{\xi}||\bar{\varpi}(s, \bar{\Im}_n(s)) - \bar{\varpi}(s, \bar{\Im}(s))|ds \\
&+ \frac{(\gimel - 1)}{B(\gimel - 1)\Gamma(\gimel)}\int_0^{\xi}(\xi - s)^{(\gimel-1)}||\bar{\varpi}(s, \bar{\Im}_n(s)) - \bar{\varpi}(s, \bar{\Im}(s))||ds,
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\gimel - 2}{B(\gimel - 1)}||\bar{\Im}_n(\xi) - \bar{\Im}(\xi)|| + \frac{(\gimel - 1)||\bar{\Im}_n(\xi) - \bar{\Im}(\xi)||}{B(\gimel - 1)\Gamma(\gimel - 1)\zeta}\left(\frac{\dot{\mho}^{(\gimel-1)}}{(\gimel - 1)} + \frac{\dot{\mho}^{\gimel}}{\gimel}\right) \\
&+ \frac{2 - \gimel}{B(\gimel - 1)\zeta}\left(\frac{\dot{\mho}^2}{2} - \dot{\mho}\right)||\bar{\varpi}(\xi, \bar{\Im}_n(\xi)) - \bar{\varpi}(\xi, \bar{\Im}(\xi))|| \\
&+ \frac{(\gimel - 1)}{B(\gimel - 1)\Gamma(\gimel)\zeta}\left(\frac{\dot{\mho}^{\gimel+1}}{\gimel + 1} + \frac{\dot{\mho}^{\gimel}}{\gimel}\right)||\bar{\varpi}(s, \bar{\Im}_n(s)) - \bar{\varpi}(s, \bar{\Im}(s))|| \\
&+ \frac{(\gimel - 1)}{B(\gimel - 1)\Gamma(\gimel)\zeta}\left(\frac{\dot{\mho}^{\gimel+1}}{\gimel + 1} + \frac{\dot{\mho}^{\gimel}}{\gimel}\right)||\bar{\varpi}(\xi, \bar{\Im}_n(\xi)) - \bar{\varpi}(\xi, \bar{\Im}(\xi))|| + \frac{\dot{\mho}^{(\gimel-1)}}{B(\gimel - 1)\Gamma(\gimel - 1)}||\bar{\Im}_n(\xi) - \bar{\Im}(\xi)|| \\
&+ \frac{2 - \gimel\dot{\mho}}{B(\gimel - 1)}||\bar{\varpi}(\xi, \bar{\Im}_n(\xi)) - \bar{\varpi}(\xi, \bar{\Im}(\xi))|| + \frac{(\gimel - 1)}{B(\gimel - 1)\Gamma(\gimel)}\left(\frac{\dot{\mho}^{\gimel}}{\gimel}\right)||\bar{\varpi}(\xi, \bar{\Im}_n(\xi)) - \bar{\varpi}, \bar{\Im}(\xi))||,
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left[\frac{\gimel - 2}{\dot{\mho}^{(\gimel-1)}} + \frac{1}{\Gamma(\gimel - 1)\zeta} + \frac{\dot{\mho}(\gimel - 1)}{(\gimel)\Gamma(\gimel - 1)\zeta} + \frac{1}{\Gamma(\gimel - 1)}\right]\frac{\dot{\mho}^{(\gimel-1)}}{B(\gimel - 1)}||\bar{\Im}_n(\xi) - \bar{\Im}(\xi)|| \\
&+ \left[\frac{2 - \gimel(2\dot{\mho}^2 - 1)}{2\zeta\dot{\mho}^{\gimel-1}} + \frac{\gimel(\gimel - 1)\dot{\mho} + (\gimel^2 - 1)}{\Gamma(\gimel + 2)\zeta} + \frac{2 - \gimel}{\dot{\mho}^{\gimel-1}} + \frac{(\gimel - 1)}{\Gamma(\gimel + 1)}\right]\frac{\dot{\mho}^{\gimel}}{B(\gimel - 1)}||\bar{\varpi}(\xi, \bar{\Im}_n(\xi)) - \bar{\varpi}(\xi, \bar{\Im}(\xi))||,
\end{aligned}
$$

Since, $f\ and\ y$ are of Caratheodory type functions, then by the Lebesgue dominated convergence theorem we have

$$
||(\dot{\mho}\bar{\Im}_n)(\xi) - (\Upsilon\bar{\Im})(\xi)|| \to 0\ as\ n \to \infty,
$$

Step 2. T maps Θ_{r_0} into itself. Let $\bar{\Im} \in \Theta_{d_0}$, by (H4), we have for each $\xi \in J$

$$
\begin{aligned}
||(\Upsilon\bar{\Im})(\xi)|| \leq\ & \frac{\daleth - 2}{B(\daleth - 1)}||\bar{\Im}|| + \frac{(\daleth - 1)||\bar{\Im}||}{B(\daleth - 1)\Gamma(\daleth - 1)\zeta}\int_0^{\dot{\mho}} \left|(\dot{\mho} - s)^{\daleth - 2} - (\dot{\mho} - s)^{(\daleth - 1)}\right| ds \\
& + \frac{2 - \daleth}{B(\daleth - 1)\zeta}\int_0^{\dot{\mho}} (\dot{\mho} - s - 1)|\bar{\varpi}(s, \bar{\Im}(s))|ds \\
& + \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth)\zeta}\int_0^{\dot{\mho}} \left((\dot{\mho} - s)^{\daleth} - (\dot{\mho} - s)^{(\daleth - 1)}\right)|\bar{\varpi}(s, \bar{\Im}(s))|ds \\
& + \frac{(\daleth - 1)||\bar{\Im}||}{B(\daleth - 1)\Gamma(\daleth - 1)}\left|\int_0^{\xi}(\xi - s)^{\daleth - 2}ds\right| + \frac{2 - \daleth}{B(\daleth - 1)}\int_0^{\xi}|\bar{\varpi}(s, \bar{\Im}(s))|ds \\
& + \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth)}\int_0^{\xi}(\xi - s)^{(\daleth - 1)}|\bar{\varpi}(s, \bar{\Im}(s))|ds, \\
||(\Upsilon\bar{\Im})(\xi)|| \leq\ & \left[\frac{\daleth - 2}{\dot{\mho}^{(\daleth - 1)}} + \frac{1}{\Gamma(\daleth - 1)\zeta} + \frac{\dot{\mho}(\daleth - 1)}{(\daleth)\Gamma(\daleth - 1)\zeta} + \frac{1}{\Gamma(\daleth - 1)}\right]\frac{\dot{\mho}^{(\daleth - 1)}}{B(\daleth - 1)}d_0 \\
& + \left[\frac{2 - \daleth(2\dot{\mho}^2 - 1)}{2\zeta\dot{\mho}^{\daleth - 1}} + \frac{\daleth(\daleth - 1)\dot{\mho} + (\daleth^2 - 1)}{\Gamma(\daleth + 2)\zeta} + \frac{2 - \daleth}{\dot{\mho}^{\daleth - 1}} + \frac{(\daleth - 1)}{\Gamma(\daleth + 1)}\right]\frac{\dot{\mho}^{\daleth}}{B(\daleth - 1)}(Md_0), \\
& ||(\Upsilon\bar{\Im})(\xi)|| \leq (\vartheta_0 + \vartheta_1 M)d_0 \leq d_0,
\end{aligned}
$$

which implies that $||(\Upsilon\bar{\Im})(\xi)|| \leq d_0$.
Step3. Show that $T(\Theta_{d_0})$ is equicontinuous. By Step 2, it is clear that $T(\Theta_{d_0}) \subset C(J, \mathbb{R})$ is bounded. Let $\xi_1, \xi_2 \in J, \ \ \xi_1 < \xi_2$ and $\bar{\Im} \in \Theta_{d_0}$, then

$$
\begin{aligned}
||(\Upsilon\bar{\Im})(\xi_2) - (\Upsilon\bar{\Im})(\xi_1)|| \leq & \frac{\daleth - 2}{B(\daleth - 1)}|\bar{\Im}(\xi_2) - \bar{\Im}(\xi_1)| + \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth - 1)}\int_{\xi_1}^{\xi_2}(\xi_2 - s)^{\daleth - 2}|\bar{\Im}(s)|ds \\
& + \frac{2 - \daleth}{B(\daleth - 1)}\int_{\xi_1}^{\xi_2}\bar{\varpi}(s, \bar{\Im}(s))ds \\
& + \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth)}\int_{\xi_1}^{\xi_2}(\xi_2 - s)^{(\daleth - 1)}|\bar{\varpi}(s, \bar{\Im}(s))|ds \\
& + \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth - 1)}\int_0^{\xi_1}\left((\xi_2 - s)^{\daleth - 2} - (\xi_1 - s)^{\daleth - 2}\right)|\bar{\Im}(s)|ds \\
& + \frac{(\daleth - 1)}{B(\daleth - 1)\Gamma(\daleth)}\int_0^{\xi_1}\left((\xi_2 - s)^{(\daleth - 1)} - (\xi_1 - s)^{(\daleth - 1)}\right)|\bar{\varpi}(s, \bar{\Im}(s))|ds
\end{aligned}
$$

$$\leq \frac{\beth-2}{B(\beth-1)}|\bar{\Im}(\xi_2)-\bar{\Im}(\xi_1)| + \frac{(\beth-1)}{B(\beth-1)\Gamma(\beth-1)}\int_{\xi_1}^{\xi_2}(\xi_2-s)^{\beth-2}(s)||\bar{\Im}(\xi)|| + \frac{2-\beth}{B(\beth-1)}\int_{\xi_1}^{\xi_2} dsM||\bar{\Im}(\xi)||$$
$$+ \frac{(\beth-1)}{B(\beth-1)\Gamma(\beth)}\int_{\xi_1}^{\xi_2}(\xi_2-s)^{(\beth-1)}dsM||\bar{\Im}(\xi)||$$
$$+ \frac{(\beth-1)}{B(\beth-1)\Gamma(\beth-1)}\int_0^{\xi_1}\left|(\xi_2-s)^{\beth-2}-(\xi_1-s)^{\beth-2}\right|ds||\bar{\Im}(\xi)||$$
$$+ \frac{(\beth-1)}{B(\beth-1)\Gamma(\beth)}\int_0^{\xi_1}\left|(\xi_2-s)^{(\beth-1)}-(\xi_1-s)^{(\beth-1)}\right|dsM||\bar{\Im}(\xi)||,$$

$$\begin{aligned}||(\Upsilon\bar{\Im})(\xi_2)-(\Upsilon\bar{\Im})(\xi_1)|| \leq & \frac{\beth-2}{B(\beth-1)}|\bar{\Im}(\xi_2)-\bar{\Im}(\xi_1)| + \frac{||\bar{\Im}(\xi)||}{B(\beth-1)\Gamma(\beth-1)}(\xi_2-\xi_1)^{(\beth-1)} \\ & + \frac{2-\beth}{B(\beth-1)}(\xi_2-\xi_1)M||\bar{\Im}(\xi)|| + \frac{(\beth-1)}{B(\beth-1)\Gamma(\beth)}\frac{M||\bar{\Im}(\xi)||}{\beth}(\xi_2-\xi_1)^{\beth} \\ & + \frac{||\bar{\Im}(\xi)||}{B(\beth-1)\Gamma(\beth-1)}\left(|(\xi_2-\xi_1)^{(\beth-1)}|-|(\xi_2)^{(\beth-1)}-(\xi_1)^{(\beth-1)}|\right) \\ & + \frac{(\beth-1)}{B(\beth-1)\Gamma(\beth)}\frac{M||\bar{\Im}(\xi)||}{\beth}\left(\left|(\xi_2-\xi_1)^{\beth}\right|-\left|(\xi_2)^{\beth}-(\xi_1)^{\beth}\right|\right),\end{aligned}$$

$$\begin{aligned}||(\Upsilon\bar{\Im})(\xi_2)-(\Upsilon\bar{\Im})(\xi_1)|| \leq & \frac{\beth-2}{B(\beth-1)}|\bar{\Im}(\xi_2)-\bar{\Im}(\xi_1)| + \frac{2||\bar{\Im}(\xi)||}{B(\beth-1)\Gamma(\beth-1)}(\xi_2-\xi_1)^{(\beth-1)} \\ & + \frac{2-\beth}{B(\beth-1)}(\xi_2-\xi_1)M||\bar{\Im}(\xi)|| + \frac{2(\beth-1)}{B(\beth-1)\Gamma(\beth)}\frac{M||\bar{\Im}(\xi)||}{\beth}(\xi_2-\xi_1)^{\beth} \\ & + \frac{||\bar{\Im}(\xi)||}{B(\beth-1)\Gamma(\beth-1)}\left(|(\xi_2)^{(\beth-1)}-(\xi_1)^{(\beth-1)}|\right) \\ & + \frac{(\beth-1)}{B(\beth-1)\Gamma(\beth)}\frac{M||\bar{\Im}(\xi)||}{\beth}\left|(\xi_2)^{\beth}-(\xi_1)^{\beth}\right|,\end{aligned}$$

As $\xi_2 \to \xi_1$, the right-hand side of the above inequality tends to zero. Then Υy is equicontinuous.
Finally, we indicate that the association holds.:
Assume that $\bar{\Psi}$ be a subset of Θd_0 such that $\bar{\Psi} = \overline{conv}(\Upsilon(\bar{\Psi})\cup 0)$. $\bar{\Psi}$ is bounded and equicontinuous, and this implies that the function $\bar{\Psi} \to v(\xi) = \mu(\bar{\Psi}\xi))$ is continuous on J.
By (**H4**), and the properties of the measure μ we have for each $\xi \in J$.

$$v(\xi) \leq \mu((\Upsilon(\bar{\Psi})(\xi)\cup 0)) \leq \mu(\Upsilon(\bar{\Psi})(\xi))$$

$$\leq \frac{\beth-2}{B(\beth-1)}z(\xi)\mu((\bar{\Psi}(\xi))+\frac{(\beth-1)}{B(\beth-1)\Gamma(\beth-1)\zeta}\int_0^{\dot{\mho}}((\dot{\mho}-s)^{\beth-2}-(\dot{\mho}-s)^{(\beth-1)})\mu((\bar{\Psi}(s))ds$$
$$+\frac{2-\beth}{B(\beth-1)\zeta}\int_0^{\dot{\mho}}(\dot{\mho}-s-1)z(s)zds\mu((\bar{\Psi}(s))ds+\frac{(\beth-1)}{B(\beth-1)\Gamma(\beth)\zeta}\int_0^{\dot{\mho}}((\dot{\mho}-s)^{\beth}-(\dot{\mho}-s)^{(\beth-1)})zds\mu((\bar{\Psi}(s))ds$$
$$-\frac{(\beth-1)}{B(\beth-1)\Gamma(\beth-1)}\int_0^{\xi}(\xi-s)^{\beth-2}\mu((\bar{\Psi}(s))ds+\frac{2-\beth}{B(\beth-1)}\int_0^{\xi}zds\mu((\bar{\Psi}(s))ds$$
$$+\frac{(\beth-1)}{B(\beth-1)\Gamma(\beth)}\int_0^{\xi}(\xi-s)^{(\beth-1)}zds\mu((\bar{\Psi}(s))ds,$$
$$\leq ||v||\left[\frac{\beth-2}{\dot{\mho}^{(\beth-1)}}+\frac{1}{\Gamma(\beth-1)\zeta}+\frac{\dot{\mho}(\beth-1)}{(\beth)\Gamma(\beth-1)\zeta}+\frac{1}{\Gamma(\beth-1)}\right]\frac{\dot{\mho}^{(\beth-1)}}{B(\beth-1)}$$
$$+||v||\Omega_2\left[\frac{2-\beth(2\dot{\mho}^2-1)}{2\zeta\dot{\mho}^{\beth-1}}+\frac{\beth(\beth-1)\dot{\mho}+(\beth^2-1)}{\Gamma(\beth+2)\zeta}+\frac{2-\beth}{\dot{\mho}^{\beth-1}}+\frac{(\beth-1)}{\Gamma(\beth+1)}\right]\frac{\dot{\mho}^{\beth}}{B(\beth-1)},$$
$$||v||\leq(\vartheta_0+\vartheta_1\Omega_2)||v||.$$

This means that

$$||v||(1-(\vartheta_0+\vartheta_1\Omega_2))\leq 0,$$

But is assumed that $(\vartheta_0+\vartheta_1\Omega_2)<1$, which implies that $||v||=0$, that is $\bar{\Psi}(\xi)=0$ for each $\xi\in J$, and then $\bar{\Psi}(\xi)$ is relatively compact in R. Now, Ascoli-Arzela theorem is applied then, we obtain $\bar{\Psi}$ is relatively compact in Θ_{d_0}. Applying now Theorem (**??**monch), we conclude that the operator Υ has a fixed point which is a solution of the problem (1. 1)-(1. 2). □

4. EXAMPLES

In this section we take an example to illustrate our results.

Example 4.1. *Let us consider the following boundary value problem*

$$\bar{\Im}'(\xi)+^{ABC}D_0^{\frac{3}{2}}\bar{\Im}(\xi)=\frac{\xi^2-6}{4(1+2e^{\xi})}\frac{|\bar{\Im}(\xi)|}{1+|\bar{\Im}(\xi)|}\qquad \xi\in[0,3] \tag{4. 4}$$

with the boundary condition

$$\begin{cases}\bar{\Im}'(0)=0.\\ \bar{\Im}(2)=\int_0^2\bar{\Im}(s)ds\end{cases} \tag{4. 5}$$

Where $\beth=\frac{4}{3}$, $B(\beth-1)=1$ *and* $\dot{\mho}=3$ *by using (H2) the result is*

$$|\bar{\varpi}(\xi,\bar{\Im}_1)-\bar{\varpi}(\xi,\bar{\Im}_2)|\leq\frac{\xi^2-6}{4(1+2e^{\xi})}|\bar{\Im}_1-\bar{\Im}_2|,$$
$$M=0.0182,$$
$$\vartheta_0=0.328$$
$$\vartheta_1=11.8401$$

Then, by Theorem(3.2) the following is obtained

$$||(\dot{\mho}\bar{\Im}_1)(\xi)-(\dot{\mho}\bar{\Im}_2)(\xi)||\leq(\vartheta_0+\vartheta_1M)||\bar{\Im}_1-\bar{\Im}_2||,\quad(\vartheta_0+\vartheta_1M)=0.5585<1.$$

Then the problem (4. 4) has a unique solution.

5. Conclusion

This study successfully establishes the existence and uniqueness of Atangana-Baleanu solutions for FDEs in the Caputo sense. By applying Mönch's fixed point theorem, we demonstrated the conditions under which solutions exist, and we employed the Banach contraction principle to showcase the uniqueness of these solutions. The theoretical insights presented are further validated through a practical example, illustrating how various fractional orders impact the behavior of the solutions. Our findings contribute to the understanding of FDEs and provide a solid foundation for Beyond its immediate findings, the work sets the stage for future investigations into more complex systems that employ AB-FDEs, encouraging further exploration into their properties and applications. This opens avenues for interdisciplinary research and fosters collaboration among mathematicians and scientists in fields where fractional calculus plays a significant role.

6. CONFLICT OF INTEREST:

The author declares no conflict of interest

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