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Sugeno capacities for the extension of exponential and hyperbolic discounting to fuzzy integrals

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Abstract. This article produces a blend between the fields of time discounting and fuzzy integrals. By exploiting the mathematical properties of the Sugeno capacities, we can produce a Choquet integral that simultaneously is subadditive and coincides with either the exponential or hyperbolic discounting additive formulas in the evaluation of individual time moments. Numerical analyses guarantee that the new valuation procedures are different from their additive counterparts.

AMS (MOS) Subject Classification Codes: 91B16; 91B06; 28E10
Key Words: Sugeno capacity, fuzzy measure, Choquet integral, exponential discounting, hyperbolic discounting.

1. Introduction

This research is at the junction of two fields, namely, time discounting (motivated by mathematical finance) and the discrete Choquet integral (motivated by the theory of aggregation operators).

In mathematical finance, time discounting refers to the process of determining the present value of future endowments such as cash flows. This is a fundamental concept because assets (such as money) received in the future are generally considered less valuable than the same assets received today, due to factors such as inflation, risk, and in the monetary case, the potential to invest that money and earn a return. Put succinctly, the basic idea is that a dollar received today is worth more than a dollar received in the future.

A natural problem that arises is the calculation of the "present value" of a future cash flow. Classical procedures make use of a certain rate, known as the "discount rate". The standard Discounted Utility model, which assumes exponential discounting, became the canonical model since its introduction by renowned economists [5, 16].

This problem is a specific setting for an aggregation operator, with particular restrictions imposed by the precise framework (which is temporal). Aggregation operators are fundamental not only in (temporal or otherwise) decision-making, but also in fields such as data analysis and information fusion. Additive aggregation embeds the type of operators mentioned above. But many important classes of aggregation operators relax or dispense with this property. The Choquet integral is a remarkable example. It enables us to incorporate interactions between the inputs (typically, characteristics of the elements, or opinions of agents or experts). Whenever we feel that the importance of a set of inputs is not simply the sum of the importance of each individual input, we can use capacities or fuzzy measures. They are non-additive measures that let us capture interactions, such as synergies or redundancies, between inputs.

1.1. **Related work.** Several experiments and studies have collectively shown that the Discounted Utility model often fails to capture the complexity of human intertemporal choices [14, 18, 20]. There is evidence that hyperbolic discounting and other models that account for time-inconsistent patterns provide a more accurate description of how people value future rewards [8, 11]. Nevertheless, paradoxes remain that require further explanations.

Additivity may be at the root of the inconvenience of existing models of time discounting. The subadditivity property seems to be more suitable than an additive expression in many different areas. For example, experimental demonstrations of the subadditivity of *probability* judgements exist [17]. Possibly the first scholars who found evidence of subadditive *pricing* were Kahneman and Knetsch [9]. In relation with our goals, Daniel Read [15] convincingly argued that time discounting is subadditive, and also that for this reason, subadditivity should supersede hyperbolic discounting.

The Choquet integral is more flexible than weighted sums (which are the particular case where the capacity is additive) or other linear aggregation methods. This is particularly useful in decision-making processes where the combined effect of multiple factors is more complex than the sum of their individual effects. And remarkably, Choquet integrals can exhibit subadditivity. Therefore if we assume that time discounting must be subadditive, then we can only agree that the Choquet integral provides an excellent opportunity to determine the present value of future endowments. A crucial step in this case is the elicitation of the right form of the capacity. The property that a capacity must satisfy to produce Choquet evaluations that are subadditive was identified by Choquet [6]. We will take advantage of this characterization, and also of other theoretical contributions about capacities, to go beyond additive discounting models without rejecting their basic premises.

In this regard, one handicap of this general approach is that 2^n values need to be specified for general capacities defined on a set with n elements. We shall address this issue by focusing on Sugeno's capacities, also called λ-fuzzy measures. This class is interesting in the first place, because the computational complexity of evaluating a capacity in this class is much lower compared to general capacities. Any λ-fuzzy measure can be fully determined by specifying its evaluation for the n singletons plus the value of the parameter λ. This means a considerable reduction of the computational burden. In the second place, because existing literature allows us to disclose the exact situations for which Choquet integrals defined from λ-fuzzy measures are subadditive.

1.2. **The goals of this research.** The main research premise of this study is easy to explain. If I know my future assignments, my perception now about how valuable they are can be affected by interactions among different periods. These interactions may take the form of either redundancies or synergies. They can be due to the presence of e.g., annual or year-end bonus in periods different from those where the endowments will be given.

The Discounted Utility and hyperbolic models mentioned above are insensitive to such cross-effects: the future value of an amount that will be obtained in the future is always updated with the same rate of discount, irrespective of other assignments in different periods.

So the challenge is: can we combine both traits –the essence of the classical evaluations in time discounting models, and the existence of interactions among periods– using a Choquet integral? And, can we do it efficiently, i.e., with a reduced computational burden?

1.3. **Outline of this research.** Section 2 revisits the two fields that inspire our research. The fundamental theoretical results that we need are stated there. Then in section 3 we give our results, both at theoretical and numerical levels. A fully developed example illustrates all the computations that are needed to solve the problem in section 4. Section 5 concludes this article.

2. Background

This section revises concepts and results that will be needed in the rest of this article. They pertain to two different fields, namely, the evaluation of temporal streams of endowments and the Choquet integral and its properties.

2.1. **Time discounting.** In the discounted utility or exponential discounting model, discounted utility assessments of choices made in the present and consecutive instants in the future, $(x_0, x_1, \ldots, x_T)$, have the following additive form: for a fixed $\beta \in (0, 1)$,

$$\sum_{t=0}^{T} \beta^t u(x_t) \tag{2. 1}$$

At moment t, x_t is chosen, and its utility at that moment is $u(x_t)$. Then a discount factor β (e.g., $\beta = 0.95$) is applied to update its "present value". Larger values of β mean a more patient attitude: in fact, the extreme case $\beta = 1$ is obviously associated with "no discount" and for this reason it is discarded.

We will not be concerned with the role of the utility u, which is independent of time, so in this article we work directly on the evaluation of $(u(x_0), \ldots, u(x_T))$. Therefore the relevant formula becomes $U_\beta(x_0, \ldots, x_T) = \sum_{t=0}^{T} \beta^t x_t$.

The hyperbolic discounting model is governed by the following additive expression:

$$H_k(x_0, \ldots, x_T) = \sum_{t=0}^{T} \frac{1}{1+kt} x_t. \tag{2. 2}$$

Now the k factor determines the degree of discounting (for example, the interest rate). The larger the value of k, the more impulsivity of the behavior.

Contrary to the case of exponential discounting, hyperbolic discounting allows for time-inconsistent patterns such as those revealed by experiments.

2.2. **Capacities and the Choquet integral: discrete versions.** Let $X = \{1, \ldots, n\}$ represent either a set of n attributes (in multi-criteria decision making) or experts (in group decision making), or the results of an event with n possible outcomes. In our context, X represents periods of time.

We need the following concept to define the indices that evaluate temporal strems in the presence of interactions among periods:

Definition 2.3 (Beliakov et al. [4], Definition 2.75)**.** *A discrete fuzzy measure (or a capacity) is a set function $\mu : 2^X \longrightarrow [0, 1]$ which is monotonic (i.e., $\mu(A) \leqslant \mu(B)$ whenever $A \subseteq B \subseteq X$) and satisfies $\mu(\varnothing) = 0$, $\mu(X) = 1$.*

The capacity satisfies additivity if it is the case that when $A, B \subseteq X$ are disjoint, then $\mu(A \cup B) = \mu(A) + \mu(B)$. Additive set functions are uniquely determined by $\mu(\{1\}), ..., \mu(\{n\})$. Thus an additive capacity is a probability measure that is determined by $\mu(\{1\}), ..., \mu(\{n\})$.

A weaker requirement is subadditivity. This property holds if whenever $A, B \subseteq X$ are disjoint then $\mu(A \cup B) \leqslant \mu(A) + \mu(B)$.

Subadditive capacities are submodular. Submodularity or strong subadditivity holds if whenever $A, B \subseteq X$, it is the case that $\mu(A \cup B) + \mu(A \cap B) \leqslant \mu(A) + \mu(B)$.

Supermodular/superadditive capacities are defined by reversing inequalities above. We do not refer to these classes of capacities in our work,

Now we define an interesting type of fuzzy measures whose main attractive is the small amount of information that we must elicit to define them. As explained above, we only need to know their evaluation on singletons, plus one conveniently chosen parameter.

Definition 2.4. *A discrete fuzzy measure μ on X is a Sugeno fuzzy measure, also known as a λ-fuzzy measure, when for a fixed $\lambda \in (-1, +\infty)$, the equality $\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\, \mu(A)\, \mu(B)$ holds for all disjoint $A, B \subseteq X$.*

The case $\lambda = 0$ produces additive fuzzy measures, hence probability measures.

When $\lambda \neq 0$, the evaluation of any subset of X follows from $\mu(\{1\}), ..., \mu(\{n\})$ with the help of the following expression, which can be found e.g., in [13, Eq. (2.4)] and [4, Eq. (2.15)]:

$$\mu(A) = \frac{1}{\lambda}\Big(\prod_{j=1}^{k}(1 + \lambda\mu(\{i_j\})) - 1\Big) \text{ when } A = \{i_1, \ldots, i_k\}. \tag{2. 3}$$

Of course, the value of λ cannot be arbitrarily chosen. The following result is well known (Leszczyński, Penczek and Grochulski [13]): the value of λ is related to $\{\mu(\{i\})\}_{i \in X}$ by the following normalization condition (for which $\lambda = 0$ is always a solution)

$$\lambda + 1 = \prod_{j=1}^{n}(1 + \lambda\mu(\{j\})). \tag{2. 4}$$

This result is described in for example, [3, section 2.9], [7, sect. 2.8.6] or [19, Theorem 4.7].

Let us now define the main analytical tool in this article.

Definition 2.5. *The discrete Choquet integral with respect to the capacity μ is $C^{\mu} : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ given by*

$$C^{\mu}(x_1, \ldots, x_n) = \sum_{i=1}^{n} \left[x_{(i)} - x_{(i-1)}\right] \mu(H_i),$$

where $\boldsymbol{x}_{\nearrow} = (x_{(1)}, \ldots, x_{(n)})$ is a non-decreasing permutation of $\boldsymbol{x} = (x_1, \ldots, x_n)$, $x_{(0)} = 0$ by convention, and $H_i = \{(i), \ldots, (n)\}$ is the set of indices corresponding to the largest $n - i + 1$ components of $\boldsymbol{x}$.

There is another equivalent definition that uses the concept of derivative of the capacity, but we do not need it here so we skip it. The reader can consult [3, section 5.2] for details and other properties of this aggregation operator.

Figure 1 helps us to visualize how the formula in Definition 2.5 evaluates a vector in a simple case with three periods, i.e., $X = \{1, 2, 3\}$.

We say that C^{μ} is subadditive when $C^{\mu}(\mathbf{x}+\mathbf{y}) \leqslant C^{\mu}(\mathbf{x}) + C^{\mu}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$. The following results are well known:

Theorem 2.6 (Choquet [6]). *C^{μ} is convex if and only if μ is submodular.*

And in this case, C^{μ} is subadditive.

The second statement is easy to prove using the positive homogeneity of the Choquet integral [3, section 5.2]. We are not aware of a self-contained, direct proof of the first equivalence in the discrete case. In addition

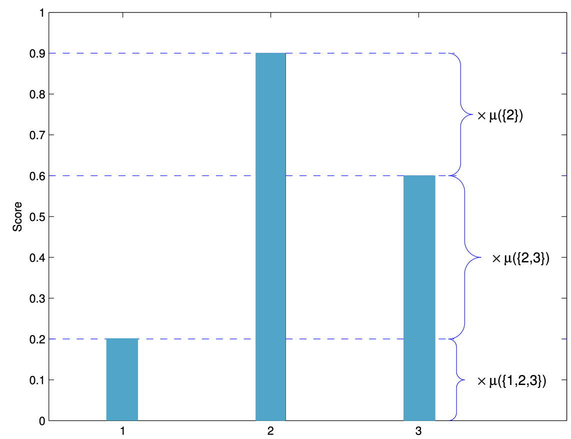

FIGURE 1. Visual interpretation of the application of the Choquet integral in an example with $X = \{1, 2, 3\}$: evaluating the vector $(0.2, 0.6, 0.9)$.

to Choquet himself, other authors considered the continuous statement, e.g., Alfonsi [2]. Many other results relating convexity and integrals exist [10, 12].

Interestingly to our purposes: in the context of the study of λ-fuzzy measures, [19, Theorem 3.6] states the following practical property.

Theorem 2.7 (Wang and Klir [19]). *Suppose that* $\{\mu(\{i\})\}_{i\in X}$ *are fixed,* $\mu(\{i\}) \in [0, 1]$ *for each* $i \in X$. *Then:*

(1) *When* $\sum_{i\in X} \mu(\{i\}) > 1$*, there is exactly one* $\lambda \in (-1, 0)$ *that satisfies the normalization condition defined by Eq. (2. 4), allowing to produce a* λ*-fuzzy measure* μ *on* X *whose values on singletons are* $\{\mu(\{i\})\}_{i\in X}$.
In this case, μ *is submodular.*
(2) *When* $\sum_{i\in X} \mu(\{i\}) < 1$*, there is exactly one* $\lambda > 0$ *that satisfies the normalization condition defined by Eq. (2. 4).*
In this case, μ *is supermodular.*

3. THE PROBLEM AND RESULTS

We are ready to state the problem we want to address formally.

For a fixed temporal horizon T formed by the periods $X = \{1, 2, \ldots, T\}$, we assume that either the exponential or hyperbolic formula define $\{\mu(\{i\})\}_{i=1}^{T}$. Hence we have a fixed horizon setup with "quanta" of time. This is only natural, as discounted sums assume quantization of time as well.

The question arises: *When can we define a Choquet integral that is subadditive from a* λ*-fuzzy measure* μ *such that its values on singletons are* $\{\mu(\{i\})\}_{i=1}^{T}$*?*

Note that when we use additive expressions ($\lambda = 0$), and we dispense with the normalization condition, then the solution that we obtain is precisely the standard formulas of the exponential or hyperbolic discounting. Hence if we solve the problem posed above, then we will be producing a subadditive expression from the same information on the evaluation of individual moments of time, but with the property for which Read [15] advocated.

This solution will produce a subadditive alternative to the exponential/hyperbolic discounting formulas, that is at the same time efficient (because it is calculated from a computationally simple capacity).

We proceed to set forth the algebraic conditions that permit to solve this question. Then we will present the results of numerical computations that solve the algebraic problems that stem from various time horizons and values of the discount rates.

3.1. **Algebraic solution.** To solve the problem stated above, the combination of Theorems 2.6 and 2.7 shows that we need to work exactly with the cases that guarantee $\sum_{i \in X} \mu(\{i\}) > 1$. When T is such that this is true, then we can assure that one $\lambda \in (-1, 0)$ satisfies the normalization condition given in Eq. (2. 4). Therefore the value of λ depends on both the model and time horizon. Also, we know that the λ-capacity that is thus defined must be submodular, by Theorem 2.7. Therefore this λ-capacity must define a subadditive Choquet integral, by Theorem 2.6.

To summarize: the normalization condition allows us to compute λ that produces a λ-capacity, and when $\sum_{i \in X} \mu(\{i\}) > 1$, the capacity that is defined yields a subadditive evaluation of the temporal vectors by the Choquet integral associated with the capacity.

The next section is dedicated to study when we can follow this steps.

3.2. **Numerical computations.** In this section we perform computer-assisted numerical experiments for the deteminarion of the values of λ that allow us to solve the problem we have posed, whenever it admits a solution. We do this separately for the hyperbolic and the exponential formulation of the problem. Of course, we must confine ourselves to a reasonable list of time horizons and parameters.

3.2.1. *The case of hyperbolic discounting.* For each temporal horizon T and rate k, the values of the λ parameter that extend the hyperbolic assessment with a λ-capacity can be computed numerically. We only need to solve Eq. (2. 4) when the values of $\{\mu(\{i\})\}_{i=1}^{T}$ are given by Eq. (2. 2). In this way, each horizon T and rate k yield a Sugeno capacity μ_k^T, which in turn produces the desired evaluation $C^{\mu_k^T}$.

Table 1 summarizes the findings for various numbers of periods. In this table, the arrow points at more impulsivity. One can observe that unless we are dealing with pathological situations showing an extreme lack of impulsivity, the problem admits a unique solution.

	$T=3$	$T=4$	$T=5$	$T=6$	$T=10$	
$k=0.95$	-0.304416	-0.581067	-0.703793	-0.772405	-0.884439	$\uparrow$
$k=0.9$	-0.377174	-0.627029	-0.737797	-0.799603	-0.9	
$k=0.5$	-0.820551	-0.902255	-0.937265	-0.95592	-0.983306	
$k=0.1$	-0.996259	-0.99898	-0.999664	-0.999875	-0.999995	
$k=0.05$	-0.999423	-0.999906	-0.999981	-0.999996	$\times$	
$k=0.01$	-0.999994	$\times$	$\times$	$\times$	$\times$	

$\times$ = too close to -1 to be useful.

TABLE 1. A summary of the values of λ obtained for the problem posed by hyperbolic discounting. Each T and k define μ_k^T, the λ-fuzzy measure associated with the value in the table.

3.2.2. *The case of exponential discounting.* Similarly to the case of section 3.2.1, for each temporal horizon T and discount rate β, the values of the λ parameter that extend the exponential assessment with a λ-capacity can be computed numerically too. To this purpose, now we solve Eq. (2. 4) when $\{\mu(\{i\})\}_{i=1}^{T}$ are given by Eq. (2. 1). In this way, each horizon T and rate β yield a Sugeno capacity $\bar{\mu}_{\beta}^{T}$, which in turn produces the desired evaluation $C^{\bar{\mu}_{\beta}^{T}}$.

Table 2 summarizes the findings for various numbers of periods. In this table, the arrow points at more patience.

We should remember that positive values are associated with superadditive behavior, for which no experimental evidence has been reported. Table 2 is therefore qualitatively different from Table 1: in the case of Table 2, both superadditive and subadditive behaviors are possible, for each number of periods. The patience threshold where the behavior changes, for each time horizon, is in the first row of Table 2. The numerical evidence seems to guarantee that for values $\beta > 0.5$, the problem admits a unique solution.

	$T=3$	$T=4$	$T=5$	$T=6$	$T=10$
Threshold	0.543689	0.51879	0.50866	0.504138	0.500245
$\beta=0.95$	-0.999288	-0.99987	-0.999971	-0.999992	$\times$
$\beta=0.9$	-0.994377	-0.998171	-0.999265	-0.999658	-0.99996
$\beta=0.55$	-0.0615814	-0.286217	-0.374087	-0.415062	-0.455633
$\beta=0.51$	$+0.402054$	$+0.09892$	-0.014924	-0.065447	-0.110392
$\beta=0.4$	$+3.31515$	$+2.6162$	$+2.39183$	$+2.30907$	$+2.25726$

Last row presented for illustration (superadditive behavior, no evidence).

TABLE 2. A summary of the values of λ obtained for the problem posed by exponential discounting. Each T and β define $\bar{\mu}_{\beta}^{T}$, the λ-fuzzy measure associated with the value in the table.

4. EXAMPLE

For both comparison and illustration, we consider a time horizon with 3 future periods. We compute the evaluations of four vectors, namely, $\mathbf{x}_1 = (14, 9, 12)$, $\mathbf{x}_2 = (3, 7, 1)$, $\mathbf{x}_3 = (6, 11, 9)$, and $\mathbf{x}_4 = (17, 16, 13)$, by four models:

(1) The hyperbolic discounting model with $k = 0.9$.
(2) The exponential discounting model with $k = 0.9$.
(3) The model developed in this article from the hyperbolic discounting model with $k = 0.9$, i.e., $C^{\mu_{0.9}^{3}}$.
(4) The model developed in this article from the exponential discounting model with $\beta = 0.9$, i.e., $C^{\bar{\mu}_{0.9}^{3}}$.

The next standard presentation defines $\mu_{0.9}^{3}$, the λ-capacity designed from $\mu(\{1\}) = \frac{1}{1+0.9\cdot 1} = 0.526316$, $\mu(\{2\}) = \frac{1}{1+0.9\cdot 2} = 0.357143$, and $\mu(\{3\}) = \frac{1}{1+0.9\cdot 3} = 0.27027$. In this case, we need to appeal to $\lambda = -0.377174$ (cf., Table 1) and so we get the full expression of $\mu_{0.9}^{h}$, namely,

$$\begin{array}{ccc} & 0 & \\ 0.812561 & 0.742934 & 0.591006 \\ 0.52631 & 0.357143 & 0.27027 \\ & 0 & \end{array}$$

The next standard presentation defines $\bar{\mu}^3_{0.9}$, the λ-capacity designed from $\mu(\{1\}) = 0.9^1 = 0.9$, $\mu(\{2\}) = 0.9^2 = 0.81$, and $\mu(\{3\}) = 0.9^3 = 0.729$: Now we need to appeal to $\lambda = -0.994377$ (cf., Table 2) and so we get the full expression of $\bar{\mu}^3_{0.9}$, namely,

$$\begin{array}{ccc} & 0 & \\ 0.985099 & 0.976589 & 0.95183 \\ 0.9 & 0.81 & 0.729 \\ & 0 & \end{array}$$

Table 3 summarizes the results of our computations. All four models coincide to recommend the ranking $\mathbf{x}_4 \succ \mathbf{x}_1 \succ \mathbf{x}_3 \succ \mathbf{x}_2$.

Vector	$(14, 9, 12)$	$(3, 7, 1)$	$(6, 11, 9)$	$(17, 16, 13)$
Hyperbolic discounting model, $k = 0.9$	13.826	4.34922	9.5189	18.1752
Exponential discounting model, $\beta = 0.9$	28.638	9.099	20.871	37.737
Extension of hyperbolic discounting model $C^{\mu^3_{0.9}}$	12.2814	4.05369	8.4873	15.964
Extension of exponential discounting model $C^{\bar{\mu}^3_{0.9}}$	13.7298	6.2102	10.4755	16.8553

TABLE 3. A summary of the evaluations produced from 4 different models for time discounting, for four vectors with a time horizon $T = 3$ and parameter 0.9.

We note that $(17, 16, 13) = (14, 9, 12) + (3, 7, 1)$. As prescribed by the subadditivity property of both $C^{\mu^3_{0.9}}$ and $C^{\bar{\mu}^3_{0.9}}$, we confirm by inspection of Table 3 that

$$15.964 = C^{\mu^3_{0.9}}(17, 16, 13) \leqslant C^{\mu^3_{0.9}}(14, 9, 12) + C^{\mu^3_{0.9}}(3, 7, 1) = 12.2814 + 4.05369$$

and

$$16.8553 = C^{\bar{\mu}^3_{0.9}}(17, 16, 13) \leqslant C^{\bar{\mu}^3_{0.9}}(14, 9, 12) + C^{\bar{\mu}^3_{0.9}}(3, 7, 1) = 13.7298 + 6.2102.$$

We must emphasize that despite the analysis above, it is not always the case that the models studied in this section produce the same rankings. To see this, we consider the vector $\mathbf{x}_5 = (7.5, 10, 8)$. Table 4 summarizes the computations of its present value, for the four models studied here. By comparison with Table 3, we note that the hyperbolic model and the extension proposed in this article coincide to recommend $\mathbf{x}_5 \succ \mathbf{x}_3$. The other two models recommend $\mathbf{x}_3 \succ \mathbf{x}_5$.

Vector	$\mathbf{x}_5 = (7.5, 10, 8)$	$\mathbf{x}_6 = (4, 6, 2)$	$\mathbf{x}_7 = (2, 7, 2)$
Hyperbolic discounting model, $k = 0.9$	9.68096	4.78866	4.09317
Exponential discounting model, $\beta = 0.9$	20.682	9.918	8.928
Extension of hyperbolic discounting model $C^{\mu^3_{0.9}}$	8.50979	4.33941	3.78571
Extension of exponential discounting model $C^{\bar{\mu}^3_{0.9}}$	9.59592	5.5902	6.05

TABLE 4. A summary of the evaluations produced from 4 different models for time discounting, for three vectors with a time horizon $T = 3$ and parameter 0.9.

There is another fact that becomes apparent from the comparison between Tables 3 and 4. Observe the contrast between $\mathbf{x}_6 = (4, 6, 2)$ and $\mathbf{x}_2 = (3, 7, 1)$. The first three criteria in the table recommend $\mathbf{x}_6 \succ \mathbf{x}_2$. However our extension of the exponential discounting model recommends the opposite. In addition, observe the comparison between $\mathbf{x}_6 = (4, 6, 2)$ and $\mathbf{x}_7 = (2, 7, 2)$. Although the exponential discounting model recommends $\mathbf{x}_6 \succ \mathbf{x}_7$, the extension of the model presented in this paper recommends the opposite. These two situations ensure that our subadditive extensions do not necessarily replicate the rankings established by the corresponding additive models of time discounting.

5. Conclusion

We found that a Choquet integral can be defined from either the exponential or hyperbolic discounting formulas, without requiring additional information. The solution hinges on the fundamental properties of the Sugeno fuzzy measures. By doing so we have found a structural difference between both models: all feasible solutions for the hyperbolic model are subadditive, whereas the exponential case allows for both subadditive and superadditive evaluations (for any fixed horizon). We have emphasized that our models are different from existing evaluations not only in the computation of present values, but also in the comparison between streams of future endowments.

A setback of the approach we have followed in this article is that the addition or removal of periods forces us to redo all calculations, since the value of λ changes. This modification obliges to recalculate the capacity, therefore the Choquet evaluation of the streams with more or fewer periods.

As a promising line for future research, we note that the problem that we posed can be defined for arbitrary values $\mu(\{1\})$, ... , $\mu(\{T\})$, i.e., for values not necessarily determined by the formulas of time discounting. It may also be interesting to consider other properties of the desired aggregation operator, instead of subadditivity. Aggregation operators have been applied in fuzzy models and its extensions [1], so itt is reasonable to expect interpretations in this framework too.

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Conflict of Interest

The author declares no conflict of interest.

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