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Bipolar Complex Fuzzy Rough Sets and Their Applications in Multicriteria Decision Making

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Abstract. Bipolar complex fuzzy set (BCFS) is a more advanced and powerful phenomenon as it consists of two-dimensional data with positive and negative impacts of an element. It can solve the data consisting of the positive and negative impacts of an element which is a bipolar fuzzy set (BFS). It also covers the two-dimensional complex data which is a complex fuzzy set (CFS). Due to these attributes, BFS and CFS are less useful in comparison with BCFS to capture vagueness, complexity, and ambiguity in the data. Furthermore, lower and upper approximations based on equivalency relations constitute another significant phenomenon known as rough set (RS). This structure is also more powerful in dealing with real-life dilemmas. Rather than comparing the RS and BCFS, we combine both phenomena to handle the complexity more powerfully to deal with such types of phenomena that are not handled by other structures. So, by combining both phenomena, we introduce a novel structure known to be bipolar complex fuzzy rough set (BCFRS) in this manuscript. After that, we define some important operations, some significant properties related to this structure, and some aggregation operators (AOs) to solve decision-making (DM) problems related to cyber security. We address a practical application of cyber security (C-S) in computing for the protection of critical data to demonstrate the usefulness of the multi-attribute DM (MADM) approach. Based on the various criteria and attributes given by the experts, we find the best and better alternative to the C-S by applying the MADM approach. We get the $\widetilde{A}_4$ as the best and finest alternative by using bipolar complex fuzzy rough (BCFR) weighted arithmetic averaging (BCFRWAA), BCFR ordered weighted arithmetic averaging (BCFROWAA), and BCFR ordered weighted geometric averaging (BCFROWGA) operators. And, by using BCFR weighted geometric averaging (BCFRWGA), we get the $\widetilde{A}_3$ as the finest alternative. Lastly, to prove the superiority, validity, and generalization of our unique established theory, we give a detailed comparative study of our established work with several prevalent theories.

Keywords: Rough set; Bipolar complex fuzzy set; Bipolar complex fuzzy relation; Bipolar complex fuzzy rough set; Aggregation operators; MADM; Cyber security.

1. Introduction

Background. The term "security" concerning computers, digital data, and networks describes the safeguards and procedures implemented to keep data and systems safe against unauthorized use, openness, modification, damage, and disturbance. Ensuring the accessibility, privacy, and security of data and resources is security's main objective. Cyber security (C-S) is the procedure of preventing malicious attacks on networks, PCs, servers, mobile devices, and other various electronic systems. IT security or electronic information security are other names for C-S. C-S stands for cyber-attacks protection system, network, devices, and data. It is achieved by applying technologies, processes, and control. Its objectives are to decrease the danger of cyber-attacks and guard against the illegal use of networks, systems, and technologies. It also involves safeguarding the networks, programs, and systems from online threats. Attackers typically use ransomware to demand money from individuals or disrupt regular corporate operations in an attempt to obtain, alter, and damage confidential information. As there are currently more technology devices than people and attackers are becoming more skilled, implementing effective C-S measures is highly challenging. Multiple layers of protection are used over the computer, networks, application, or data that one wishes to keep safe in a successful C-S method. To effectively defend against cyber-attacks, an organization's people, procedures, and technology must work in collaboration with one another. Further, it serves as a defense against destructive hacker and cyber-criminal attacks against internet-connected devices and services. Businesses employ these techniques to guard against financial losses, identity theft, phishing schemes, and data breaches. To defend digital data from various crimes, several C-S techniques are employed. Within the framework of the C-S systems, digital transformation and C-S challenges for business resilience were defined by Saeed et al. [42]. C-S technologies essential in the digital transformation era were suggested by Ohkubo [32]. Okoye et al. [33] examined the review of C-S challenges for securing financial data storage. A systematic review of the C-S was proposed by Perwej et al. [37] . A survey of C-S for digital manufacturing was invented by Mahesh et al. [25].

The DM procedure known as MADM involves assessing and choosing alternatives based on many factors. It was made to help people or organizations make difficult decisions by considering different elements and their relative importance. Decisions cannot often be made simply on a single criterion or factor process in real-world circumstances. Rather, several factors are involved, including cost, worth, hazards, long-term viability, and so forth. MADM enables decision-makers to simultaneously evaluate and compare several options depending on several dimensions. This gives a thorough overview of the possibilities and assists in choosing the best one. There are several stakeholders with various interests and goals in many DM circumstances. By considering their priorities and weights for various aspects, MADM enables the assimilation of various stakeholders' perspectives. By considering several factors, it makes it easier to estimate the risks associated with various possibilities. Several MADM and multicriteria DM (MCDM) strategies are used to handle DM difficulties such as the ""Multi-attribute border approximation area comparison"(MABAC) approach introduced by Pamucar and Cirovic [35] as a way to solve MCGDM difficulties. An extended EDAS approach for fuzzy MCDM is described by Ghorabaee et al. [13]. Opricovic and Tzeng [34] provided a comparison between VIKOR and "Technique for Order Preference by Similarity to Ideal Solution" (TOPSIS). Marbini and Tavana [15] interpreted an extension of the ELECTRE I approach for group DM in a fuzzy context. The "Multi-Objective Optimization by Ratio Analysis" (MOORA) method and its application to privatization in a transition economy are explained by Brauers and Zavadskas [9]. The PROMETHEE method was developed by Brans et al. [8] as a means of choosing and ranking projects.

In 1965, a famous mathematician Zadeh [50] investigated a novel concept which is the extension of the crisp set known to be fuzzy set (FS). As the Degree of membership is more common than strict boundaries in many circumstances, this framework provides a flexible way to describe such uncertainty. This concept was used by many researchers in different fields such as an advanced review of FS theory introduced by Zimmermann [52]. The application FS in artificial intelligence was developed by Kandel and Schneider [21]. Applications of FS theory were established by Maiers and Sherif [29]. Still, there are several shortcomings and restrictions with FS that it cannot handle the non-membership degree. An extended fuzzy structure known as intuitionistic FS (IFS) developed by Atanassov [5] can handle the DOM and non-membership degree. This structure extends the FS because it can handle the membership degree as well as the non-membership degree. When a given attribute has both positive and negative features, the DOM cannot fully convey ambiguity. The positive and negative aspects of membership are not identified by FS. To get over these obstacles and challenges, Zhang [51] investigated the idea of bipolar FS (BFS). This structure's primary characteristic is its ability to express membership in two ways, combining the positive and negative DOM. Conflicting information can be expressed using the BF structure. Further in the model of BFS, an extension of BFS was proposed by Chen et al. [10]. BF Dombi AOs and BF prioritized Dombi AOs in the MADM technique developed by Jana et al. [19] and Jana et al. [20] respectively. Wei et al. [45] established BF Hamacher AOs.

Complex FS (CFS) is another structure that modifies FS; it was first described in polar form by Ramot et al. [39]. Some application domains where FS might not be sufficient can be covered by CFS. It is useful for capturing uncertainty in a variety of real-world scenarios due to its flexibility. FS cannot handle the data in two dimensions. For this purpose, CFS is used to handle such type of data in two dimensions. Following that, the notion of CFS in cartesian form was presented by Tamir et al. [44]. A systemic review of CFS was established by Yasdanbakhsh and Dick [48]. Complex fuzzy (CF) arithmetic and geometric AOs were investigated by Bi et al. [7] and Bi et al. [6] respectively. A more generalized and advanced concept is BCFS which is used to handle uncertainty and contradictory information in a more advanced manner. It combines the concepts of bipolarity and complexity to handle both positive and negative evaluations as well as interdependencies between different elements. Each element is associated with a complex number that consists of a pair of values representing positive and negative membership respectively. BCFS is more advanced and generalized to handle complexity and ambiguity as compared to the FS, BFS, and CFS. A novel and most verified idea of BCFS was given by Mahmood and Ur Rehman [26]. Additionally, Mahmood et al. [28] proposed utilizing BCFS to classify AOs. The MADM technique based on Dombi AOs under BCF information was introduced by Mahmood and Ur Rehman [27].

Another powerful phenomenon which is known to be rough set (RS), consisting of lower and upper approximation based on equivalence relation was investigated by Pawlak [36]. This structure is also more powerful in dealing with real-life DM dilemmas. This method has various applications in machine learning and artificial intelligence. One of RS's main advantages is that, unlike the DOM in FS theory, it does not require extra information. Yao [47] investigated probabilistic RS approximations. Another important concept that is more advanced than simple RS is fuzzy RS (FRS) which was introduced by Dubois and Prade [12]. The advantages of FS theory and RST are combined in FRS to provide a more flexible approach to handling complicated and unpredictable information. Along with fuzzy lower and upper approximations, it presents the idea of fuzzy equivalence relation. Further, a comparative study of FRS was established by Radzikowska and Kerre [38]. Yeung et al. [49] worked on the generalization of FRS. Another important concept that is the extension of FRS is Bipolar FRS (BFRS) introduced by Yang et al. [46]. It accomplishes this by representing uncertainty via bipolarity as opposed to just one DOM. By taking both positive and negative membership into account with roughness, BFRS offers a more thorough depiction of uncertainty. Han et al. [14] worked on bipolar-valued rough FS and its application in decision information systems. In addition to this, Complex FRS (CFRS) is also the extension of simple FRS that combines concepts from CFS theory, and RST. For managing uncertainty and complexity across several domains, it offers a more advanced and comprehensive method. This concept was introduced by Khan et al. [22] in 2023 in polar form.

Motivation of the paper. Investigating the relationship between FS and RS is the main goal of the researcher nowadays. By merging FS theory and RS, the researcher has made several advancements such as FRS [12], BFRS [46], CFRS [22], intuitionistic FRS [11], picture FRS [1], and so on. The aforementioned structures highlight the significance of fuzzy structure and RS. It implies that the RST can connect several structures with itself to generate novel structures and development. A positive and negative membership using the two-dimensional method is particularly important in real-world scenarios. The concept of BCFS is crucial for handling such a scenario. For example, a doctor wants to prepare a medicine product by adding some quantity of alcohol. Then, they have to measure the positive and negative effects of medicine and the positive and negative reactions of society to medicine. To solve such a dilemma, the BCF structure is the best structure to solve such problems. Simple FS, BFS, and CFS fail to maintain such a type of dilemma and cannot solve such a dilemma. Only the BCF structure is the best approach to solve such types of dilemmas where the information consists of positive and negative aspects with two dimensions. Also, RST on the other hand, is helpful in knowledge representation and decision analysis for managing vague and incomplete data. When there is a lack of precision and clarity in the given information, RST offers a framework to deal with such types of dilemmas. Representing information in terms of lower and upper approximation is the core concept of the RS model. BCFRS can handle the two-dimensional data with positive and negative aspects in terms of lower and upper approximation. The existing structures such as FS, FRS, CFS, CFRS, BFS, BFRS, RS, and BCFS can't handle the data in the BCFR model. But our proposed structure not only handles the data in the BCFR model but also has the capability of handling the data in the model of such FS, FRS, CFS, CFRS, BFS, BFRS, RS, and BCFS, which is the beauty and supremacy of our investigated structure. It is evident from the description above that no data framework provides the pros and cons of both positive and negative membership in two dimensions with lower and upper approximation simultaneously. All aforementioned structures including FS, FRS, BFS, BFRS, CFS, CFRS, and BCFS, were generalized by the created theory of BCFRS. The following Table 1 shows that our proposal is more generalized, up-to-date, and superior as compared to the FS, FRS, BFS, BFRS, CFS, CFRS, and BCFS.

TABLE 1. Comparison of Developed structure with prevalent existing structures.

Structures	**Fuzziness**	**Fuzzy Roughness**	**Bipolarity**	**Bipolar Fuzzy Roughness**	**Two Dimensional Data**	**Complex Fuzzy Roughness**	**Bipolarity In two dimensions**	**Bipolar Complex fuzzy with Roughness**
FS [50]	✓	××××	××××	××××	××××	××××	××××	××××
FRS [12]	✓	✓✓✓✓	××××	××××	××××	××××	××××	××××
BFS [51]	✓	××××	✓✓✓✓	××××	××××	××××	××××	××××
BFRS [46]	✓	✓✓✓✓	✓✓✓✓	✓✓✓✓	××××	××××	××××	××××
CFS [44]	✓	××××	××××	××××	✓✓✓✓	××××	××××	××××
CFRS [22]	✓	✓✓✓✓	××××	××××	✓✓✓✓	✓✓✓✓	××××	××××
BCFS [26]	✓	××××	✓✓✓✓	××××	✓✓✓✓	××××	✓✓✓✓	××××
BCFRS	✓	✓✓✓✓	✓✓✓✓	✓✓✓✓	✓✓✓✓	✓✓✓✓	✓✓✓✓	✓✓✓✓

The effectiveness, relevancy, and validity of the investigated structure are displayed in Table.1. Our investigated structure can further degenerate to FS, FRS, BFS, BFRS, CFS, CFRS, and BCFS. This indicates that the data in the models of FS, FRS, BFS, BFRS, CFS, CFRS, and BCFS can be handled and solved by our suggested model.

Contribution of the paper. The summarized novelties of the papers are as follows:

- To maintain the advantages of both the BCFS and RS model simultaneously, we construct the theory of BCFRS in this study. Decision makers have greater space with this technique, which also gives them the ability to manage complicated, unreliable, imprecise, and unclear information.
- We examine the BCF relation as well as several operations for BCFRS, including union, intersection, complement, sum, product, and scalar multiplication.
- For our aim of ranking and contrasting various BCFR numbers (BCFRNs), we invented the score and accuracy function.
- Additionally, we deploy the BCF rough weighted arithmetic averaging (BCFRWAA), BCF rough ordered weighted arithmetic averaging (BCFROWAA), BCF rough weighted geometric averaging (BCFRWGA), and BCF rough ordered weighted geometric averaging (BCFROWGA) aggregation operators related to the novel theory.
- A case study of "Selection of best Cyber Security alternative" is discussed in which a committee of experts evaluates the worth of different alternatives of C-S based on different attributes that are handled by the MADM technique.
- A comparative analysis of the novel technique and novel developed structure is discussed to reveal their supremacy and authenticity.

Structure of the Paper. The sequence of the whole manuscript is as under to show the proper discipline of the manuscript.

In Section**2**, we revised the main notion of RS, BCFS, t-norms, and t-conorms with various examples. Section **3** consists of the developed notion of BCF relation, BCFRS with example. We also developed the operation of union, intersection, complement, addition, multiplication, power, and scalar multiplication of BCFRNs. For ranking and comparison of different BCFRNs, the score and accuracy function are defined in this section. Various AO techniques referred to as BCFRWAA, BCFROWAA, BCFRWGA, and BCFROWGA operators for BCFRNs are used for the combination of different types of information into a single set. We demonstrated various properties of the diagnosed operators such as idempotency, monotonicity, and boundedness for the applicability of the proposed work.

In Section **4**, we developed the algorithm for the MADM technique for DM dilemmas. After that, we apply the MADM technique to real-life issues of the C-S system for the protection of digital data. After applying this technique we get the optimal alternative of the C-S system for the protection of network, computer, confidential, and digital data. Section **5** ,consists of a detailed comparison of the investigated structure with various existing structures to show the supremacy, generality, validity, and authenticity of investigated work.

At last, we conclude and summarize the whole literature in Section **6**.

2. Preliminaries

In this section, we revise the basic notions of IFS, RS and BCFS. We also review the basic ideas of t-norm and t-conorm with their properties and discuss various examples of t-norm and t-conorm to better understand these triangular norms.

Definitions 1: [5] Let, U be the universal set (US). An intuitionistic fuzzy set (IFS) is of the form:

$$\chi = (\nu, \overline{O}(\nu), \mathcal{F}(\nu), \nu \in U) \tag{1}$$

where $\overline{O}(\nu)$ represents the membership degree and $\mathcal{F}(\nu)$ represents the non-membership degree of ν, provided that $0 \leq \overline{O}(\nu) + \mathcal{F}(\nu) \leq 1$.

Definition 2:[36] Let, U be the US and R_e represent an equivalence relation. Then, the pair (U, R_e) is referred to as an approximation space (AS). For any non-empty subset $C \subseteq U$, the lower and upper approximations of C with respect to AS (U, R_e) are represented by $\underline{R_e}(C)$ and $\overline{R_e}(C)$, respectively, which are defined as:

$$\underline{R_e}(C) = \{\nu \in U \mid [\nu]_{R_e} \subseteq C\ \} \tag{2}$$

$$\overline{R_e}(C) = \{\nu \in U \mid [\nu]_{R_e} \cap C \neq \emptyset\} \tag{3}$$

The pair $(\underline{R_e}(C), \overline{R_e}(C))$ is called rough set (RS), with $\underline{R_e}(C) \neq \overline{R_e}(C)$.

Definition 3:[26] Let U be the US. A bipolar complex fuzzy set (BCFS) is of the form:

$$\chi = (\nu, \beta^{+}(\nu), \gamma^{-}(\nu), \nu \in U) \tag{4}$$

where $\beta^{+}(\nu) = (u^{+}(\nu) + iv^{+}(\nu), \nu \in U)$ and $\gamma^{-}(\nu) = (u^{-}(\nu) + iv^{-}(\nu), \nu \in U)$ represent the positive and negative membership degrees with $u^{+}(\nu), v^{+}(\nu) : U \to [0,1]$ and $u^{-}(\nu), v^{-}(\nu) : U \to [-1,0]$. For simplicity, we write $\chi = (u^{+} + iv^{+}, u^{-} + iv^{-})$, which represents the bipolar complex fuzzy numbers (BCFNs).

Definition 4:[27] The mathematical form of the score value for BCFNs is described as:

$$H_{SF}(\chi) = \frac{1}{4}(2 + u^{+} + v^{+} + u^{-} + v^{-}), \quad H_{SF} \in [0,1] \tag{5}$$

Definition 5:[27] The mathematical form of the accuracy value for BCFNs is described as:

$$H_{AF}(\chi) = \frac{(u^{+} + v^{+} - u^{-} - v^{-})}{4}, \quad H_{AF} \in [0,1] \tag{6}$$

Definition 6:[27] The fundamental operational laws for two BCFNs $\chi_1 = (u_1^{+} + iv_1^{+}, u_1^{-} + iv_1^{-})$ and $\chi_2 = (u_2^{+} + iv_2^{+}, u_2^{-} + iv_2^{-})$ and $\mathbb{A} > 0$ are stated as:

(1) $\chi_1 \oplus \chi_2 = \begin{pmatrix} u_1^{+} + u_2^{+} - u_1^{+}u_2^{+} + i(v_1^{+} + v_2^{+} - v_1^{+}v_2^{+}) \\ -(u_1^{-} + u_2^{-}) + i(-(v_1^{-} + v_2^{-})) \end{pmatrix}$

(2) $\chi_1 \otimes \chi_2 = \begin{pmatrix} u_1^{+}u_2^{+} + i(v_1^{+}v_2^{+}) \\ u_1^{-} + u_2^{-} + u_1^{-}u_2^{-} + i(v_1^{-} + v_2^{-} + v_1^{-}v_2^{-}) \end{pmatrix}$

(3) $\mathbb{A}\chi_1 = \begin{pmatrix} 1 - (1 - u_1^{+})^{\mathbb{A}} + i(1 - (1 - v_1^{+})^{\mathbb{A}}) \\ -|u_1^{-}|^{\mathbb{A}} + i(-|v_1^{-}|^{\mathbb{A}}) \end{pmatrix}$

(4) $(\chi_1)^{\mathbb{A}} = \begin{pmatrix} (u_1^{+})^{\mathbb{A}} + i(v_1^{+})^{\mathbb{A}} \\ -1 + (1 + u_1^{-})^{\mathbb{A}} + i\left(-1 + (1 + v_1^{-})^{\mathbb{A}}\right) \end{pmatrix}$

Definition 7:[31] A map $\Theta : [0,1] \times [0,1] \to [0,1]$ is interpreted as a t-norm if it satisfies:

- Symmetry: $\Theta(x_1, x_2) = \Theta(x_2, x_1)$
- Monotonicity: $\Theta(x_1, x_2) \leq \Theta(x_1, x_3)$ if $x_2 \leq x_3$
- Associativity: $\Theta(x_1, \Theta(x_2, x_3)) = \Theta(\Theta(x_1, x_2), x_3)$

- One identity: $\Theta(x_1, 1) = x_1$

for all $x_1, x_2, x_3 \in [0, 1]$.

Example 1: Underneath are popular examples of t-norms:

- Product t-norm: $\Theta_{pr}(x_1, x_2) = x_1 \cdot x_2$
- Minimum t-norm: $\Theta_{Wi}(x_1, x_2) = \min(x_1, x_2)$
- Drastic t-conorm:

$$\Theta_{Dr}(x_1, x_2) = \begin{cases} x_1 & \text{if } x_2 = 1 \\ x_2 & \text{if } x_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

- Lukasiewicz t-conorm:

$$\Theta_{LK}(x_1, x_2) = \max(x_1 + x_2 - 1, 0)$$

Definition 8: [24] A map $X : [0, 1] \times [0, 1] \to [0, 1]$ is interpreted as a t-conorm if it satisfies:

- Symmetry: $X(x_1, x_2) = X(x_2, x_1)$
- Monotonicity: $X(x_1, x_2) \leq X(x_1, x_3)$ if $x_2 \leq x_3$
- Associativity: $X(x_1, X(x_2, x_3)) = X(X(x_1, x_2), x_3)$
- One identity: $X(x_1, 0) = x_1$
  for all $x_1, x_2, x_3 \in [0, 1]$.

Example 2: Below are popular examples of t-conorms:

- Probabilistic t-conorm: $X_{pr}(x_1, x_2) = x_1 + x_2 - x_1 \cdot x_2$
- Minimum t-conorm: $X_{Wi}(x_1, x_2) = \max(x_1, x_2)$
- Drastic t-conorm:

$$X_{Dr}(x_1, x_2) = \begin{cases} x_1 & \text{if } x_2 = 1 \\ x_2 & \text{if } x_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

- Lukasiewicz t-conorm:

$$X_{LK}(x_1, x_2) = \min(x_1 + x_2, 1)$$

3. Construction of Bipolar Complex Fuzzy Rough Set

In this section, we investigate the main notion of BCF relation. We develop the main notion of BCFRS along with examples to better understand and implement the proposed novel definitions. Further, we study the fundamental operational rules for BCFNs. For ranking and comparison of different BCFRNs, the score and accuracy functions are defined in this section. Various aggregation operators referred to as BCFRWAA, BCFROWAA, BCFRWGA, and BCFROWGA are used for the combination of different types of information into a single set. We demonstrate various properties of the diagnosed operators such as idempotency, monotonicity, and boundedness for the applicability of the proposed work.

Definition 9: Let U be the universal set. A BCF subset R_{BCF} of $U \times U$ is given by:

$$R_{BCF} = \{(\nu, s), \beta^+(\nu, s), \gamma^-(\nu, s) \mid \beta^+(\nu, s) = (u^+(\nu, s) + iv^+(\nu, s)), \gamma^-(\nu, s) = (u^-(\nu, s) + iv^-(\nu, s))\}$$

where $u^+(\nu, s), v^+(\nu, s) \in [0, 1]$ and $u^-(\nu, s), v^-(\nu, s) \in [-1, 0]$, $\beta^+ : U \times U \to [0, 1] + i[0, 1]$ and $\gamma^- : U \times U \to [-1, 0] + i[-1, 0]$. Then, R_{BCF} is called a BCF relation.

Definition 10: Let U be the universal set and R_{BCF} be a BCF relation over U, i.e., $R_{BCF} \in BCF(U \times U)$. Then, the pair (U, R_{BCF}) is called the BCF approximation space (BCFAS). For any subset $C \subseteq$

$BCF(U)$, the lower and upper approximations of C with respect to BCFAS (U, R_{BCF}) are represented by $\underline{(R_{BCF})}(C)$ and $\overline{(R_{BCF})}(C)$, which are defined as:

$$\underline{(R_{BCF})}(C) = \{(\nu, \beta^{+}_{\underline{(R_{BCF})}(C)}(\nu), \gamma^{-}_{\underline{(R_{BCF})}(C)}(\nu)) \mid \nu \in U\} \tag{7}$$

$$\overline{(R_{BCF})}(C) = \{(\nu, \beta^{+}_{\overline{(R_{BCF})}(C)}(\nu), \gamma^{-}_{\overline{(R_{BCF})}(C)}(\nu)) \mid \nu \in U\} \tag{8}$$

$$\beta^{+}_{\underline{(R_{BCF})}(C)}(\nu) = \bigwedge_{r\in U} \left[1 - \mu^{+}_{R_{BCF}}(\nu, r) \vee \mu^{+}_{C}(r)\right] + i \bigwedge_{r\in U} \left[1 - \upsilon^{+}_{R_{BCF}}(\nu, r) \vee \upsilon^{+}_{C}(r)\right] = (\underline{\pi^{+}} + i\underline{\phi^{+}}),$$

$$\gamma^{-}_{\underline{(R_{BCF})}(C)}(\nu) = \bigwedge_{r\in U} \left[-1 - \mu^{-}_{R_{BCF}}(\nu, r) \vee \mu^{-}_{C}(r)\right] + i \bigwedge_{r\in U} \left[-1 - \upsilon^{-}_{R_{BCF}}(\nu, r) \vee \upsilon^{-}_{C}(r)\right] = (\underline{\pi^{-}} + i\underline{\phi^{-}})$$

$$\beta^{+}_{\overline{(R_{BCF})}(C)}(\nu) = \bigvee_{r\in U} \left[\mu^{+}_{R_{BCF}}(\nu, r) \wedge \mu^{+}_{C}(r)\right] + i \bigvee_{r\in U} \left[\upsilon^{+}_{R_{BCF}}(\nu, r) \wedge \upsilon^{+}_{C}(r)\right] =, (\overline{\pi^{+}} + i\overline{\phi^{+}})$$

$$\gamma^{-}_{\overline{(R_{BCF})}(C)}(\nu) = \bigvee_{r\in U} \left[\mu^{-}_{R_{BCF}}(\nu, r) \wedge \mu^{-}_{C}(r)\right] + i \bigvee_{r\in U} \left[\upsilon^{-}_{R_{BCF}}(\nu, r) \wedge \upsilon^{-}_{C}(r)\right] = (\overline{\pi^{-}} + i\overline{\phi^{-}})$$

The pair $(\underline{(R_{BCF})}(C), \overline{(R_{BCF})}(C))$ is called a BCF rough set (BCFRS) with respect to (U, R_{BCF}) and $\underline{(R_{BCF})}, \overline{(R_{BCF})} : BCF(U) \to BCF(U)$ are referred to as lower and upper approximation operators. For simplicity, we can say that $\chi_{BCFRN} = ((\underline{\pi^{+}} + i\underline{\phi^{+}}), (\underline{\pi^{-}} + i\underline{\phi^{-}}), (\overline{\pi^{+}} + i\overline{\phi^{+}}), (\overline{\pi^{-}} + i\overline{\phi^{-}}))$ represents the BCFRN.

Example 3: Let $U = \{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5\}$ be the universal set and BCFR is defined in Table 2 as follows:

TABLE 2. BCF Relation

R_{BCF}	ν_1	ν_2	ν_3	ν_4	ν_5
ν_1	$\begin{pmatrix} 0.23 + i0.41 \\ -0.11 - i0.33 \end{pmatrix}$	$\begin{pmatrix} 0.18 + i0.82 \\ -0.49 - i0.55 \end{pmatrix}$	$\begin{pmatrix} 0.21 + i0.96 \\ -0.19 - i0.43 \end{pmatrix}$	$\begin{pmatrix} 0.76 + i0.31 \\ -0.70 - i0.74 \end{pmatrix}$	$\begin{pmatrix} 0.20 + i0.94 \\ -0.88 - i0.57 \end{pmatrix}$
ν_2	$\begin{pmatrix} 0.33 + i0.87 \\ -0.27 - i0.39 \end{pmatrix}$	$\begin{pmatrix} 0.55 + i0.81 \\ -0.06 - i0.69 \end{pmatrix}$	$\begin{pmatrix} 0.29 + i0.73 \\ -0.08 - i0.84 \end{pmatrix}$	$\begin{pmatrix} 0.41 + i0.59 \\ -0.75 - i0.92 \end{pmatrix}$	$\begin{pmatrix} 0.66 + i0.11 \\ -0.94 - i0.22 \end{pmatrix}$
ν_3	$\begin{pmatrix} 0.51 + i0.77 \\ -0.91 - i0.71 \end{pmatrix}$	$\begin{pmatrix} 0.09 + i0.56 \\ -0.68 - i0.91 \end{pmatrix}$	$\begin{pmatrix} 0.21 + i0.95 \\ -0.44 - i0.54 \end{pmatrix}$	$\begin{pmatrix} 0.60 + i0.68 \\ -0.30 - i0.51 \end{pmatrix}$	$\begin{pmatrix} 0.23 + i0.64 \\ -0.75 - i0.21 \end{pmatrix}$
ν_4	$\begin{pmatrix} 0.14 + i0.88 \\ -0.12 - i0.99 \end{pmatrix}$	$\begin{pmatrix} 0.44 + i0.22 \\ -0.53 - i0.50 \end{pmatrix}$	$\begin{pmatrix} 0.10 + i0.67 \\ -0.19 - i0.73 \end{pmatrix}$	$\begin{pmatrix} 0.67 + i0.41 \\ -0.09 - i0.83 \end{pmatrix}$	$\begin{pmatrix} 0.99 + i0.44 \\ -0.29 - i0.39 \end{pmatrix}$
ν_5	$\begin{pmatrix} 0.11 + i0.51 \\ -0.37 - i0.92 \end{pmatrix}$	$\begin{pmatrix} 0.81 + i0.49 \\ -0.37 - i0.80 \end{pmatrix}$	$\begin{pmatrix} 0.86 + i0.40 \\ -0.17 - i0.30 \end{pmatrix}$	$\begin{pmatrix} 0.59 + i0.33 \\ -0.14 - i0.81 \end{pmatrix}$	$\begin{pmatrix} 0.58 + i0.37 \\ -0.92 - i0.31 \end{pmatrix}$

Now, we assume that $C = \{(\nu_1, 0.45+i0.21, -0.56-i0.41), (\nu_2, 0.55+i0.49, -0.76-i0.61), (\nu_3, 0.92+i0.53, -0.84 - i0.09), (\nu_4, 0.54 + i0.26, -0.94 - i0.35), (\nu_5, 0.71 + i0.66, -0.18 - i0.43)\}$ be BCFS over U. Then the lower and upper approximations are given by:

$$\underline{R_{BCF}}(C) = \{(\nu_1, 0.08 + i0.04, -0.89 - i0.91), (\nu_2, 0.08 + i0.13, -0.94 - i0.91),$$
$$(\nu_3, 0.08 + i0.05, -0.82 - i0.91), (\nu_4, 0.01 + i0.12, -0.91 - i0.91), (\nu_5, 0.08 + i0.34, -0.86 - i0.91)\}$$
$$\overline{R_{BCF}}(C) = \{(\nu_1, 0.54 + i0.66, -0.56 - i0.41), (\nu_2, 0.66 + i0.53, -0.56 - i0.41),$$
$$(\nu_3, 0.54 + i0.64, -0.75 - i0.43), (\nu_4, 0.71 + i0.53, -0.29 - i0.43), (\nu_5, 0.86 + i0.49, -0.56 - i0.30)\}$$

Hence, it is clear that the pair $\left(\underline{R_{BCF}}(C), \overline{R_{BCF}}(C)\right)$ is a BCFRS.

3.1. **Operations for BCFRS.** In this part, we study the fundamental operational rules for BCFNs. Here, we also define the score and accuracy functions that help us compare two BCFRNs.

Definition 11 For two BCFRSs $\chi_{1\text{BCFRS}} = \{(\upsilon, (\underline{\pi_1^+(\upsilon)} + i\underline{\phi_1^+(\upsilon)}, \underline{\pi_1^-(\upsilon)} + i\underline{\phi_1^-(\upsilon)}), (\overline{\pi_1^+(\upsilon)} + i\overline{\phi_1^+(\upsilon)}, \overline{\pi_1^-(\upsilon)} + i\overline{\phi_1^-(\upsilon)})) \mid \upsilon \in U\}$ and $\chi_{2\text{BCFRS}} = \{(\upsilon, (\underline{\pi_2^+(\upsilon)} + i\underline{\phi_2^+(\upsilon)}, \underline{\pi_2^-(\upsilon)} + i\underline{\phi_2^-(\upsilon)}), (\overline{\pi_2^+(\upsilon)} + i\overline{\phi_2^+(\upsilon)}, \overline{\pi_2^-(\upsilon)} + i\overline{\phi_2^-(\upsilon)})) \mid \upsilon \in U\}$, their union, intersection, and complement are defined by:

(1) $\chi_{1\text{BCFRS}} \cup \chi_{2\text{BCFRS}} = \{(\upsilon, (\max(\underline{\pi_1^+(\upsilon)}, \underline{\pi_2^+(\upsilon)}) + i\max(\underline{\phi_1^+(\upsilon)}, \underline{\phi_2^+(\upsilon)}), \min(\underline{\pi_1^-(\upsilon)}, \underline{\pi_2^-(\upsilon)}) + i\min(\underline{\phi_1^-(\upsilon)}, \underline{\phi_2^-(\upsilon)}), \max(\overline{\pi_1^+(\upsilon)}, \overline{\pi_2^+(\upsilon)}) + i\max(\overline{\phi_1^+(\upsilon)}, \overline{\phi_2^+(\upsilon)}), \min(\overline{\pi_1^-(\upsilon)}, \overline{\pi_2^-(\upsilon)}) + i\min(\overline{\phi_1^-(\upsilon)}, \overline{\phi_2^-(\upsilon)})) \mid \upsilon \in U\}$ (9)

(2) $\chi_{1\text{BCFRS}} \cap \chi_{2\text{BCFRS}} = \{(\upsilon, (\min(\underline{\pi_1^+(\upsilon)}, \underline{\pi_2^+(\upsilon)}) + i\min(\underline{\phi_1^+(\upsilon)}, \underline{\phi_2^+(\upsilon)}), \max(\underline{\pi_1^-(\upsilon)}, \underline{\pi_2^-(\upsilon)}) + i\max(\underline{\phi_1^-(\upsilon)}, \underline{\phi_2^-(\upsilon)}), \min(\overline{\pi_1^+(\upsilon)}, \overline{\pi_2^+(\upsilon)}) + i\min(\overline{\phi_1^+(\upsilon)}, \overline{\phi_2^+(\upsilon)}), \max(\overline{\pi_1^-(\upsilon)}, \overline{\pi_2^-(\upsilon)}) + i\max(\overline{\phi_1^-(\upsilon)}, \overline{\phi_2^-(\upsilon)})) \mid \upsilon \in U\}$ (10)

(3) $(\chi_{1\text{BCFRS}})^C = \{(\upsilon, (1 - \underline{\pi_1^+(\upsilon)}) + i(1 - \underline{\phi_1^+(\upsilon)}), (-1 - \underline{\pi_1^-(\upsilon)}) + i(-1 - \underline{\phi_1^-(\upsilon)}), (1 - \overline{\pi_1^+(\upsilon)}) + i(1 - \overline{\phi_1^+(\upsilon)}), (-1 - \overline{\pi_1^-(\upsilon)}) + i(-1 - \overline{\phi_1^-(\upsilon)})) \mid \upsilon \in U\}$ (11)

Definition 12: The fundamental operational laws for two BCFRNs

$$\chi_{1\text{BCFRS}} = \left(\left(\underline{\pi_1^+} + i\underline{\phi_1^+}, \underline{\pi_1^-} + i\underline{\phi_1^-}\right), \left(\overline{\pi_1^+} + i\overline{\phi_1^+}, \overline{\pi_1^-} + i\overline{\phi_1^-}\right)\right)$$

and

$$\chi_{2\text{BCFRS}} = \left(\left(\underline{\pi_2^+} + i\underline{\phi_2^+}, \underline{\pi_2^-} + i\underline{\phi_2^-}\right), \left(\overline{\pi_2^+} + i\overline{\phi_2^+}, \overline{\pi_2^-} + i\overline{\phi_2^-}\right)\right)$$

and $\aleph > 0$, are stated as:

1. $\chi_{1\text{BCFRN}} \oplus \chi_{2\text{BCFRN}} =$

$$\begin{pmatrix} \underline{\pi_1^+} + \underline{\pi_2^+} - \underline{\pi_1^+}\,\underline{\pi_2^+} + i\left(\underline{\phi_1^+} + \underline{\phi_2^+} - \underline{\phi_1^+}\,\underline{\phi_2^+}\right), \\ -\left(\underline{\pi_1^-}\,\underline{\pi_2^-}\right) + i\left(-\left(\underline{\phi_1^-}\,\underline{\phi_2^-}\right)\right), \\ \overline{\pi_1^+} + \overline{\pi_2^+} - \overline{\pi_1^+}\,\overline{\pi_2^+} + i\left(\overline{\phi_1^+} + \overline{\phi_2^+} - \overline{\phi_1^+}\,\overline{\phi_2^+}\right), \\ -\left(\overline{\pi_1^-}\,\overline{\pi_2^-}\right) + i\left(-\left(\overline{\phi_1^-}\,\overline{\phi_2^-}\right)\right) \end{pmatrix} \quad (12)$$

2. $\chi_{1\text{BCFRN}} \otimes \chi_{2\text{BCFRN}} =$

$$\begin{pmatrix} \underline{\pi_1^+}\,\underline{\pi_2^+} + i\left(\underline{\phi_1^+}\,\underline{\phi_2^+}\right), \\ \underline{\pi_1^-} + \underline{\pi_2^-} + \underline{\pi_1^-}\,\underline{\pi_2^-} + i\left(\underline{\phi_1^-} + \overline{\phi_2^-} + \underline{\phi_1^-}\,\underline{\phi_2^-}\right), \\ \overline{\pi_1^+} + \overline{\pi_2^+} + i\left(\overline{\phi_1^+}\,\overline{\phi_1^+}\right), \\ \overline{\pi_1^-} + \overline{\pi_2^-} + \overline{\pi_1^-}\,\overline{\pi_2^-} + i\left(\overline{\phi_1^-} + \overline{\phi_2^-} + \overline{\phi_1^-}\,\overline{\phi_2^-}\right) \end{pmatrix} \quad (13)$$

3. $\aleph\,\chi_{1\text{BCFRN}}$ =

$$\left(\begin{array}{c} 1-\left(1-\underline{\pi_1^+}\right)^{\aleph}+i\left(1-\left(1-\underline{\phi_1^+}\right)^{\aleph}\right), \\ -\left|\underline{\pi_1^-}\right|^{\aleph}+i\left(-\left|\underline{\phi_1^-}\right|^{\aleph}\right), \\ 1-\left(1-\overline{\pi_1^+}\right)^{\aleph}+i\left(1-\left(1-\overline{\phi_1^+}\right)^{\aleph}\right), \\ -\left|\overline{\pi_1^-}\right|^{\aleph}+i\left(-\left|\overline{\phi_1^-}\right|^{\aleph}\right) \end{array}\right) \tag{14}$$

4. $(\chi_{1\text{BCFRN}})^{\aleph}$ =

$$\left(\begin{array}{c} \left(\underline{\pi_1^+}\right)^{\aleph}+i\left(\underline{\phi_1^+}\right)^{\aleph}, \\ -1+\left(1+\underline{\pi_1^-}\right)^{\aleph}+i\left(-1+\left(1+\underline{\phi_1^-}\right)^{\aleph}\right), \\ \left(\overline{\pi_1^+}\right)^{\aleph}+i\left(\overline{\phi_1^+}\right)^{\aleph}, \\ -1+\left(1+\overline{\pi_1^-}\right)^{\aleph}+i\left(-1+\left(1+\overline{\phi_1^-}\right)^{\aleph}\right) \end{array}\right) \tag{15}$$

Definition 13 The mathematical form of the score value is described as:

$$H_{\text{SF}}(\chi_{\text{BCFRN}})=\frac{1}{8}\left(4+\underline{\pi^+}+\underline{\phi^+}+\underline{\pi^-}+\underline{\phi^-}+\overline{\pi^+}+\overline{\phi^+}+\overline{\pi^-}+\overline{\phi^-}\right),\quad H_{\text{SF}}\in[0,1] \tag{16}$$

Definition 14 The mathematical form of the accuracy value is described as:

$$H_{\text{AF}}(\chi_{\text{BCFRN}})=\frac{\left(\underline{\pi^+}+\underline{\phi^+}-\underline{\pi^-}-\underline{\phi^-}+\overline{\pi^+}+\overline{\phi^+}-\overline{\pi^-}-\overline{\phi^-}\right)}{8},\quad H_{\text{AF}}\in[0,1] \tag{17}$$

Definition 15 For two BCFRNs $\chi_{1\text{BCFRN}}=\left((\underline{\pi_1^+}+i\underline{\phi_1^+},\underline{\pi_1^-}+i\underline{\phi_1^-}),(\overline{\pi_1^+}+i\overline{\phi_1^+},\overline{\pi_1^-}+i\overline{\phi_1^-})\right)$ and $\chi_{2\text{BCFRN}}=\left((\underline{\pi_2^+}+i\underline{\phi_2^+},\underline{\pi_2^-}+i\underline{\phi_2^-}),(\overline{\pi_2^+}+i\overline{\phi_2^+},\overline{\pi_2^-}+i\overline{\phi_2^-})\right)$, we get:

(1) If $H_{\text{SF}}(\chi_{1\text{BCFRN}})<H_{\text{SF}}(\chi_{2\text{BCFRN}})$, then $\chi_{1\text{BCFRN}}<\chi_{2\text{BCFRN}}$
(2) If $H_{\text{SF}}(\chi_{1\text{BCFRN}})>H_{\text{SF}}(\chi_{2\text{BCFRN}})$, then $\chi_{1\text{BCFRN}}>\chi_{2\text{BCFRN}}$
(3) If $H_{\text{SF}}(\chi_{1\text{BCFRN}})=H_{\text{SF}}(\chi_{2\text{BCFRN}})$, then
   (a) If $H_{\text{AF}}(\chi_{1\text{BCFRN}})<H_{\text{AF}}(\chi_{2\text{BCFRN}})$, then $\chi_{1\text{BCFRN}}<\chi_{2\text{BCFRN}}$
   (b) If $H_{\text{AF}}(\chi_{1\text{BCFRN}})>H_{\text{AF}}(\chi_{2\text{BCFRN}})$, then $\chi_{1\text{BCFRN}}>\chi_{2\text{BCFRN}}$
   (c) If $H_{\text{AF}}(\chi_{1\text{BCFRN}})=H_{\text{AF}}(\chi_{2\text{BCFRN}})$, then $\chi_{1\text{BCFRN}}=\chi_{2\text{BCFRN}}$

3.2. **BCFR Arithmetic and Geometric AOs.** In this framework, various AO techniques referred to as BCFRWAA, BCFROWAA, BCFRWGA, and BCFROWGA operators for BCFRNs are used for the combination of different types of information into a single set. We demonstrated various properties of the diagnosed operators such as idempotency, monotonicity, and boundedness for the applicability of the proposed work. Throughout this section, $\tau_{\varpi}=\{\tau_{\varpi_1},\tau_{\varpi_2},\ldots,\tau_{\varpi_\eta}\}$ represents the weight vectors (WVs) such that $\tau_{\varpi_{\hat{j}}}$ must belong to $[0,1]$ and $\sum_{\hat{j}=1}^{\eta}\tau_{\varpi_{\hat{j}}}=1$.

Definition 16 Let, $\chi_{\hat{j}\text{BCFRN}} = \left((\underline{\pi_{\hat{j}}^{+}} + i\underline{\phi_{\hat{j}}^{+}}, \underline{\pi_{\hat{j}}^{-}} + i\underline{\phi_{\hat{j}}^{-}}), (\overline{\pi_{\hat{j}}^{+}} + i\overline{\phi_{\hat{j}}^{+}}, \overline{\pi_{\hat{j}}^{-}} + i\overline{\phi_{\hat{j}}^{-}}) \right)$ be the family of BCFRNs where $\hat{j} = 1, 2, 3, \ldots, \eta$. Then the notion of the BCFRWA operator is defined by:

$$\text{BCFRWAA}(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}) = \bigoplus_{\hat{j}=1}^{\eta} \tau_{\varpi_{\hat{j}}} \chi_{\hat{j}\text{BCFRN}} = \tau_{\varpi_1} \chi_{1\text{BCFRN}} \oplus \cdots \oplus \tau_{\varpi_\eta} \chi_{\eta\text{BCFRN}} \quad (18)$$

Theorem 1: Let,

$$\chi_{\hat{j}\text{BCFRN}} = \left(\left(\underline{\pi_{\hat{j}}^{+}} + i\underline{\phi_{\hat{j}}^{+}}, \underline{\pi_{\hat{j}}^{-}} + i\underline{\phi_{\hat{j}}^{-}} \right), \left(\overline{\pi_{\hat{j}}^{+}} + i\overline{\phi_{\hat{j}}^{+}}, \overline{\pi_{\hat{j}}^{-}} + i\overline{\phi_{\hat{j}}^{-}} \right) \right)$$

be the family of BCFRNs where $\hat{j} = 1, 2, 3, \ldots, \eta$. Then we can see that the aggregated result obtained by using the definition (16) is again BCFRN and is given by:

$$\text{BCFRWAA}\left(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}\right) =$$

$$\begin{pmatrix} 1 - \prod_{\hat{j}=1}^{\eta} \left(1 - \underline{\pi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}} + i\left(1 - \prod_{\hat{j}=1}^{\eta} \left(1 - \underline{\phi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}}\right), -\prod_{\hat{j}=1}^{\eta} \left|\underline{\pi_{\hat{j}}^{-}}\right|^{\tau_{\varpi_{\hat{j}}}} + i\left(-\prod_{\hat{j}=1}^{\eta} \left|\underline{\phi_{\hat{j}}^{-}}\right|^{\tau_{\varpi_{\hat{j}}}}\right) \\ 1 - \prod_{\hat{j}=1}^{\eta} \left(1 - \overline{\pi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}} + i\left(1 - \prod_{\hat{j}=1}^{\eta} \left(1 - \overline{\phi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}}\right), -\prod_{\hat{j}=1}^{\eta} \left|\overline{\pi_{\hat{j}}^{-}}\right|^{\tau_{\varpi_{\hat{j}}}} + i\left(-\prod_{\hat{j}=1}^{\eta} \left|\overline{\phi_{\hat{j}}^{-}}\right|^{\tau_{\varpi_{\hat{j}}}}\right) \end{pmatrix}$$

Proof: We proved the equation(19) using mathematical induction. For this, we assume for $\hat{j} = 2$,

$$\tau_{\varpi_1} \chi_{1\text{BCFRN}} = \begin{pmatrix} 1 - \left(1 - \underline{\pi_1^{+}}\right)^{\tau_{\varpi_1}} + i\left(1 - \left(1 - \underline{\phi_1^{+}}\right)^{\tau_{\varpi_1}}\right), -\left|\underline{\pi_1^{-}}\right|^{\tau_{\varpi_1}} + i\left(-\left|\underline{\phi_1^{-}}\right|^{\tau_{\varpi_1}}\right) \\ 1 - \left(1 - \overline{\pi_1^{+}}\right)^{\tau_{\varpi_1}} + i\left(1 - \left(1 - \overline{\phi_1^{+}}\right)^{\tau_{\varpi_1}}\right), -\left|\overline{\pi_1^{-}}\right|^{\tau_{\varpi_1}} + i\left(-\left|\overline{\phi_1^{-}}\right|^{\tau_{\varpi_1}}\right) \end{pmatrix}$$

$$\tau_{\varpi_2} \chi_{2\text{BCFRN}} = \begin{pmatrix} 1 - \left(1 - \underline{\pi_2^{+}}\right)^{\tau_{\varpi_2}} + i\left(1 - \left(1 - \underline{\phi_2^{+}}\right)^{\tau_{\varpi_2}}\right), -\left|\underline{\pi_2^{-}}\right|^{\tau_{\varpi_2}} + i\left(-\left|\underline{\phi_2^{-}}\right|^{\tau_{\varpi_2}}\right) \\ 1 - \left(1 - \overline{\pi_2^{+}}\right)^{\tau_{\varpi_2}} + i\left(1 - \left(1 - \overline{\phi_2^{+}}\right)^{\tau_{\varpi_2}}\right), -\left|\overline{\pi_2^{-}}\right|^{\tau_{\varpi_2}} + i\left(-\left|\overline{\phi_2^{-}}\right|^{\tau_{\varpi_2}}\right) \end{pmatrix}$$

$$\tau_{\varpi_1} \chi_{1\text{BCFRN}} \oplus \tau_{\varpi_2} \chi_{2\text{BCFRN}} = \begin{pmatrix} \left(1 - \left(1 - \underline{\pi_1^{+}}\right)^{\tau_{\varpi_1}} \left(1 - \underline{\pi_2^{+}}\right)^{\tau_{\varpi_2}}\right) + i\left(1 - \left(1 - \underline{\phi_1^{+}}\right)^{\tau_{\varpi_1}} \left(1 - \underline{\phi_2^{+}}\right)^{\tau_{\varpi_2}}\right), \\ -\left(\left|\underline{\pi_1^{-}}\right|^{\tau_{\varpi_1}} \left|\underline{\pi_2^{-}}\right|^{\tau_{\varpi_2}}\right) + i\left(-\left|\underline{\phi_1^{-}}\right|^{\tau_{\varpi_1}} \left|\underline{\phi_2^{-}}\right|^{\tau_{\varpi_2}}\right) \\ \left(1 - \left(1 - \overline{\pi_1^{+}}\right)^{\tau_{\varpi_1}} \left(1 - \overline{\pi_2^{+}}\right)^{\tau_{\varpi_2}}\right) + i\left(1 - \left(1 - \overline{\phi_1^{+}}\right)^{\tau_{\varpi_1}} \left(1 - \overline{\phi_2^{+}}\right)^{\tau_{\varpi_2}}\right), \\ -\left(\left|\overline{\pi_1^{-}}\right|^{\tau_{\varpi_1}} \left|\overline{\pi_2^{-}}\right|^{\tau_{\varpi_2}}\right) + i\left(-\left|\overline{\phi_1^{-}}\right|^{\tau_{\varpi_1}} \left|\overline{\phi_2^{-}}\right|^{\tau_{\varpi_2}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \prod_{\hat{j}=1}^{2} \left(1 - \underline{\pi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}} + i\left(1 - \prod_{\hat{j}=1}^{2} \left(1 - \underline{\phi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}}\right), -\prod_{\hat{j}=1}^{2} \left|\underline{\pi_{\hat{j}}^{-}}\right|^{\tau_{\varpi_{\hat{j}}}} + i\left(-\prod_{\hat{j}=1}^{2} \left|\underline{\phi_{\hat{j}}^{-}}\right|^{\tau_{\varpi_{\hat{j}}}}\right) \\ 1 - \prod_{\hat{j}=1}^{2} \left(1 - \overline{\pi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}} + i\left(1 - \prod_{\hat{j}=1}^{2} \left(1 - \overline{\phi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}}\right), -\prod_{\hat{j}=1}^{2} \left|\overline{\pi_{\hat{j}}^{-}}\right|^{\tau_{\varpi_{\hat{j}}}} + i\left(-\prod_{\hat{j}=1}^{2} \left|\overline{\phi_{\hat{j}}^{-}}\right|^{\tau_{\varpi_{\hat{j}}}}\right) \end{pmatrix}$$

Assuming that eq (19) above holds for i.e. $\hat{j} = \eta$,

$$\text{BCFRWAA}\left(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}\right) =$$

$$\begin{pmatrix} 1-\prod_{\hat{j}=1}^{\eta}\left(1-\underline{\pi}_{\hat{j}}^{+}\right)^{\tau_{\varpi_{\hat{j}}}}+i\left(1-\prod_{\hat{j}=1}^{\eta}\left(1-\underline{\phi}_{\hat{j}}^{+}\right)^{\tau_{\varpi_{\hat{j}}}}\right),-\prod_{\hat{j}=1}^{\eta}\left|\underline{\pi}_{\hat{j}}^{-}\right|^{\tau_{\varpi_{\hat{j}}}}+i\left(-\prod_{\hat{j}=1}^{\eta}\left|\underline{\phi}_{\hat{j}}^{-}\right|^{\tau_{\varpi_{\hat{j}}}}\right) \\ 1-\prod_{\hat{j}=1}^{\eta}\left(1-\overline{\pi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}}+i\left(1-\prod_{\hat{j}=1}^{\eta}\left(1-\overline{\phi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}}\right),-\prod_{\hat{j}=1}^{\eta}\left|\overline{\pi_{\hat{j}}^{-}}\right|^{\tau_{\varpi_{\hat{j}}}}+i\left(-\prod_{\hat{j}=1}^{\eta}\left|\overline{\phi_{\hat{j}}^{-}}\right|^{\tau_{\varpi_{\hat{j}}}}\right) \end{pmatrix}$$

We now established the equation for $\hat{j}=\eta+1$:

$$\text{BCFRWAA}\left(\chi_{1\text{BCFRN}},\chi_{2\text{BCFRN}},\ldots,\chi_{(\eta+1)\text{BCFRN}}\right)=\text{BCFRWAA}\left(\chi_{1\text{BCFRN}},\chi_{2\text{BCFRN}},\ldots,\chi_{\eta\text{BCFRN}}\right)\oplus\chi_{(\eta+1)\text{BCFRN}}$$

$$=\begin{pmatrix} \left(1-\prod_{\hat{j}=1}^{\eta}\left(1-\underline{\pi}_{\hat{j}}^{+}\right)^{\tau_{\varpi_{\hat{j}}}}\right)\left(1-\underline{\pi}_{\eta+1}^{+}\right)^{\tau_{\varpi_{\eta+1}}}+i\left(1-\prod_{\hat{j}=1}^{\eta}\left(1-\underline{\phi}_{\hat{j}}^{+}\right)^{\tau_{\varpi_{\hat{j}}}}\right)\left(1-\underline{\phi}_{\eta+1}^{+}\right)^{\tau_{\varpi_{\eta+1}}}, \\ -\prod_{\hat{j}=1}^{\eta}\left|\underline{\pi}_{\hat{j}}^{-}\right|^{\tau_{\varpi_{\hat{j}}}}\left|\underline{\pi}_{\eta+1}^{-}\right|^{\tau_{\varpi_{\eta+1}}}+i\left(-\prod_{\hat{j}=1}^{\eta}\left|\underline{\phi}_{\hat{j}}^{-}\right|^{\tau_{\varpi_{\hat{j}}}}\left|\underline{\phi}_{\eta+1}^{-}\right|^{\tau_{\varpi_{\eta+1}}}\right) \\ \left(1-\prod_{\hat{j}=1}^{\eta}\left(1-\overline{\pi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}}\right)\left(1-\overline{\pi_{\eta+1}^{+}}\right)^{\tau_{\varpi_{\eta+1}}}+i\left(1-\prod_{\hat{j}=1}^{\eta}\left(1-\overline{\phi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}}\right)\left(1-\overline{\phi_{\eta+1}^{+}}\right)^{\tau_{\varpi_{\eta+1}}}, \\ -\prod_{\hat{j}=1}^{\eta}\left|\overline{\pi_{\hat{j}}^{-}}\right|^{\tau_{\varpi_{\hat{j}}}}\left|\overline{\pi_{\eta+1}^{-}}\right|^{\tau_{\varpi_{\eta+1}}}+i\left(-\prod_{\hat{j}=1}^{\eta}\left|\overline{\phi_{\hat{j}}^{-}}\right|^{\tau_{\varpi_{\hat{j}}}}\left|\overline{\phi_{\eta+1}^{-}}\right|^{\tau_{\varpi_{\eta+1}}}\right) \end{pmatrix}$$

$$=\begin{pmatrix} 1-\prod_{\hat{j}=1}^{\eta+1}\left(1-\underline{\pi}_{\hat{j}}^{+}\right)^{\tau_{\varpi_{\hat{j}}}}+i\left(1-\prod_{\hat{j}=1}^{\eta+1}\left(1-\underline{\phi}_{\hat{j}}^{+}\right)^{\tau_{\varpi_{\hat{j}}}}\right),-\prod_{\hat{j}=1}^{\eta+1}\left|\underline{\pi}_{\hat{j}}^{-}\right|^{\tau_{\varpi_{\hat{j}}}}+i\left(-\prod_{\hat{j}=1}^{\eta+1}\left|\underline{\phi}_{\hat{j}}^{-}\right|^{\tau_{\varpi_{\hat{j}}}}\right) \\ 1-\prod_{\hat{j}=1}^{\eta+1}\left(1-\overline{\pi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}}+i\left(1-\prod_{\hat{j}=1}^{\eta+1}\left(1-\overline{\phi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}}\right),-\prod_{\hat{j}=1}^{\eta+1}\left|\overline{\pi_{\hat{j}}^{-}}\right|^{\tau_{\varpi_{\hat{j}}}}+i\left(-\prod_{\hat{j}=1}^{\eta+1}\left|\overline{\phi_{\hat{j}}^{-}}\right|^{\tau_{\varpi_{\hat{j}}}}\right) \end{pmatrix}$$

This implies that the above eq (19) is true for $\eta>0$.

Theorem 2. [Idempotency property] Let,

$$\chi_{\hat{j}\text{BCFRN}}=\left(\left(\underline{\pi}_{\hat{j}}^{\ +}+i\underline{\phi}_{\hat{j}}^{\ +},\underline{\pi}_{\hat{j}}^{\ -}+i\underline{\phi}_{\hat{j}}^{\ -}\right),\left(\overline{\pi_{\hat{j}}^{+}}+i\overline{\phi_{\hat{j}}^{+}},\overline{\pi_{\hat{j}}^{-}}+i\overline{\phi_{\hat{j}}^{+}}\right)\right)$$

be the family of BCFRNs, where $\hat{j}=1,2,3,\ldots,\eta$. If

$$\chi_{\hat{j}\text{BCFRN}}=\chi_{\text{BCFRN}}\,\forall\,\hat{j},\text{ i.e. }\underline{\pi}_{\hat{j}}^{+}=\underline{\pi}^{+},\underline{\phi}_{\hat{j}}^{+}=\underline{\phi}^{+},\underline{\pi}_{\hat{j}}^{-}=\underline{\pi}^{-},\underline{\phi}_{\hat{j}}^{-}=\underline{\phi}^{-},\overline{\pi_{\hat{j}}^{+}}=\overline{\pi^{+}},\overline{\phi_{\hat{j}}^{+}}=\overline{\phi^{+}},\overline{\pi_{\hat{j}}^{-}}=\overline{\pi^{-}},\overline{\phi_{\hat{j}}^{-}}=\overline{\phi^{-}}\forall\,\hat{j},$$

Then,

$$BCFRWAA\left(\chi_{1\text{BCFRN}},\chi_{2\text{BCFRN}},\ldots,\chi_{\eta\text{BCFRN}}\right)=\chi_{\text{BCFRN}}\quad(20)$$

Proof:

Assuming that $\chi_{\hat{j}\text{BCFRN}}=\chi_{\text{BCFRN}}$, i.e.,

$$\underline{\pi}_{\hat{j}}^{+}=\underline{\pi}^{+},\underline{\phi}_{\hat{j}}^{+}=\underline{\phi}^{+},\underline{\pi}_{\hat{j}}^{-}=\underline{\pi}^{-},\underline{\phi}_{\hat{j}}^{-}=\underline{\phi}^{-},\overline{\pi_{\hat{j}}^{+}}=\overline{\pi^{+}},\overline{\phi_{\hat{j}}^{+}}=\overline{\phi^{+}},\overline{\pi_{\hat{j}}^{-}}=\overline{\pi^{-}},\overline{\phi_{\hat{j}}^{-}}=\overline{\phi^{-}}\,\forall\,\hat{j},\text{ then}$$

$$BCFRWA\left(\chi_{1\text{BCFRN}},\chi_{2\text{BCFRN}},\ldots,\chi_{\eta\text{BCFRN}}\right)=\begin{pmatrix} 1-\prod_{\hat{j}=1}^{\eta}(1-\underline{\pi}^{+})^{\tau\omega_{\hat{j}}}+i\left(1-\prod_{\hat{j}=1}^{\eta}(1-\underline{\phi}^{+})^{\tau\omega_{\hat{j}}}\right), \\ -\prod_{\hat{j}=1}^{\eta}|\underline{\pi}^{-}|^{\tau\omega_{\hat{j}}}+i\left(-\prod_{\hat{j}=1}^{\eta}|\underline{\phi}^{-}|^{\tau\omega_{\hat{j}}}\right), \\ 1-\prod_{\hat{j}=1}^{\eta}(1-\overline{\pi^{+}})^{\tau\omega_{\hat{j}}}+i\left(1-\prod_{\hat{j}=1}^{\eta}(1-\overline{\phi^{+}})^{\tau\omega_{\hat{j}}}\right), \\ -\prod_{\hat{j}=1}^{\eta}|\overline{\pi^{-}}|^{\tau\omega_{\hat{j}}}+i\left(-\prod_{\hat{j}=1}^{\eta}|\overline{\phi^{-}}|^{\tau\omega_{\hat{j}}}\right) \end{pmatrix}$$

$$=\begin{pmatrix} 1-(1-\underline{\pi}^{+})^{\sum_{\hat{j}=1}^{\eta}\tau\omega_{\hat{j}}}+i\left(1-(1-\underline{\phi}^{+})^{\sum_{\hat{j}=1}^{\eta}\tau\omega_{\hat{j}}}\right), \\ -|\underline{\pi}^{-}|^{\sum_{\hat{j}=1}^{\eta}\tau\omega_{\hat{j}}}+i\left(-|\underline{\phi}^{-}|^{\sum_{\hat{j}=1}^{\eta}\tau\omega_{\hat{j}}}\right), \\ 1-(1-\overline{\pi^{+}})^{\sum_{\hat{j}=1}^{\eta}\tau\omega_{\hat{j}}}+i\left(1-(1-\overline{\phi^{+}})^{\sum_{\hat{j}=1}^{\eta}\tau\omega_{\hat{j}}}\right), \\ -|\overline{\pi^{-}}|^{\sum_{\hat{j}=1}^{\eta}\tau\omega_{\hat{j}}}+i\left(-|\overline{\phi^{-}}|^{\sum_{\hat{j}=1}^{\eta}\tau\omega_{\hat{j}}}\right) \end{pmatrix}$$

$$=\begin{pmatrix} 1-(1-\underline{\pi}^{+})+i\left(1-(1-\underline{\phi}^{+})\right), \\ \underline{\pi}^{-}+i\underline{\phi}^{-}, \\ 1-(1-\overline{\pi^{+}})+i\left(1-(1-\overline{\phi^{+}})\right), \\ \overline{\pi^{-}}+i\overline{\phi^{-}} \end{pmatrix}$$

$$=\begin{pmatrix} \underline{\pi}^{+}+i\underline{\phi}^{+}, \\ \underline{\pi}^{-}+i\underline{\phi}^{-}, \\ \overline{\pi^{+}}+i\overline{\phi^{+}}, \\ \overline{\pi^{-}}+i\overline{\phi^{-}} \end{pmatrix}=\chi_{\text{BCFRN}}$$

Theorem 3 (Monotonicity property): Let,

$$\chi_{\hat{j}\text{BCFRN}}=\left(\left(\underline{\pi_{\hat{j}}^{+}}+i\underline{\phi_{\hat{j}}^{+}},\underline{\pi_{\hat{j}}^{-}}+i\underline{\phi_{\hat{j}}^{-}}\right),\left(\overline{\pi_{\hat{j}}^{+}}+i\overline{\phi_{\hat{j}}^{+}},\overline{\pi_{\hat{j}}^{-}}+i\overline{\phi_{\hat{j}}^{-}}\right)\right)$$

and

$$\chi^{*}_{\hat{j}\text{BCFRN}}=\left(\left(\underline{\pi_{\hat{j}}^{*+}}+i\underline{\phi_{\hat{j}}^{*+}},\underline{\pi_{\hat{j}}^{*-}}+i\underline{\phi_{\hat{j}}^{*-}}\right),\left(\overline{\pi_{\hat{j}}^{*+}}+i\overline{\phi_{\hat{j}}^{*+}},\overline{\pi_{\hat{j}}^{*-}}+i\overline{\phi_{\hat{j}}^{*-}}\right)\right)$$

be two collections of BCFRNs, where $\hat{j}=1,2,3,\ldots,\eta$. If

$$\underline{\pi_{\hat{j}}^{+}}\leq\underline{\pi_{\hat{j}}^{*+}},\quad \underline{\phi_{\hat{j}}^{+}}\leq\underline{\phi_{\hat{j}}^{*+}},\quad \underline{\pi_{\hat{j}}^{-}}\leq\underline{\pi_{\hat{j}}^{*+}},\quad \underline{\phi_{\hat{j}}^{-}}\leq\underline{\phi_{\hat{j}}^{*-}}$$

$$\overline{\pi_{\hat{j}}^{+}}\leq\overline{\pi_{\hat{j}}^{*+}},\quad \overline{\phi_{\hat{j}}^{+}}\leq\overline{\phi_{\hat{j}}^{*+}},\quad \overline{\pi_{\hat{j}}^{-}}\leq\overline{\pi_{\hat{j}}^{*-}}\quad \text{and}\quad \overline{\phi_{\hat{j}}^{-}}\leq\overline{\phi_{\hat{j}}^{*-}}\quad \forall\hat{j},$$

then

$$BCFRWAA(\chi_{1\text{BCFRN}},\chi_{2\text{BCFRN}},\ldots,\chi_{\eta\text{BCFRN}})\leq BCFRWAA(\chi^{*}_{1\text{BCFRN}},\chi^{*}_{2\text{BCFRN}},\ldots,\chi^{*}_{\eta\text{BCFRN}}). \quad (21)$$

Proof: Suppose, we assume that

$$\underline{\pi_{\hat{j}}^{+}}\leq\underline{\pi_{\hat{j}}^{*+}},\quad \underline{\phi_{\hat{j}}^{+}}\leq\underline{\phi_{\hat{j}}^{*+}},\quad \underline{\pi_{\hat{j}}^{-}}\leq\underline{\pi_{\hat{j}}^{*+}},\quad \underline{\phi_{\hat{j}}^{-}}\leq\underline{\phi_{\hat{j}}^{*-}}$$

$$\overline{\pi_{\hat{j}}^{+}}\leq\overline{\pi_{\hat{j}}^{*+}},\quad \overline{\phi_{\hat{j}}^{+}}\leq\overline{\phi_{\hat{j}}^{*+}},\quad \overline{\pi_{\hat{j}}^{-}}\leq\overline{\pi_{\hat{j}}^{*-}}\quad \text{and}\quad \overline{\phi_{\hat{j}}^{-}}\leq\overline{\phi_{\hat{j}}^{*-}}\quad \forall\hat{j}.$$

Then,

$$1-\underline{\pi_{\hat{j}}^{+}}\geq 1-\underline{\pi_{\hat{j}}^{*+}}$$

Thus, we have

$$\prod_{\hat{j}=1}^{\eta}\left(1-\underline{\pi_{\hat{j}}^{+}}\right)^{\tau_{\varpi\hat{j}}} \geq \prod_{\hat{j}=1}^{\eta}\left(1-\underline{\pi_{\hat{j}}^{*+}}\right)^{\tau_{\varpi\hat{j}}}$$

$$1-\prod_{\hat{j}=1}^{\eta}\left(1-\underline{\pi_{\hat{j}}^{+}}\right)^{\tau_{\varpi\hat{j}}} \leq 1-\prod_{\hat{j}=1}^{\eta}\left(1-\underline{\pi_{\hat{j}}^{*+}}\right)^{\tau_{\varpi\hat{j}}}$$

Also,

$$\prod_{\hat{j}=1}^{\eta}\left(1-\overline{\pi_{\hat{j}}^{+}}\right)^{\tau_{\varpi\hat{j}}} \geq \prod_{\hat{j}=1}^{\eta}\left(1-\overline{\pi_{\hat{j}}^{*+}}\right)^{\tau_{\varpi\hat{j}}}$$

$$1-\prod_{\hat{j}=1}^{\eta}\left(1-\overline{\pi_{\hat{j}}^{+}}\right)^{\tau_{\varpi\hat{j}}} \leq 1-\prod_{\hat{j}=1}^{\eta}\left(1-\overline{\pi_{\hat{j}}^{*+}}\right)^{\tau_{\varpi\hat{j}}}$$

Similarly,

$$1-\prod_{\hat{j}=1}^{\eta}\left(1-\underline{\phi_{\hat{j}}^{+}}\right)^{\tau_{\varpi\hat{j}}} \leq 1-\prod_{\hat{j}=1}^{\eta}\left(1-\underline{\phi_{\hat{j}}^{*+}}\right)^{\tau_{\varpi\hat{j}}}$$

$$1-\prod_{\hat{j}=1}^{\eta}\left(1-\overline{\phi_{\hat{j}}^{+}}\right)^{\tau_{\varpi\hat{j}}} \leq 1-\prod_{\hat{j}=1}^{\eta}\left(1-\overline{\phi_{\hat{j}}^{*+}}\right)^{\tau_{\varpi\hat{j}}}$$

Further,

$$-\prod_{\hat{j}=1}^{\eta}\left|\underline{\pi_{\hat{j}}^{-}}\right|^{\tau_{\varpi\hat{j}}} \geq -\prod_{\hat{j}=1}^{\eta}\left|\underline{\pi_{\hat{j}}^{*-}}\right|^{\tau_{\varpi\hat{j}}}$$

$$-\prod_{\hat{j}=1}^{\eta}\left|\overline{\pi_{\hat{j}}^{-}}\right|^{\tau_{\varpi\hat{j}}} \geq -\prod_{\hat{j}=1}^{\eta}\left|\overline{\pi_{\hat{j}}^{*-}}\right|^{\tau_{\varpi\hat{j}}}$$

Also,

$$-\prod_{\hat{j}=1}^{\eta}\left|\underline{\phi_{\hat{j}}^{-}}\right|^{\tau_{\varpi\hat{j}}} \geq -\prod_{\hat{j}=1}^{\eta}\left|\underline{\phi_{\hat{j}}^{*-}}\right|^{\tau_{\varpi\hat{j}}}$$

$$-\prod_{\hat{j}=1}^{\eta}\left|\overline{\phi_{\hat{j}}^{-}}\right|^{\tau_{\varpi\hat{j}}} \geq -\prod_{\hat{j}=1}^{\eta}\left|\overline{\phi_{\hat{j}}^{*-}}\right|^{\tau_{\varpi\hat{j}}}$$

Hence, by the above inequalities, we get our desired result, i.e.,

$$BCFRWAA(\chi_{1\text{BCFRN}},\chi_{2\text{BCFRN}},\ldots,\chi_{\eta\text{BCFRN}}) \leq BCFRWAA(\chi^{*}_{1\text{BCFRN}},\chi^{*}_{2\text{BCFRN}},\ldots,\chi^{*}_{\eta\text{BCFRN}}).$$

Theorem 4 (Boundedness property): Let,

$$\chi_{\hat{j}\text{BCFRN}} = \left(\ (\underline{\pi_{\hat{j}}^{+}} + i\underline{\phi_{\hat{j}}^{+}}, \underline{\pi_{\hat{j}}^{-}} + i\underline{\pi_{\hat{j}}^{-}}), \quad (\overline{\pi_{\hat{j}}^{+}} + i\overline{\phi_{\hat{j}}^{+}}, \overline{\pi_{\hat{j}}^{-}} + i\overline{\pi_{\hat{j}}^{-}}) \ \right)$$

be the family of BCFRNs where $\hat{j} = 1, 2, 3, \ldots, \eta$, and assume that

$$\chi^{-}_{\hat{j}\text{BCFRN}} = \begin{pmatrix} \min_{\hat{j}}\{\underline{\pi_{\hat{j}}^{+}}\} + i\min_{\hat{j}}\{\underline{\phi_{\hat{j}}^{+}}\}, \min_{\hat{j}}\{\underline{\pi_{\hat{j}}^{-}}\} + i\min_{\hat{j}}\{\underline{\phi_{\hat{j}}^{-}}\}, \\ \min_{\hat{j}}\{\overline{\pi_{\hat{j}}^{+}}\} + i\min_{\hat{j}}\{\overline{\phi_{\hat{j}}^{+}}\}, \min_{\hat{j}}\{\overline{\pi_{\hat{j}}^{-}}\} + i\min_{\hat{j}}\{\overline{\phi_{\hat{j}}^{-}}\} \end{pmatrix}$$

and

$$\chi^{+}_{\hat{j}\text{BCFRN}} = \begin{pmatrix} \max_{\hat{j}}\{\underline{\pi^{+}_{\hat{j}}}\} + i\max_{\hat{j}}\{\underline{\phi^{+}_{\hat{j}}}\}, \max_{\hat{j}}\{\underline{\pi^{-}_{\hat{j}}}\} + i\max_{\hat{j}}\{\underline{\phi^{-}_{\hat{j}}}\}, \\ \max_{\hat{j}}\{\overline{\pi^{+}_{\hat{j}}}\} + i\max_{\hat{j}}\{\overline{\phi^{+}_{\hat{j}}}\}, \max_{\hat{j}}\{\overline{\pi^{-}_{\hat{j}}}\} + i\max_{\hat{j}}\{\overline{\phi^{-}_{\hat{j}}}\} \end{pmatrix}$$

then

$$\chi^{-}_{\hat{j}\text{BCFRN}} \leq BCFRWAA(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}) \leq \chi^{+}_{\hat{j}\text{BCFRN}} \quad (22)$$

Proof: We can rapidly and simply prove the required theorem by utilizing the properties of idempotency and monotonicity.

$$BCFRWAA(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}) \leq BCFRWAA(\chi^{+}_{1\text{BCFRN}}, \chi^{+}_{2\text{BCFRN}}, \ldots, \chi^{+}_{\eta\text{BCFRN}}) = \chi^{+}_{\hat{j}BCFRN}$$

$$BCFRWAA(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}) \geq BCFRWAA(\chi^{-}_{1\text{BCFRN}}, \chi^{-}_{2\text{BCFRN}}, \ldots, \chi^{-}_{\eta\text{BCFRN}}) = \chi^{-}_{\hat{j}\text{BCFRN}}$$

Hence, using the equation above, we arrive at

$$\chi^{-}_{\hat{j}\text{BCFRN}} \leq BCFRWAA(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}) \leq \chi^{+}_{\hat{j}\text{BCFRN}}$$

Definition 17: Let,

$$\chi_{\hat{j}\text{BCFRN}} = \left(\left(\underline{\pi^{+}_{\hat{j}}} + i\underline{\phi^{+}_{\hat{j}}}, \underline{\pi^{-}_{\hat{j}}} + i\underline{\phi^{-}_{\hat{j}}}\right), \left(\overline{\pi^{+}_{\hat{j}}} + i\overline{\phi^{+}_{\hat{j}}}, \overline{\pi^{-}_{\hat{j}}} + i\overline{\phi^{-}_{\hat{j}}}\right)\right)$$

be the family of BCFRNs where $\hat{j} = 1, 2, 3, \ldots, \eta$, then the notion of the BCFROWAA operator is defined by:

$$BCFROWAA(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}) = \bigoplus_{\hat{j}=1}^{\eta} \tau_{\omega_{\hat{j}}} \chi_{\hat{O}_{\hat{j}\text{BCFRN}}}$$
$$= \tau_{\omega_1}(\chi_{\hat{O}_{1\text{BCFRN}}}) \oplus \tau_{\omega_2}(\chi_{\hat{O}_{2\text{BCFRN}}}) \oplus \ldots \oplus \tau_{\omega_\eta}(\chi_{\hat{O}_{\eta\text{BCFRN}}}) \quad (23)$$

where $(\hat{O}_1, \hat{O}_2, \ldots, \hat{O}_\eta)$ are the permutation of $(1, 2, 3, \ldots, \eta)$ such that

$$\chi_{\hat{O}_{(\hat{j}-1)\text{BCFRN}}} \geq \chi_{\hat{O}_{\hat{j}\text{BCFRN}}} \quad \forall \hat{j}.$$

Theorem 5: Let,

$$\chi_{\hat{j}\text{BCFRN}} = \left(\left(\underline{\pi^{+}_{\hat{j}}} + i\underline{\phi^{+}_{\hat{j}}}, \underline{\pi^{-}_{\hat{j}}} + i\underline{\phi^{-}_{\hat{j}}}\right), \left(\overline{\pi^{+}_{\hat{j}}} + i\overline{\phi^{+}_{\hat{j}}}, \overline{\pi^{-}_{\hat{j}}} + i\overline{\phi^{-}_{\hat{j}}}\right)\right)$$

be the family of BCFRNs where $\hat{j} = 1, 2, 3, \ldots, \eta$. Then we can see that the aggregated result obtained by using the definition (17) is again BCFRN and is given by:

$$\text{BCFROWAA}\left(\chi_{1\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}\right) = \begin{pmatrix} 1 - \prod_{\hat{j}=1}^{\eta}\left(1 - \underline{\pi^{+}_{\hat{O}(\hat{j})}}\right)^{\tau_{\omega_{\hat{j}}}} + i\left(1 - \prod_{\hat{j}=1}^{\eta}\left(1 - \underline{\phi^{+}_{\hat{O}(\hat{j})}}\right)^{\tau_{\omega_{\hat{j}}}}\right), \\ -\prod_{\hat{j}=1}^{\eta}\left|\underline{\pi^{-}_{\hat{O}(\hat{j})}}\right|^{\tau_{\omega_{\hat{j}}}} + i\left(-\prod_{\hat{j}=1}^{\eta}\left|\underline{\phi^{-}_{\hat{O}(\hat{j})}}\right|^{\tau_{\omega_{\hat{j}}}}\right), \\ 1 - \prod_{\hat{j}=1}^{\eta}\left(1 - \overline{\pi^{+}_{\hat{O}(\hat{j})}}\right)^{\tau_{\omega_{\hat{j}}}} + i\left(1 - \prod_{\hat{j}=1}^{\eta}\left(1 - \overline{\phi^{+}_{\hat{O}(\hat{j})}}\right)^{\tau_{\omega_{\hat{j}}}}\right), \\ -\prod_{\hat{j}=1}^{\eta}\left|\overline{\pi^{-}_{\hat{O}(\hat{j})}}\right|^{\tau_{\omega_{\hat{j}}}} + i\left(-\prod_{\hat{j}=1}^{\eta}\left|\overline{\phi^{-}_{\hat{O}(\hat{j})}}\right|^{\tau_{\omega_{\hat{j}}}}\right) \end{pmatrix} \quad (24)$$

Proof: Trivial.

Theorem 6 (Idempotency property): Let

$$\chi_{\hat{j}\text{BCFRN}} = \left(\left(\underline{\pi_{\hat{j}}^{+}} + i\underline{\phi_{\hat{j}}^{+}}, \underline{\pi_{\hat{j}}^{-}} + i\underline{\phi_{\hat{j}}^{-}}\right), \left(\overline{\pi_{\hat{j}}^{+}} + i\overline{\phi_{\hat{j}}^{+}}, \overline{\pi_{\hat{j}}^{-}} + i\overline{\phi_{\hat{j}}^{-}}\right)\right)$$

be the family of BCFRNs where $\hat{j} = 1, 2, 3, \ldots, \eta$. If $\chi_{\hat{j}\text{BCFRN}} = \chi_{\text{BCFRN}}\ \forall\, \hat{j}$, i.e., $\underline{\pi_{\hat{j}}^{+}} = \underline{\pi}^{+}$, $\underline{\phi_{\hat{j}}^{+}} = \underline{\phi}^{+}$, $\underline{\pi_{\hat{j}}^{-}} = \underline{\pi}^{-}$, $\underline{\phi_{\hat{j}}^{-}} = \underline{\phi}^{-}$, $\overline{\pi_{\hat{j}}^{+}} = \overline{\pi^{+}}$, $\overline{\phi_{\hat{j}}^{+}} = \overline{\phi^{+}}$, $\overline{\pi_{\hat{j}}^{-}} = \overline{\pi^{-}}$, $\overline{\phi_{\hat{j}}^{-}} = \overline{\phi^{-}}\ \forall\, \hat{j}$, then

$$\text{BCFROWAA}\left(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}\right) = \chi_{\text{BCFRN}} \tag{25}$$

Proof: Trivial.

Theorem 7 (Monotonicity property): Let,

$$\chi_{\hat{j}\text{BCFRN}} = \left(\left(\underline{\pi_{\hat{j}}^{+}} + i\underline{\phi_{\hat{j}}^{+}}, \underline{\pi_{\hat{j}}^{-}} + i\underline{\phi_{\hat{j}}^{-}}\right), \left(\overline{\pi_{\hat{j}}^{+}} + i\overline{\phi_{\hat{j}}^{+}}, \overline{\pi_{\hat{j}}^{-}} + i\overline{\phi_{\hat{j}}^{-}}\right)\right)$$

and

$$\chi^{*}_{\hat{j}\text{BCFRN}} = \left(\left(\underline{\pi_{\hat{j}}^{*+}} + i\underline{\phi_{\hat{j}}^{*+}}, \underline{\pi_{\hat{j}}^{*-}} + i\underline{\phi_{\hat{j}}^{*-}}\right), \left(\overline{\pi_{\hat{j}}^{*+}} + i\overline{\phi_{\hat{j}}^{*+}}, \overline{\pi_{\hat{j}}^{*-}} + i\overline{\phi_{\hat{j}}^{*-}}\right)\right)$$

be two collections of BCFRNs, where $\hat{j} = 1, 2, 3, \ldots, \eta$. If

$$\underline{\pi_{\hat{j}}^{+}} \le \underline{\pi_{\hat{j}}^{*+}}, \quad \underline{\phi_{\hat{j}}^{+}} \le \underline{\phi_{\hat{j}}^{*+}}, \quad \underline{\pi_{\hat{j}}^{-}} \le \underline{\pi_{\hat{j}}^{*+}}, \quad \underline{\phi_{\hat{j}}^{-}} \le \underline{\phi_{\hat{j}}^{*-}}$$

$$\overline{\pi_{\hat{j}}^{+}} \le \overline{\pi_{\hat{j}}^{*+}}, \quad \overline{\phi_{\hat{j}}^{+}} \le \overline{\phi_{\hat{j}}^{*+}}, \quad \overline{\pi_{\hat{j}}^{-}} \le \overline{\pi_{\hat{j}}^{*-}} \quad \text{and} \quad \overline{\phi_{\hat{j}}^{-}} \le \overline{\phi_{\hat{j}}^{*-}} \quad \forall \hat{j},$$

then

$$BCFROWAA(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}) \le BCFROWAA(\chi^{*}_{1\text{BCFRN}}, \chi^{*}_{2\text{BCFRN}}, \ldots, \chi^{*}_{\eta\text{BCFRN}}). \tag{26}$$

Proof: Trivial.

Theorem 8 (Boundedness property): Let,

$$\chi_{\hat{j}\text{BCFRN}} = \left((\underline{\pi_{\hat{j}}^{+}} + i\underline{\phi_{\hat{j}}^{+}}, \underline{\pi_{\hat{j}}^{-}} + i\underline{\pi_{\hat{j}}^{-}}), \quad (\overline{\pi_{\hat{j}}^{+}} + i\overline{\phi_{\hat{j}}^{+}}, \overline{\pi_{\hat{j}}^{-}} + i\overline{\pi_{\hat{j}}^{-}}) \right)$$

be the family of BCFRNs where $\hat{j} = 1, 2, 3, \ldots, \eta$, and assume that

$$\chi^{-}_{\hat{j}\text{BCFRN}} = \begin{pmatrix} \min\limits_{\hat{j}}\{\underline{\pi_{\hat{j}}^{+}}\} + i\min\limits_{\hat{j}}\{\underline{\phi_{\hat{j}}^{+}}\}, \min\limits_{\hat{j}}\{\underline{\pi_{\hat{j}}^{-}}\} + i\min\limits_{\hat{j}}\{\underline{\phi_{\hat{j}}^{-}}\}, \\ \min\limits_{\hat{j}}\{\overline{\pi_{\hat{j}}^{+}}\} + i\min\limits_{\hat{j}}\{\overline{\phi_{\hat{j}}^{+}}\}, \min\limits_{\hat{j}}\{\overline{\pi_{\hat{j}}^{-}}\} + i\min\limits_{\hat{j}}\{\overline{\phi_{\hat{j}}^{-}}\} \end{pmatrix}$$

and

$$\chi^{+}_{\hat{j}\text{BCFRN}} = \begin{pmatrix} \max\limits_{\hat{j}}\{\underline{\pi_{\hat{j}}^{+}}\} + i\max\limits_{\hat{j}}\{\underline{\phi_{\hat{j}}^{+}}\}, \max\limits_{\hat{j}}\{\underline{\pi_{\hat{j}}^{-}}\} + i\max\limits_{\hat{j}}\{\underline{\phi_{\hat{j}}^{-}}\}, \\ \max\limits_{\hat{j}}\{\overline{\pi_{\hat{j}}^{+}}\} + i\max\limits_{\hat{j}}\{\overline{\phi_{\hat{j}}^{+}}\}, \max\limits_{\hat{j}}\{\overline{\pi_{\hat{j}}^{-}}\} + i\max\limits_{\hat{j}}\{\overline{\phi_{\hat{j}}^{-}}\} \end{pmatrix}$$

then

$$\chi^{-}_{\hat{j}\text{BCFRN}} \le BCFROWAA(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}) \le \chi^{+}_{\hat{j}\text{BCFRN}} \tag{27}$$

Proof: Trivial.

Definition 18 Let,

$$\chi_{\hat{j}\text{BCFRN}} = \left(\left(\underline{\pi_{\hat{j}}^{+}} + i\underline{\phi_{\hat{j}}^{+}}, \underline{\pi_{\hat{j}}^{-}} + i\underline{\phi_{\hat{j}}^{-}}\right), \left(\overline{\pi_{\hat{j}}^{+}} + i\overline{\phi_{\hat{j}}^{+}}, \overline{\pi_{\hat{j}}^{-}} + i\overline{\phi_{\hat{j}}^{-}}\right)\right)$$

be the family of BCFRNs where $\hat{j} = 1, 2, 3, \ldots, \eta$. Then the notion of BCFRWGA aggregation operator is defined by:

$$\text{BCFRWGA}\left(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}\right) = \bigotimes_{\hat{j}=1}^{\eta}\left(\chi_{\hat{j}\text{BCFRN}}\right)^{\tau_{\varpi_{\hat{j}}}} = \left(\chi_{1\text{BCFRN}}\right)^{\tau_{\varpi_1}} \otimes \cdots \otimes \left(\chi_{\eta\text{BCFRN}}\right)^{\tau_{\varpi_\eta}} \quad (28)$$

Theorem 9: Let

$$\chi_{\hat{j}\text{BCFRN}} = \left(\left(\underline{\pi_{\hat{j}}^{+}} + i\underline{\phi_{\hat{j}}^{+}}, \underline{\pi_{\hat{j}}^{-}} + i\underline{\phi_{\hat{j}}^{-}}\right), \left(\overline{\pi_{\hat{j}}^{+}} + i\overline{\phi_{\hat{j}}^{+}}, \overline{\pi_{\hat{j}}^{-}} + i\overline{\phi_{\hat{j}}^{-}}\right)\right)$$

be the family of BCFRNs where $\hat{j} = 1, 2, 3, \ldots, \eta$. Then we can see that the aggregated result obtained by using the definition (18) is again BCFRN and is given by:

$$\text{BCFRWGA}\left(\chi_{1\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}\right) = \begin{pmatrix} \prod_{\hat{j}=1}^{\eta}\left(\underline{\pi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}} + i\prod_{\hat{j}=1}^{\eta}\left(\underline{\phi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}}, \\ -1 + \prod_{\hat{j}=1}^{\eta}\left(1 + \underline{\pi_{\hat{j}}^{-}}\right)^{\tau_{\varpi_{\hat{j}}}} + i\left(-1 + \prod_{\hat{j}=1}^{\eta}\left(1 + \underline{\phi_{\hat{j}}^{-}}\right)^{\tau_{\varpi_{\hat{j}}}}\right), \\ \prod_{\hat{j}=1}^{\eta}\left(\overline{\pi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}} + i\prod_{\hat{j}=1}^{\eta}\left(\overline{\phi_{\hat{j}}^{+}}\right)^{\tau_{\varpi_{\hat{j}}}}, \\ -1 + \prod_{\hat{j}=1}^{\eta}\left(1 + \overline{\pi_{\hat{j}}^{-}}\right)^{\tau_{\varpi_{\hat{j}}}} + i\left(-1 + \prod_{\hat{j}=1}^{\eta}\left(1 + \overline{\phi_{\hat{j}}^{-}}\right)^{\tau_{\varpi_{\hat{j}}}}\right) \end{pmatrix} \quad (29)$$

Proof: We prove equation (29) using mathematical induction. Assume for $\hat{j} = 2$:

$$\left(\chi_{1\text{BCFRN}}\right)^{\tau_{\varpi_1}} = \begin{pmatrix} \left(\underline{\pi_{1}^{+}}\right)^{\tau_{\varpi_1}} + i\left(\underline{\phi_{1}^{+}}\right)^{\tau_{\varpi_1}}, \\ -1 + \left(1 + \underline{\pi_{1}^{-}}\right)^{\tau_{\varpi_1}} + i\left(-1 + \left(1 + \underline{\phi_{1}^{-}}\right)^{\tau_{\varpi_1}}\right), \\ \left(\overline{\pi_{1}^{+}}\right)^{\tau_{\varpi_1}} + i\left(\overline{\phi_{1}^{+}}\right)^{\tau_{\varpi_1}}, \\ -1 + \left(1 + \overline{\pi_{1}^{-}}\right)^{\tau_{\varpi_1}} + i\left(-1 + \left(1 + \overline{\phi_{1}^{-}}\right)^{\tau_{\varpi_1}}\right) \end{pmatrix},$$

$$\left(\chi_{2\text{BCFRN}}\right)^{\tau_{\varpi_2}} = \begin{pmatrix} \left(\underline{\pi_{2}^{+}}\right)^{\tau_{\varpi_2}} + i\left(\underline{\phi_{2}^{+}}\right)^{\tau_{\varpi_2}}, \\ -1 + \left(1 + \underline{\pi_{2}^{-}}\right)^{\tau_{\varpi_2}} + i\left(-1 + \left(1 + \underline{\phi_{2}^{-}}\right)^{\tau_{\varpi_2}}\right), \\ \left(\overline{\pi_{2}^{+}}\right)^{\tau_{\varpi_2}} + i\left(\overline{\phi_{2}^{+}}\right)^{\tau_{\varpi_2}}, \\ -1 + \left(1 + \overline{\pi_{2}^{-}}\right)^{\tau_{\varpi_2}} + i\left(-1 + \left(1 + \overline{\phi_{2}^{-}}\right)^{\tau_{\varpi_2}}\right) \end{pmatrix},$$

$$(\chi_{1\text{BCFRN}})^{\tau_{\varpi_1}} \otimes (\chi_{2\text{BCFRN}})^{\tau_{\varpi_2}} = \begin{pmatrix} \left(\underline{\pi_1^+}\right)^{\tau_{\varpi_1}} \left(\underline{\pi_2^+}\right)^{\tau_{\varpi_2}} + i\left(\underline{\phi_1^+}\right)^{\tau_{\varpi_1}} \left(\underline{\phi_2^+}\right)^{\tau_{\varpi_2}}, \\ -1 + \left(1+\underline{\pi_1^-}\right)^{\tau_{\varpi_1}} \left(1+\underline{\pi_2^-}\right)^{\tau_{\varpi_2}} + i\left(-1 + \left(1+\underline{\phi_1^-}\right)^{\tau_{\varpi_1}} \left(1+\underline{\phi_2^-}\right)^{\tau_{\varpi_2}}\right), \\ \left(\overline{\pi_1^+}\right)^{\tau_{\varpi_1}} \left(\overline{\pi_2^+}\right)^{\tau_{\varpi_2}} + i\left(\overline{\phi_1^+}\right)^{\tau_{\varpi_1}} \left(\overline{\phi_2^+}\right)^{\tau_{\varpi_2}}, \\ -1 + \left(1+\overline{\pi_1^-}\right)^{\tau_{\varpi_1}} \left(1+\overline{\pi_2^-}\right)^{\tau_{\varpi_2}} + i\left(-1 + \left(1+\overline{\phi_1^-}\right)^{\tau_{\varpi_1}} \left(1+\overline{\phi_2^-}\right)^{\tau_{\varpi_2}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \prod_{\hat{j}=1}^{2} \left(\underline{\pi_{\hat{j}}^+}\right)^{\tau_{\varpi_{\hat{j}}}} + i\prod_{\hat{j}=1}^{2} \left(\underline{\phi_{\hat{j}}^+}\right)^{\tau_{\varpi_{\hat{j}}}}, \\ -1 + \prod_{\hat{j}=1}^{2} \left(1+\underline{\pi_{\hat{j}}^-}\right)^{\tau_{\varpi_{\hat{j}}}} + i\left(-1 + \prod_{\hat{j}=1}^{2} \left(1+\underline{\phi_{\hat{j}}^-}\right)^{\tau_{\varpi_{\hat{j}}}}\right), \\ \prod_{\hat{j}=1}^{2} \left(\overline{\pi_{\hat{j}}^+}\right)^{\tau_{\varpi_{\hat{j}}}} + i\prod_{\hat{j}=1}^{2} \left(\overline{\phi_{\hat{j}}^+}\right)^{\tau_{\varpi_{\hat{j}}}}, \\ -1 + \prod_{\hat{j}=1}^{2} \left(1+\overline{\pi_{\hat{j}}^-}\right)^{\tau_{\varpi_{\hat{j}}}} + i\left(-1 + \prod_{\hat{j}=1}^{2} \left(1+\overline{\phi_{\hat{j}}^-}\right)^{\tau_{\varpi_{\hat{j}}}}\right) \end{pmatrix}.$$

Now, assume that equation (29) holds for $\hat{j} = \eta$, i.e.,

$$\text{BCFRWGA}\left(\chi_{1\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}\right) = \begin{pmatrix} \prod_{\hat{j}=1}^{\eta} \left(\underline{\pi_{\hat{j}}^+}\right)^{\tau_{\varpi_{\hat{j}}}} + i\prod_{\hat{j}=1}^{\eta} \left(\underline{\phi_{\hat{j}}^+}\right)^{\tau_{\varpi_{\hat{j}}}}, \\ -1 + \prod_{\hat{j}=1}^{\eta} \left(1+\underline{\pi_{\hat{j}}^-}\right)^{\tau_{\varpi_{\hat{j}}}} + i\left(-1 + \prod_{\hat{j}=1}^{\eta} \left(1+\underline{\phi_{\hat{j}}^-}\right)^{\tau_{\varpi_{\hat{j}}}}\right), \\ \prod_{\hat{j}=1}^{\eta} \left(\overline{\pi_{\hat{j}}^+}\right)^{\tau_{\varpi_{\hat{j}}}} + i\prod_{\hat{j}=1}^{\eta} \left(\overline{\phi_{\hat{j}}^+}\right)^{\tau_{\varpi_{\hat{j}}}}, \\ -1 + \prod_{\hat{j}=1}^{\eta} \left(1+\overline{\pi_{\hat{j}}^-}\right)^{\tau_{\varpi_{\hat{j}}}} + i\left(-1 + \prod_{\hat{j}=1}^{\eta} \left(1+\overline{\phi_{\hat{j}}^-}\right)^{\tau_{\varpi_{\hat{j}}}}\right) \end{pmatrix}$$

For $\hat{j} = \eta + 1$, and using (28), we have:

$$\text{BCFRWGA}\left(\chi_{1\text{BCFRN}}, \ldots, \chi_{(\eta+1)\text{BCFRN}}\right) = \begin{pmatrix} \prod_{\hat{j}=1}^{\eta+1} \left(\underline{\pi_{\hat{j}}^+}\right)^{\tau_{\varpi_{\hat{j}}}} + i\prod_{\hat{j}=1}^{\eta+1} \left(\underline{\phi_{\hat{j}}^+}\right)^{\tau_{\varpi_{\hat{j}}}}, \\ -1 + \prod_{\hat{j}=1}^{\eta+1} \left(1+\underline{\pi_{\hat{j}}^-}\right)^{\tau_{\varpi_{\hat{j}}}} + i\left(-1 + \prod_{\hat{j}=1}^{\eta+1} \left(1+\underline{\phi_{\hat{j}}^-}\right)^{\tau_{\varpi_{\hat{j}}}}\right), \\ \prod_{\hat{j}=1}^{\eta+1} \left(\overline{\pi_{\hat{j}}^+}\right)^{\tau_{\varpi_{\hat{j}}}} + i\prod_{\hat{j}=1}^{\eta+1} \left(\overline{\phi_{\hat{j}}^+}\right)^{\tau_{\varpi_{\hat{j}}}}, \\ -1 + \prod_{\hat{j}=1}^{\eta+1} \left(1+\overline{\pi_{\hat{j}}^-}\right)^{\tau_{\varpi_{\hat{j}}}} + i\left(-1 + \prod_{\hat{j}=1}^{\eta+1} \left(1+\overline{\phi_{\hat{j}}^-}\right)^{\tau_{\varpi_{\hat{j}}}}\right) \end{pmatrix}$$

This implies that the above eq (29) is true for $\eta > 0$

.

Theorem 10 (Idempotency property): Let,

$$\chi_{\hat{j}\text{BCFRN}} = \left(\left(\underline{\pi_{\hat{j}}^+} + i\underline{\phi_{\hat{j}}^+}, \underline{\pi_{\hat{j}}^-} + i\underline{\phi_{\hat{j}}^-}\right), \left(\overline{\pi_{\hat{j}}^+} + i\overline{\phi_{\hat{j}}^+}, \overline{\pi_{\hat{j}}^-} + i\overline{\phi_{\hat{j}}^-}\right)\right)$$

be the family of BCFRNs where $\hat{j} = 1, 2, 3, \ldots, \eta$. If $\chi_{\hat{j}\text{BCFRN}} = \chi_{\text{BCFRN}}\ \forall\ \hat{j}$, i.e., $\underline{\pi_{\hat{j}}^+} = \underline{\pi}^+$, $\underline{\phi_{\hat{j}}^+} = \underline{\phi}^+$, $\underline{\pi_{\hat{j}}^-} = \underline{\pi}^-$, $\underline{\phi_{\hat{j}}^-} = \underline{\phi}^-$, $\overline{\pi_{\hat{j}}^+} = \overline{\pi^+}$, $\overline{\phi_{\hat{j}}^+} = \overline{\phi^+}$, $\overline{\pi_{\hat{j}}^-} = \overline{\pi^-}$, $\overline{\phi_{\hat{j}}^-} = \overline{\phi^-}\ \forall\ \hat{j}$, then

$$\text{BCFRWGA}\left(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}\right) = \chi_{\text{BCFRN}} \quad (30)$$

Proof: Trivial.

Theorem 11 (Monotonicity property): Let,

$$\chi_{\hat{j}\text{BCFRN}} = \left(\left(\underline{\pi_{\hat{j}}^{+}} + i\underline{\phi_{\hat{j}}^{+}}, \underline{\pi_{\hat{j}}^{-}} + i\underline{\phi_{\hat{j}}^{-}}\right), \left(\overline{\pi_{\hat{j}}^{+}} + i\overline{\phi_{\hat{j}}^{+}}, \overline{\pi_{\hat{j}}^{-}} + i\overline{\phi_{\hat{j}}^{-}}\right)\right)$$

and

$$\chi_{\hat{j}\text{BCFRN}}^{*} = \left(\left(\underline{\pi_{\hat{j}}^{*+}} + i\underline{\phi_{\hat{j}}^{*+}}, \underline{\pi_{\hat{j}}^{*-}} + i\underline{\phi_{\hat{j}}^{*-}}\right), \left(\overline{\pi_{\hat{j}}^{*+}} + i\overline{\phi_{\hat{j}}^{*+}}, \overline{\pi_{\hat{j}}^{*-}} + i\overline{\phi_{\hat{j}}^{*-}}\right)\right)$$

be two collections of BCFRNs, where $\hat{j} = 1, 2, 3, \ldots, \eta$. If

$$\underline{\pi_{\hat{j}}^{+}} \leq \underline{\pi_{\hat{j}}^{*+}}, \quad \underline{\phi_{\hat{j}}^{+}} \leq \underline{\phi_{\hat{j}}^{*+}}, \quad \underline{\pi_{\hat{j}}^{-}} \leq \underline{\pi_{\hat{j}}^{*+}}, \quad \underline{\phi_{\hat{j}}^{-}} \leq \underline{\phi_{\hat{j}}^{*-}}$$
$$\overline{\pi_{\hat{j}}^{+}} \leq \overline{\pi_{\hat{j}}^{*+}}, \quad \overline{\phi_{\hat{j}}^{+}} \leq \overline{\phi_{\hat{j}}^{*+}}, \quad \overline{\pi_{\hat{j}}^{-}} \leq \overline{\pi_{\hat{j}}^{*-}} \quad \text{and} \quad \overline{\phi_{\hat{j}}^{-}} \leq \overline{\phi_{\hat{j}}^{*-}} \quad \forall \hat{j},$$

then

$$BCFRWGA(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}) \leq BCFRWGA(\chi_{1\text{BCFRN}}^{*}, \chi_{2\text{BCFRN}}^{*}, \ldots, \chi_{\eta\text{BCFRN}}^{*}). \quad (31)$$

Proof: Trivial.

Theorem 12 (Boundedness property): Let,

$$\chi_{\hat{j}\text{BCFRN}} = \left((\underline{\pi_{\hat{j}}^{+}} + i\underline{\phi_{\hat{j}}^{+}}, \underline{\pi_{\hat{j}}^{-}} + i\underline{\pi_{\hat{j}}^{-}}), \quad (\overline{\pi_{\hat{j}}^{+}} + i\overline{\phi_{\hat{j}}^{+}}, \overline{\pi_{\hat{j}}^{-}} + i\overline{\pi_{\hat{j}}^{-}}) \right)$$

be the family of BCFRNs where $\hat{j} = 1, 2, 3, \ldots, \eta$, and assume that

$$\chi_{\hat{j}\text{BCFRN}}^{-} = \begin{pmatrix} \min_{\hat{j}}\{\underline{\pi_{\hat{j}}^{+}}\} + i\min_{\hat{j}}\{\underline{\phi_{\hat{j}}^{+}}\}, \min_{\hat{j}}\{\underline{\pi_{\hat{j}}^{-}}\} + i\min_{\hat{j}}\{\underline{\phi_{\hat{j}}^{-}}\}, \\ \min_{\hat{j}}\{\overline{\pi_{\hat{j}}^{+}}\} + i\min_{\hat{j}}\{\overline{\phi_{\hat{j}}^{+}}\}, \min_{\hat{j}}\{\overline{\pi_{\hat{j}}^{-}}\} + i\min_{\hat{j}}\{\overline{\phi_{\hat{j}}^{-}}\} \end{pmatrix}$$

and

$$\chi_{\hat{j}\text{BCFRN}}^{+} = \begin{pmatrix} \max_{\hat{j}}\{\underline{\pi_{\hat{j}}^{+}}\} + i\max_{\hat{j}}\{\underline{\phi_{\hat{j}}^{+}}\}, \max_{\hat{j}}\{\underline{\pi_{\hat{j}}^{-}}\} + i\max_{\hat{j}}\{\underline{\phi_{\hat{j}}^{-}}\}, \\ \max_{\hat{j}}\{\overline{\pi_{\hat{j}}^{+}}\} + i\max_{\hat{j}}\{\overline{\phi_{\hat{j}}^{+}}\}, \max_{\hat{j}}\{\overline{\pi_{\hat{j}}^{-}}\} + i\max_{\hat{j}}\{\overline{\phi_{\hat{j}}^{-}}\} \end{pmatrix}$$

then

$$\chi_{\hat{j}\text{BCFRN}}^{-} \leq BCFRWGA(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}) \leq \chi_{\hat{j}\text{BCFRN}}^{+} \quad (32)$$

Proof: Trivial.

Definition 19: Let,

$$\chi_{\hat{j}\text{BCFRN}} = \left(\left(\underline{\pi_{\hat{j}}^{+}} + i\underline{\phi_{\hat{j}}^{+}}, \underline{\pi_{\hat{j}}^{-}} + i\underline{\phi_{\hat{j}}^{-}}\right), \left(\overline{\pi_{\hat{j}}^{+}} + i\overline{\phi_{\hat{j}}^{+}}, \overline{\pi_{\hat{j}}^{-}} + i\overline{\phi_{\hat{j}}^{-}}\right)\right)$$

be the family of BCFRNs where $\hat{j} = 1, 2, 3, \ldots, \eta$, then the notion of the BCFROWGA operator is defined by:

$$BCFROWGA(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}) = \bigotimes_{\hat{j}=1}^{\eta} \tau_{\omega_{\hat{j}}} \chi_{\hat{O}_{\hat{j}\text{BCFRN}}}$$
$$= \tau_{\omega_1}(\chi_{\hat{O}_{1\text{BCFRN}}}) \otimes \tau_{\omega_2}(\chi_{\hat{O}_{2\text{BCFRN}}}) \otimes \ldots \otimes \tau_{\omega_\eta}(\chi_{\hat{O}_{\eta\text{BCFRN}}}) \quad (33)$$

where $(\hat{O}_1, \hat{O}_2, \ldots, \hat{O}_\eta)$ are the permutation of $(1, 2, 3, \ldots, \eta)$ such that

$$\chi_{\hat{O}_{(\hat{j}-1)\text{BCFRN}}} \geq \chi_{\hat{O}_{\hat{j}\text{BCFRN}}} \quad \forall \hat{j}.$$

Theorem 13: Let,

$$\chi_{\hat{j}\text{BCFRN}} = \left(\left(\underline{\pi_{\hat{j}}^{+}} + i\underline{\phi_{\hat{j}}^{+}}, \underline{\pi_{\hat{j}}^{-}} + i\underline{\phi_{\hat{j}}^{-}}\right), \left(\overline{\pi_{\hat{j}}^{+}} + i\overline{\phi_{\hat{j}}^{+}}, \overline{\pi_{\hat{j}}^{-}} + i\overline{\phi_{\hat{j}}^{-}}\right)\right)$$

be the family of BCFRNs where $\hat{j} = 1, 2, 3, \ldots, \eta$. Then we can see that the aggregated result obtained by using the definition (19) is again BCFRN and is given by:

$$\text{BCFROWGA}\left(\chi_{1\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}\right) = \left(\begin{array}{c} \prod_{\hat{j}=1}^{\eta}\left(\underline{\pi_{\hat{O}(\hat{j})}^{+}}\right)^{\tau_{\omega_{\hat{j}}}} + i\left(\prod_{\hat{j}=1}^{\eta}\left(\underline{\phi_{\hat{O}(\hat{j})}^{+}}\right)^{\tau_{\omega_{\hat{j}}}}\right), \\ -1 + \prod_{\hat{j}=1}^{\eta}\left(1 + \underline{\pi_{\hat{O}(\hat{j})}^{-}}\right)^{\tau_{\omega_{\hat{j}}}} + i\left(-1 + \prod_{\hat{j}=1}^{\eta}\left(1 + \underline{\phi_{\hat{O}(\hat{j})}^{-}}\right)^{\tau_{\omega_{\hat{j}}}}\right), \\ \prod_{\hat{j}=1}^{\eta}\left(\overline{\pi_{\hat{O}(\hat{j})}^{+}}\right)^{\tau_{\omega_{\hat{j}}}} + i\left(\prod_{\hat{j}=1}^{\eta}\left(\overline{\phi_{\hat{O}(\hat{j})}^{+}}\right)^{\tau_{\omega_{\hat{j}}}}\right), \\ -1 + \prod_{\hat{j}=1}^{\eta}\left(1 + \overline{\pi_{\hat{O}(\hat{j})}^{-}}\right)^{\tau_{\omega_{\hat{j}}}} + i\left(-1 + \prod_{\hat{j}=1}^{\eta}\left(1 + \overline{\phi_{\hat{O}(\hat{j})}^{-}}\right)^{\tau_{\omega_{\hat{j}}}}\right) \end{array}\right)$$

(34)

Theorem 14 (Idempotency property): Let,

$$\chi_{\hat{j}\text{BCFRN}} = \left(\left(\underline{\pi_{\hat{j}}^{+}} + i\underline{\phi_{\hat{j}}^{+}}, \underline{\pi_{\hat{j}}^{-}} + i\underline{\phi_{\hat{j}}^{-}}\right), \left(\overline{\pi_{\hat{j}}^{+}} + i\overline{\phi_{\hat{j}}^{+}}, \overline{\pi_{\hat{j}}^{-}} + i\overline{\phi_{\hat{j}}^{-}}\right)\right)$$

be the family of BCFRNs where $\hat{j} = 1, 2, 3, \ldots, \eta$. If $\chi_{\hat{j}\text{BCFRN}} = \chi_{\text{BCFRN}}\ \forall\ \hat{j}$, i.e., $\underline{\pi_{\hat{j}}^{+}} = \underline{\pi}^{+}$, $\underline{\phi_{\hat{j}}^{+}} = \underline{\phi}^{+}$, $\underline{\pi_{\hat{j}}^{-}} = \underline{\pi}^{-}$, $\underline{\phi_{\hat{j}}^{-}} = \underline{\phi}^{-}$, $\overline{\pi_{\hat{j}}^{+}} = \overline{\pi^{+}}$, $\overline{\phi_{\hat{j}}^{+}} = \overline{\phi^{+}}$, $\overline{\pi_{\hat{j}}^{-}} = \overline{\pi^{-}}$, $\overline{\phi_{\hat{j}}^{-}} = \overline{\phi^{-}}\ \forall\ \hat{j}$, then

$$\text{BCFROWGA}\left(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}\right) = \chi_{\text{BCFRN}} \quad (35)$$

Proof: Trivial.

Theorem 15 (Monotonicity property): Let,

$$\chi_{\hat{j}\text{BCFRN}} = \left(\left(\underline{\pi_{\hat{j}}^{+}} + i\underline{\phi_{\hat{j}}^{+}}, \underline{\pi_{\hat{j}}^{-}} + i\underline{\phi_{\hat{j}}^{-}}\right), \left(\overline{\pi_{\hat{j}}^{+}} + i\overline{\phi_{\hat{j}}^{+}}, \overline{\pi_{\hat{j}}^{-}} + i\overline{\phi_{\hat{j}}^{-}}\right)\right)$$

and

$$\chi_{\hat{j}\text{BCFRN}}^{*} = \left(\left(\underline{\pi_{\hat{j}}^{*+}} + i\underline{\phi_{\hat{j}}^{*+}}, \underline{\pi_{\hat{j}}^{*-}} + i\underline{\phi_{\hat{j}}^{*-}}\right), \left(\overline{\pi_{\hat{j}}^{*+}} + i\overline{\phi_{\hat{j}}^{*+}}, \overline{\pi_{\hat{j}}^{*-}} + i\overline{\phi_{\hat{j}}^{*-}}\right)\right)$$

be two collections of BCFRNs, where $\hat{j} = 1, 2, 3, \ldots, \eta$. If

$$\underline{\pi_{\hat{j}}^{+}} \leq \underline{\pi_{\hat{j}}^{*+}}, \quad \underline{\phi_{\hat{j}}^{+}} \leq \underline{\phi_{\hat{j}}^{*+}}, \quad \underline{\pi_{\hat{j}}^{-}} \leq \underline{\pi_{\hat{j}}^{*+}}, \quad \underline{\phi_{\hat{j}}^{-}} \leq \underline{\phi_{\hat{j}}^{*-}}$$

$$\overline{\pi_{\hat{j}}^{+}} \leq \overline{\pi_{\hat{j}}^{*+}}, \quad \overline{\phi_{\hat{j}}^{+}} \leq \overline{\phi_{\hat{j}}^{*+}}, \quad \overline{\pi_{\hat{j}}^{-}} \leq \overline{\pi_{\hat{j}}^{*-}} \quad \text{and} \quad \overline{\phi_{\hat{j}}^{-}} \leq \overline{\phi_{\hat{j}}^{*-}} \quad \forall \hat{j},$$

then

$$BCFROWGA(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}) \leq BCFROWGA(\chi_{1\text{BCFRN}}^{*}, \chi_{2\text{BCFRN}}^{*}, \ldots, \chi_{\eta\text{BCFRN}}^{*}). \quad (36)$$

Proof: Trivial.

Theorem 16 (Boundedness property): Let,

$$\chi_{\hat{j}\text{BCFRN}} = \left(\left(\underline{\pi_{\hat{j}}^{+}} + i\underline{\phi_{\hat{j}}^{+}}, \underline{\pi_{\hat{j}}^{-}} + i\underline{\pi_{\hat{j}}^{-}}\right), \quad \left(\overline{\pi_{\hat{j}}^{+}} + i\overline{\phi_{\hat{j}}^{+}}, \overline{\pi_{\hat{j}}^{-}} + i\overline{\pi_{\hat{j}}^{-}}\right) \right)$$

be the family of BCFRNs where $\hat{j} = 1, 2, 3, \ldots, \eta$, and assume that

$$\chi_{\hat{j}\text{BCFRN}}^{-} = \begin{pmatrix} \min\limits_{\hat{j}}\{\underline{\pi_{\hat{j}}^{+}}\} + i\min\limits_{\hat{j}}\{\underline{\phi_{\hat{j}}^{+}}\}, \min\limits_{\hat{j}}\{\underline{\pi_{\hat{j}}^{-}}\} + i\min\limits_{\hat{j}}\{\underline{\phi_{\hat{j}}^{-}}\}, \\ \min\limits_{\hat{j}}\{\overline{\pi_{\hat{j}}^{+}}\} + i\min\limits_{\hat{j}}\{\overline{\phi_{\hat{j}}^{+}}\}, \min\limits_{\hat{j}}\{\overline{\pi_{\hat{j}}^{-}}\} + i\min\limits_{\hat{j}}\{\overline{\phi_{\hat{j}}^{-}}\} \end{pmatrix}$$

and

$$\chi_{\hat{j}\text{BCFRN}}^{+} = \begin{pmatrix} \max\limits_{\hat{j}}\{\underline{\pi_{\hat{j}}^{+}}\} + i\max\limits_{\hat{j}}\{\underline{\phi_{\hat{j}}^{+}}\}, \max\limits_{\hat{j}}\{\underline{\pi_{\hat{j}}^{-}}\} + i\max\limits_{\hat{j}}\{\underline{\phi_{\hat{j}}^{-}}\}, \\ \max\limits_{\hat{j}}\{\overline{\pi_{\hat{j}}^{+}}\} + i\max\limits_{\hat{j}}\{\overline{\phi_{\hat{j}}^{+}}\}, \max\limits_{\hat{j}}\{\overline{\pi_{\hat{j}}^{-}}\} + i\max\limits_{\hat{j}}\{\overline{\phi_{\hat{j}}^{-}}\} \end{pmatrix}$$

then

$$\chi_{\hat{j}\text{BCFRN}}^{-} \leq BCFROWGA(\chi_{1\text{BCFRN}}, \chi_{2\text{BCFRN}}, \ldots, \chi_{\eta\text{BCFRN}}) \leq \chi_{\hat{j}\text{BCFRN}}^{+} \quad (37)$$

Proof: Trivial.

4. MADM TECHNIQUE IN THE ENVIRONMENT OF BCFRS

Several aspects of our personal and professional lives are affected by decision-making (DM), which is an essential component of real-life settings. It is essential to solving issues related to everyday life. It includes determining problems, assessing various solutions, and selecting the best plan of action to deal with the situation. The MADM approach provides an organized structure for managing complexity and ambiguity in diverse real-world scenarios, where several variables or parameters are required. Using the MADM technique based on diagnosed operators is the fundamental concept of this theory. In this section, first, we define the algorithm of the MADM technique, and then based on this algorithm we apply the MADM technique in a Cyber Security system to select the best alternative of C-S system to secure the network and digital data, etc.

4.1. **Algorithm of MADM.** To continue the process of DM by assessing different real-world problems in the light of identified work, we suppose that the collection of alternatives $\tilde{\mathcal{A}} = (\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_m)$ and their attributes $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n)$ with $\tau_\omega = \{\tau_{\omega_1}, \tau_{\omega_2}, \ldots, \tau_{\omega_n}\}$ represents the weight vectors (WVs) such that τ_{ω_j} must belong to $[0, 1]$ and $\sum_{j=1}^{n} \tau_{\omega_j} = 1$. We recommend using several types of data in the form of a BCFRN context to illustrate the matrix such as $\chi_{\text{BCFRN}} = (\underline{\pi}^{+} + i\underline{\phi}^{+}, \underline{\pi}^{-} + i\underline{\phi}^{-}, \underline{\pi}^{+} + i\overline{\phi}^{+}, \overline{\pi}^{-} + i\overline{\phi}^{-})$. We set up a few procedures for assessing the aforementioned dilemmas to increase the significance of the proposed theory.

Step 1: Sort the data and identify if it is cost or benefit-type data. Normalize it if it is of the cost type by using

$$\tilde{N}_{\text{normalize}} = \begin{cases} (\underline{\pi}^{+} + i\underline{\phi}^{+}, \underline{\pi}^{-} + i\underline{\phi}^{-}, \\ \overline{\pi^{+}} + i\overline{\phi^{+}}, \overline{\pi^{-}} + i\overline{\phi^{-}}) & \text{Benefit} \\ (1 - \underline{\pi}^{+} + i(1 - \underline{\phi}^{+}), -1 + \underline{\pi}^{-} + i(-1 + \underline{\phi}^{-}), 1 - \overline{\pi^{+}} + i(1 - \overline{\phi^{+}}), \\ -1 + \overline{\pi^{-}} + i(-1 + \overline{\phi^{-}})) & \text{Cost} \end{cases}$$

Step 2: Determine the singleton values by evaluating the data using the BCFRWAA, BCFROWAA, BCFRWGA, and BCFROWGA operators.

Step 3: The SVs of the aggregated data are evaluated in this step:

Step 4: Determine which alternatives are optimal by evaluating the ranking values in the light of the obtained SVs.

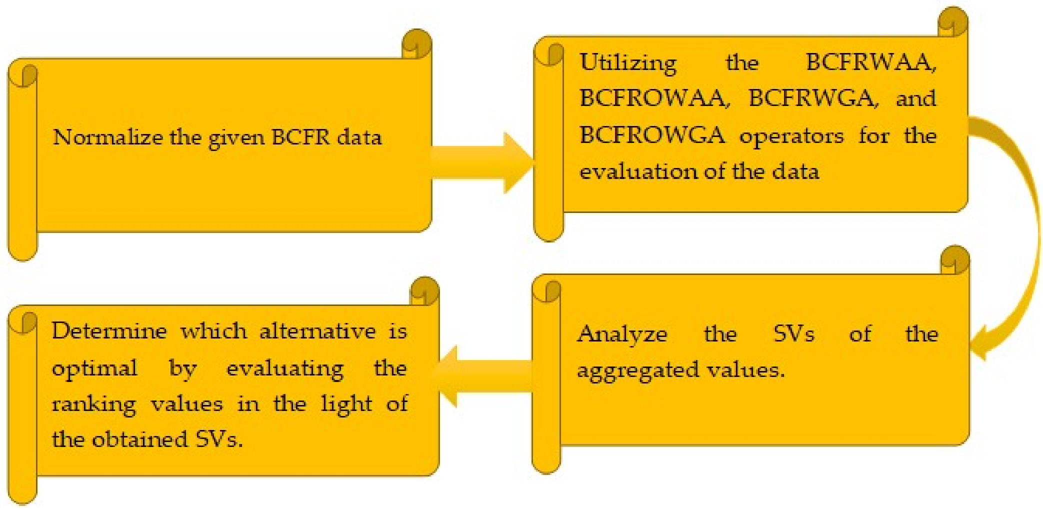

FIGURE 1. Frame Diagram for MADM

4.2. **Application.** It is crucial for protecting private data, keeping systems operating, and stopping harmful or unauthorized activity that may threaten the availability and integrity of digital resources. Cyber security (C-S) is the term utilized to secure computers, digital data, and networks. The process of preventing illegal access, violations, harm, and robbery from happening to computer networks, data, and other systems is referred to as C-S. C-S stands for cyber-attacks protection system, network, devices, and data. It is achieved by applying technologies, processes, and control. Its objectives are to decrease the danger of cyber-attacks and guard against the illegal use of networks, systems, and technologies. It also involves safeguarding the networks, programs, and systems from online threats. Attackers typically use ransomware to demand money from individuals or disrupt regular corporate operations in an attempt to obtain, alter, and damage confidential information. As there are currently more technology devices than people and attackers are becoming more skilled, implementing effective C-S measures is highly challenging. Multiple layers of protection are used over the computer, networks, application, or data that one wishes to keep safe in a successful C-S method. To effectively defend against cyber-attacks, an organization's people, procedures, and technology must work in collaboration with one another. Further, it serves as a defense against destructive hacker and cyber-criminal attacks against internet-connected devices and services. Businesses employ these techniques to guard against financial losses, identity theft, phishing schemes, and data breaches. To defend digital data from various crimes, several C-S techniques are employed. The following are a few kinds of C-S:

1. Application Security: Keeping software and devices safe from threats is the main goal of application security. Well before a program or device is implemented, successful security must be considered throughout the design phase. Unauthorized usage and access to apps and related data are prevented by application security. This is because the majority of vulnerabilities are introduced during the stages of development and publication. Numerous C-S techniques are used in application security to assist in finding vulnerabilities throughout the design and development stages.

2. Cloud Security: Protecting cloud-based resources and services, such as systems, data, and apps is the main goal of cloud security. For the most part, cloud service providers and businesses share responsibility for managing cloud security. Under this shared responsibility framework, enterprises secure their data in the cloud while cloud service providers manage the protection of the cloud environment.

3. Critical Infrastructure Security: Critical infrastructure security involves protecting these assets from physical and cyber threats. Cyber security measures for critical infrastructure include network monitoring, intrusion detection systems, and incident response plans to prevent disruptions and ensure the resilience of essential services.

4. Data Security: Data security involves protecting sensitive information from unauthorized access, disclosure, alteration, or destruction. This includes implementing encryption, access controls, and data backup strategies. Data security is critical in preventing data breaches, identity theft, and the loss of valuable or confidential information. Compliance with data protection regulations is also an essential aspect of data security.

To evaluate the above alternatives of cyber security, we have different attributes which are as follows:

1. Authentication: Authentication is the process of verifying the identity of a user, system, or entity. It ensures that individuals or systems are who they claim to be before granting access to resources. For example, passwords, biometrics including fingerprint and facial recognition, two-factor authentication, and smart cards.

2. Authorization: Authorization is the process of granting or denying access rights and permissions to users, systems, or applications. It determines what actions or resources an authenticated user, system, or application is permitted to access or perform. Authorization ensures that users only have access to the resources and data that they are allowed to use.

3. Confidentiality: Confidentiality is the protection of information from unauthorized access or disclosure. It ensures that sensitive data is kept secret and only accessible to those with the proper authorization. Different methods such as encryption, data classification, and access control are used for the protection of information from unauthorized access.

4. Availability: Availability ensures that information and resources are accessible and usable when needed. It guarantees that systems and data are available for authorized users, even in the face of disruptions or attacks. Different measures such as redundancy, backup systems, disaster recovery planning, and proactive measuring are used to ensure the resources and information are accessible and usable when needed.

4.3. **Numerical Example.** Suppose a manager of a company wants to enhance the cyber security of his organization. He wishes to compare the performance of four different alternatives/types of C-S methods to select the best ones. These four types of cyber security are as follows:

$\tilde{A}_1$: Application Security
$\tilde{A}_2$: Cloud Security
$\tilde{A}_3$: Critical Infrastructure Security
$\tilde{A}_4$: Data Security

To check the worth and reliability of different types of cyber security methods, different attributes are used which are as follows:

$\mathcal{S}_1$: Authentication
$\mathcal{S}_2$: Authorization
$\mathcal{S}_3$: Confidentiality

$\mathcal{S}_4$: Availability

For this evaluation, the company's manager hires a team of experts and the hired experts provide the weight vector to every attribute $\{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4\}$ as $(0.40, 0.25, 0.20, 0.15)$ according to their choice and preference. The hired team of experts gave their decision matrix in the form of BCFRN, which is shown in Table **3**.

TABLE 3. Decision matrix given by experts in the form of BCFR setting

Alternatives	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_4$
$\tilde{A}_1$	$\begin{pmatrix} (0.76 + i0.81, \\ -0.75 - i0.79, \\ 0.81 + i0.23, \\ -0.92 - i0.65) \end{pmatrix}$	$\begin{pmatrix} (0.44 + i0.56, \\ -0.76 - i0.63, \\ 0.12 + i0.24, \\ -0.93 - i0.42) \end{pmatrix}$	$\begin{pmatrix} 0.24 + i0.52, \\ -0.13 - i0.97, \\ 0.43 + i0.49, \\ -0.85 - i0.39) \end{pmatrix}$ (	$\begin{pmatrix} (0.27 + i0.59, \\ -0.64 - i0.87, \\ 0.15 + i0.61, \\ -0.90 - i0.91) \end{pmatrix}$
$\tilde{A}_2$	$\begin{pmatrix} (0.08 + i0.86, \\ -0.19 - i0.79, \\ 0.34 + i0.53, \\ -0.92 - i0.86) \end{pmatrix}$	$\begin{pmatrix} (0.10 + i0.61, \\ -0.35 - i0.47, \\ 0.39 + i0.59, \\ -0.78 - i0.89) \end{pmatrix}$	$\begin{pmatrix} (0.29 + i0.16, \\ -0.97 - i0.78, \\ 0.19 + i0.67, \\ -0.69 - i0.51) \end{pmatrix}$	$\begin{pmatrix} (0.25 + i0.49, \\ -0.98 - i0.68, \\ 0.23 + i0.57, \\ -0.71 - i0.09) \end{pmatrix}$
$\tilde{A}_3$	$\begin{pmatrix} (0.18 + i0.20, \\ -0.15 - i0.85, \\ 0.61 + i0.59, \\ -0.77 - i0.44) \end{pmatrix}$	$\begin{pmatrix} (0.54 + i0.19, \\ -0.11 - i0.71, \\ 0.67 + i0.09, \\ -0.18 - i0.93) \end{pmatrix}$	$\begin{pmatrix} (0.45 + i0.51, \\ -0.27 - i0.72, \\ 0.83 + i0.31, \\ -0.12 - i0.88) \end{pmatrix}$	$\begin{pmatrix} (0.29 + i0.77, \\ -0.77 - i0.90, \\ 0.73 + i0.39, \\ -0.87 - i0.61) \end{pmatrix}$
$\tilde{A}_4$	$\begin{pmatrix} (0.22 + i0.47, \\ -0.90 - i0.47, \\ 0.88 + i0.26, \\ -0.68 - i0.99) \end{pmatrix}$	$\begin{pmatrix} (0.16 + i0.45, \\ -0.83 - i0.57, \\ 0.33 + i0.66, \\ -0.55 - i0.31) \end{pmatrix}$	$\begin{pmatrix} (0.37 + i0.59, \\ -0.29 - i0.11, \\ 0.42 + i0.67, \\ -0.78 - i0.09) \end{pmatrix}$	$\begin{pmatrix} (0.60 + i0.32, \\ -0.54 - i0.66, \\ 0.81 + i0.89, \\ -0.75 - i0.90) \end{pmatrix}$

Step 1: After normalizing the data, we get the following values given in Table 4.

TABLE 4. Normalized matrix in the form of BCFR setting

Alternatives	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_4$
$\tilde{A}_1$	$\begin{pmatrix}(0.76+i0.81,\\-0.75-i0.79,\\0.81+i0.23,\\-0.92-i0.65)\end{pmatrix}$	$\begin{pmatrix}(0.44+i0.56,\\-0.76-i0.63,\\0.12+i0.24,\\-0.93-i0.42)\end{pmatrix}$	$\begin{pmatrix}(0.24+i0.52,\\-0.13-i0.97,\\0.43+i0.49,\\-0.85-i0.39)\end{pmatrix}$	$\begin{pmatrix}(0.27+i0.59,\\-0.64-i0.87,\\0.15+i0.61,\\-0.90-i0.91)\end{pmatrix}$
$\tilde{A}_2$	$\begin{pmatrix}(0.08+i0.86,\\-0.19-i0.79,\\0.34+i0.53,\\-0.92-i0.86)\end{pmatrix}$	$\begin{pmatrix}(0.10+i0.61,\\-0.35-i0.47,\\0.39+i0.59,\\-0.78-i0.89)\end{pmatrix}$	$\begin{pmatrix}(0.29+i0.16,\\-0.97-i0.78,\\0.19+i0.67,\\-0.69-i0.51)\end{pmatrix}$	$\begin{pmatrix}(0.25+i0.49,\\-0.98-i0.68,\\0.23+i0.57,\\-0.71-i0.09)\end{pmatrix}$
$\tilde{A}_3$	$\begin{pmatrix}(0.18+i0.20,\\-0.15-i0.85,\\0.61+i0.59,\\-0.77-i0.44)\end{pmatrix}$	$\begin{pmatrix}(0.54+i0.19,\\-0.11-i0.71,\\0.67+i0.09,\\-0.18-i0.93)\end{pmatrix}$	$\begin{pmatrix}(0.45+i0.51,\\-0.27-i0.72,\\0.83+i0.31,\\-0.12-i0.88)\end{pmatrix}$	$\begin{pmatrix}(0.29+i0.77,\\-0.77-i0.90,\\0.73+i0.39,\\-0.87-i0.61)\end{pmatrix}$
$\tilde{A}_4$	$\begin{pmatrix}(0.22+i0.47,\\-0.90-i0.47,\\0.88+i0.26,\\-0.68-i0.99)\end{pmatrix}$	$\begin{pmatrix}(0.16+i0.45,\\-0.83-i0.57,\\0.33+i0.66,\\-0.55-i0.31)\end{pmatrix}$	$\begin{pmatrix}(0.37+i0.59,\\-0.29-i0.11,\\0.42+i0.67,\\-0.78-i0.09)\end{pmatrix}$	$\begin{pmatrix}(0.60+i0.32,\\-0.54-i0.66,\\0.81+i0.89,\\-0.75-i0.90)\end{pmatrix}$

Step 2: Apply the BCFRWAA, BCFROWAA, BCFRWGA, and BCFROWGA operators to determine the singleton values as stated in Table 5.

TABLE 5. The Aggregated values were obtained by using Aggregation Operators

Operators	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$
BCFRWAA	$\begin{pmatrix}(0.56+i0.68,\\-0.52-i0.79,\\0.57+i0.36,\\-0.91-i0.55)\end{pmatrix}$	$\begin{pmatrix}(0.16+i0.69,\\-0.39-i0.68,\\0.31+i0.58,\\-0.80-i0.56)\end{pmatrix}$	$\begin{pmatrix}(0.36+i0.40,\\-0.20-i0.79,\\0.70+i0.41,\\-0.38-i0.64)\end{pmatrix}$	$\begin{pmatrix}(0.31+i0.47,\\-0.65-i0.39,\\0.73+i0.61,\\-0.67-i0.45)\end{pmatrix}$
BCFROWAA	$\begin{pmatrix}(0.55+i0.68,\\-0.47-i0.81,\\0.57+i0.37,\\-0.90-i0.55)\end{pmatrix}$	$\begin{pmatrix}(0.17+i0.62,\\-0.47-i0.62,\\0.31+i0.59,\\-0.77-i0.46)\end{pmatrix}$	$\begin{pmatrix}(0.41+i0.45,\\-0.22-i0.77,\\0.75+i0.35,\\-0.26-i0.74)\end{pmatrix}$	$\begin{pmatrix}(0.43+i0.45,\\-0.54-i0.39,\\0.70+i0.76,\\-0.70-i0.41)\end{pmatrix}$
BCFRWGA	$\begin{pmatrix}(0.45+i0.64,\\-0.66-i0.85,\\0.34+i0.31,\\-0.91-i0.64)\end{pmatrix}$	$\begin{pmatrix}(0.13+i0.52,\\-0.77-i0.72,\\0.30+i0.58,\\-0.84-i0.78)\end{pmatrix}$	$\begin{pmatrix}(0.31+i0.29,\\-0.31-i0.81,\\0.68+i0.30,\\-0.62-i0.77)\end{pmatrix}$	$\begin{pmatrix}(0.26+i0.46,\\-0.79-i0.48,\\0.59+i0.48,\\-0.69-i0.84)\end{pmatrix}$
BCFROWGA	$\begin{pmatrix}(0.44+i0.64,\\-0.64-i0.87,\\0.37+i0.32,\\-0.91-i0.64)\end{pmatrix}$	$\begin{pmatrix}(0.14+i0.51,\\-0.82-i0.66,\\0.30+i0.58,\\-0.80-i0.76)\end{pmatrix}$	$\begin{pmatrix}(0.37+i0.35,\\-0.34-i0.79,\\0.73+i0.27,\\-0.50-i0.83)\end{pmatrix}$	$\begin{pmatrix}(0.35+i0.42,\\-0.67-i0.52,\\0.58+i0.65,\\-0.72-i0.41)\end{pmatrix}$

Step 3: The SVs of the aggregated values are assessed in this step, as shown in Table 6. The pictorial representation of the Table 6 is shown in Fig. 2:

TABLE 6. SVs of the Aggregated Values

Operators	$H_{SF}(\tilde{A}_1)$	$H_{SF}(\tilde{A}_2)$	$H_{SF}(\tilde{A}_3)$	$H_{SF}(\tilde{A}_4)$
BCFRWAA	0.43	0.41	0.48	0.49
BCFROWAA	0.43	0.42	0.50	0.53
BCFRWGA	0.34	0.30	0.38	0.37
BCFROWGA	0.34	0.31	0.41	0.46

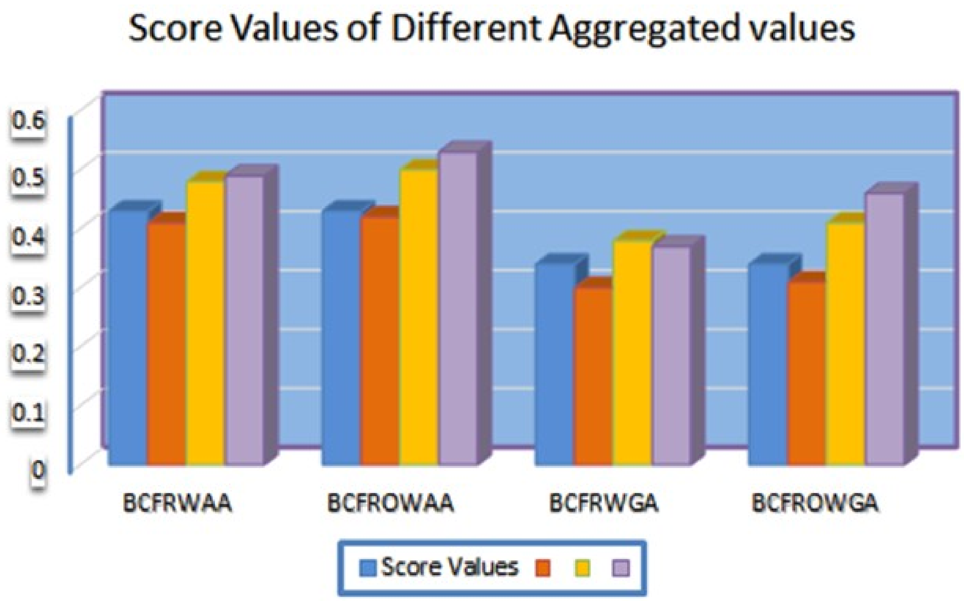

FIGURE 2. Score Values of Different Aggregated Values

Step 4: After ranking the different values of alternative $\mathbf{H_{SF}}(\tilde{\mathbf{A}})$, we get the following optimal solutions which are shown in Table 7 as under:

TABLE 7. Ranking the Different Values of Alternatives $H_{SF}(\tilde{A})$

Operators	Ranking
BCFRWAA	$\tilde{A}_4 > \tilde{A}_3 > \tilde{A}_1 > \tilde{A}_2$
BCFROWAA	$\tilde{A}_4 > \tilde{A}_3 > \tilde{A}_1 > \tilde{A}_2$
BCFRWGA	$\tilde{A}_3 > \tilde{A}_4 > \tilde{A}_1 > \tilde{A}_2$
BCFROWGA	$\tilde{A}_4 > \tilde{A}_3 > \tilde{A}_1 > \tilde{A}_2$

The Table 7 shows that $\tilde{A}_4$ is the best and finest alternative by using BCFRWAA, BCFROWAA, and BCFROWGA operators. By using BCFRWGA, $\tilde{A}_3$ is identified as the finest alternative.

5. Comparative Analysis

Comparative analysis is the compulsory section of the manuscript. This is because it helps to provide a context for the study by comparing the established research findings with existing literature and prevalent studies. This allows readers to understand the background and evolution of the topic. By comparing different variables, groups, or cases, researchers can identify patterns and trends that may not be apparent in isolated studies. This helps in drawing more robust conclusions and making connections between variables. We compare our proposed method with an existing method to enhance the reliability of the research findings. It is because, when we compare our results with existing methodologies, it adds a layer of validation of our investigated work. Consistency and inconsistency with previous findings can be important for establishing the credibility of the research.

5.1. **Structure-wise Comparative Analysis:** First of all, we compare our established structure with prevalent defined structures such as FS, FRS, BFS, BFRS, CFS, CFRS, and BCFS to ensure the generality, supremacy, advancement, and comprehension of the established structure. The comparison of the established structures with previously defined structures are also shown in Table 8 which are as follows:

Table 8. Comparison of Developed structure with prevalent existing structures.

Structures	**Fuzzi ness**	**Fuzzy Rough ness**	**Bipolarity**	**Bipolar Fuzzy Rough ness**	**Two Dimensional Data**	**Complex Fuzzy Rough ness**	**Bipolarity In two dimensions**	**Bipolar Complex fuzzy with Roughness**
FS [50]	✓	××××	××××	××××	××××	××××	××××	××××
FRS [12]	✓	✓✓✓✓	××××	××××	××××	××××	××××	××××
BFS [51]	✓	××××	✓✓✓✓	××××	××××	××××	××××	××××
BFRS [46]	✓	✓✓✓✓	✓✓✓✓	✓✓✓✓	××××	××××	××××	××××
CFS [44]	✓	××××	××××	××××	✓✓✓✓	××××	××××	××××
CFRS [22]	✓	✓✓✓✓	××××	××××	✓✓✓✓	✓✓✓✓	××××	××××
BCFS [26]	✓	××××	✓✓✓✓	××××	✓✓✓✓	××××	✓✓✓✓	××××
BCFRS	✓	✓✓✓✓	✓✓✓✓	✓✓✓✓	✓✓✓✓	✓✓✓✓	✓✓✓✓	✓✓✓✓

1. We start the comparison of our investigated novel structure with the fuzzy structure. FS can handle fuzzy data and assign a membership to each element in the interval $[0, 1]$. It is more general and can capture the uncertainty and ambiguity of the data by assigning a membership value to each element compared to the crisp set. However, FS can only handle data in a fuzzy structure and cannot tackle data in the structures of FRS, BFS, BFRS, CFS, CFRS, BCFS, and BCFRS. In contrast, our investigated structure is more generalized and advanced as compared to FS and can handle data in all such scenarios. We observe from the developed structure for BCFRS.

$$\Big((\underline{\pi_j^+} + i\underline{\phi_j^+}, \underline{\pi_j^-} + i\underline{\phi_j^-}), (\overline{\pi_j^+} + i\overline{\phi_j^+}, \overline{\pi_j^-} + i\overline{\phi_j^-})\Big)$$

that when we take the lower and upper approximation with positive and negative membership as equal and by ignoring the imaginary part of the set, the investigated structure reduces to FS. This indicates that the established structure can generalize FS theory, which highlights the beauty of the developed notion.

2. Next, we compare our investigated structure with FRS. FRS is more advanced and generalized than FS because it includes membership functions with lower and upper approximations. Due to the roughness of the data, FRS can deal with incomplete and uncertain data more accurately than simple FS. However, FRS also has drawbacks and cannot handle data in the forms of BFS, BFRS, CFS, CFRS, BCFS, and BCFRS. Our investigated structure is more generalized and advanced compared to FRS and can handle data in all such scenarios. From the analysis of the established structure, we see that if we take the positive and negative membership as equal and ignore the imaginary part of the data, the investigated structure reduces to FRS. This shows that the established structure can generalize FRS theory, demonstrating the beauty of the developed notion.

3. The other structure that is more generalized than FS is BFS. BFS consists of positive and negative memberships of a function. BFS can address the positive and negative aspects of certain data that FS cannot handle. However, BFS does not tackle data in the scenarios of BFRS, CFS, CFRS, BCFS, and BCFRS. Our proposed structure is more advanced and general than BFS and can handle data in all such scenarios. From the analysis of the established structure, we see that if we take the lower and upper approximations of the data as equal and ignore the imaginary part, our proposed structure can degenerate into BFS. This indicates that the established structure can generalize BFS theory, illustrating the beauty of the developed notion.

4. BFRS is more advanced and general compared to FS, FRS, and BFS. BFRS includes positive and negative memberships of a function with lower and upper approximations. Due to the roughness of the data, BFRS can handle incomplete and uncertain data more accurately than simple BFS. However, BFRS cannot handle data in the context of CFS, CFRS, BCFS, and BCFRS, which is a shortcoming. Our investigated work can handle data in the BFR structure as well as in the environments of CFS, CFRS, BCFS, and BCFRS. Our investigated structure generalizes BFRS by the fact that if we remove the imaginary part from the lower and upper approximations of our proposed structure, it can be reduced to BFRS. This demonstrates the supremacy, beauty, and advancement of the proposed structure.

5. CFS generalizes FS by considering the membership of the function in two dimensions. CFS can handle two-dimensional data and tackle uncertainty and ambiguity more accurately compared to FS. However, CFS has limitations in handling information in the environments of BFS, BFRS, CFRS, BCFS, and BCFRS. Our established BCFRS structure generalizes CFS and can handle data in all the mentioned formats. Our investigated structure generalizes CFS by the fact that if we take the lower and upper approximations with positive and negative memberships as equal, then our structure can degenerate into CFS. This highlights the supremacy, beauty, and advancement of the proposed structure.

6. CFRS is a generalization of FS and CFS. CFRS can handle two-dimensional data with lower and upper approximations. Due to its roughness, CFRS can tackle data more authentically compared to simple CFS. However, CFRS cannot handle data in the contexts of BFS, BFRS, BCFS, and BCFRS, which is a limitation. Our established BCFRS structure generalizes CFRS and can handle data in all the mentioned formats. Our investigated structure generalizes CFRS by the fact that if we take the positive and negative memberships as equal, our proposed structure can reduce to CFRS. This demonstrates the supremacy, beauty, and advancement of the proposed structure.

7. At last, we compare our proposed novel structure with BCFS. BCFS is a more advanced and strong structure in comparison with FS, CFS, and BFS. BCFS can hold the uncertainty and ambiguity of the data more accurately than above mentioned structure. But, still, BCFS has some issues that this structure can't tackle the information in FRS, BFRS, CFRS, and BCFRS. However, our established BCFRS structure is the generalization of CFRS and holds the data in the form of all formats mentioned above. Our investigated structure is the generalization of CFRS by the fact that if take the lower and upper approximation as equal, then our investigated novel structure can reduce to BCFS. This demonstrates how superior, advanced, and innovative is the suggested structure.

5.2. **Comparative Analysis in the Context of AOs.** We now compare our investigated AOs which is BCFRWAA, BCFROWAA, BCFRWGA, and BCFROWGA with several existing AOs. This comparison shows the authenticity and validity of our proposed novel AOs in the BCFR model. The following Table 9 show the comparison of the established AOs with several existing AOs in different models.

TABLE 9. The comparison of established AOs with several existing AOs in different models

AOs	$H_{SF}(\tilde{A}_1)$	$H_{SF}(\tilde{A}_2)$	$H_{SF}(\tilde{A}_3)$	$H_{SF}(\tilde{A}_4)$	**Ranking**
BF Dombi AOs [19]	××××	××××	××××	××××	××××
BF Hamacher AOs [45]	××××	××××	××××	××××	××××
CF Arithmetic AOs [7]	××××	××××	××××	××××	××××
CF Geometric AOs [6]	××××	××××	××××	××××	××××
CF Rough AOs [22]	××××	××××	××××	××××	××××
BCF Arithmetic AOs [28]	××××	××××	××××	××××	××××
BCF Geometric AOs [28]	××××	××××	××××	××××	××××
BCF Dombi AOs [27]	××××	××××	××××	××××	××××
BCFRWAA	0.43	0.41	0.48	0.49	$\tilde{A}_4 > \tilde{A}_3 > \tilde{A}_1 > \tilde{A}_2$
BCFROWAA	0.43	0.42	0.50	0.53	$\tilde{A}_4 > \tilde{A}_3 > \tilde{A}_1 > \tilde{A}_2$
BCFRWGA	0.34	0.30	0.38	0.37	$\tilde{A}_3 > \tilde{A}_4 > \tilde{A}_1 > \tilde{A}_2$
BCFROWGA	0.34	0.31	0.41	0.46	$\tilde{A}_4 > \tilde{A}_3 > \tilde{A}_1 > \tilde{A}_2$

We start the comparison of our investigated AOs with other prevalent AOs in the following paragraphs.

1. Jana et al. [19] constructed BF Dombi AOs in the context of the BF model. These AOs can aggregate data in the BF model. However, they have limitations and cannot handle data in the BFRS, CFS, CFRS, BCFS, and BCFRS models. Our investigated AOs are more advanced and reliable, capable of handling data in the BF, FRS, BFRS, CFS, CFRS, BCFS, and BCFRS models. Therefore, our novel AOs are superior to the work of Jana et al. [19].

2. BF Hamacher AOs were established by Wei et al.[45] and constructed BF Hamacher AOs in the BF structure. Data from the BF structure can be processed and compiled by these AOs.However, the data in the BFRS, CFS, CFRS, BCFS, and BCFRS models could not be handled by the suggested AOs of Wei et al.[45] . Our investigated AOs not only tackle the information in the BF environment but also tackle the information in FRS, CFRS, CFS, BFRS, BCFS, and BCFRS models. So, due to these attributes, our established novel AOs are far better than the existing work of Wei et al. [45] .

3. CFR AOs were investigated by Khan et al.[22] in the CFR model. These AOs are more generalized and advanced than simple CF AOs. They can tackle and aggregate the two-dimensional data with lower and upper approximations. Due to the roughness involved in the structure, it can tackle the data more authentically in comparison with simple CFS. However, still CFR AOs have some drawbacks in that they can solve and aggregate the data in the scenario of BFS, BFRS, BCFS, and BCFRS. However, our investigated AOs are more advanced and reliable as compared to CFR AOs. Our investigated AOs not only tackle the information in the CFR environment but also tackle the information in FRS, CFRS, BFS, BFRS, BCFS, and BCFRS models. So, due to these attributes, our established novel AOs are far better than the existing work of Khan et al.[22].

4. Mahmood et al.[28] constructed the BCF Arithmetic and Geometric AOs. The established AOs of Mahmood et al. [28] are superior, generalized, and advanced as compared to the prevalent structures such as FS, BFS, and CFS. They not only tackle the information in the BCF model but also tackle the data in above mentioned models. However, still these AOs have some drawbacks and limitations that they cannot solve and tackle the information in FRS, BFRS, CFRS, and BCFRS. Due to these weaknesses, these AOs are not so generalized and advanced in comparison with our established AOs. Our established not only tackles and aggregates the data in the BCF model but also aggregates the data in FRS, BFRS, CFRS, and BFRS. So, due to these attributes, our established novel AOs are far better than the existing work of Mahmood et al.[28]

5. BCF Dombi AOs were investigated by Mahmood and Ur Rehman [27] in the BCF model. As these AOs are investigated in the BCF model, they are more advanced and generalized in aggregating the data not only in the BCF model but also in the FS, BFS, and CFS models. But, still, these AOs have some limitations that they can solve and aggregate the data in FRS, BFRS, CFRS, and BCFRS. Due to these weaknesses, these AOs are not so generalized and advanced in comparison with our established AOs. Our established not only tackles and aggregates the data in the BCF model but also aggregates the data in FRS, BFRS, CFRS, and BFRS. So, due to these attributes, our established novel AOs are far better than the existing work of Mahmood and Ur Rehman [27] .

6. Since all the above-mentioned AOs have some limitations and drawbacks, they cannot tackle the data in many structures, especially in the structure of BCFRS. However, our established AOs in the model of BCFRS can tackle the data in all previous existing models as shown in above Table 9. It not only tackles the information in BCFRS but also finds the best optimal solution which is also shown in the above Table 9. All the existing AOs fail to rank and find the optimal solution in the BCFR model. But our established novel AOs can rank the data and find the best optimal solution in any model including FS, FRS, BFS, BFRS, CFS, CFRS, and BCFS. So, our established novel AOs give more space to data analysts and experts to tackle the undefined, vague, ambiguous data more accurately as compared to the existing AOs.

6. Conclusion

In this manuscript, we discussed the C-S phenomenon. C-S stands for cyber-attacks protection system, network, devices, and data. It is achieved by applying technologies, processes, and control. Its objectives are to decrease the danger of cyber-attacks and guard against the illegal use of networks, systems, and technologies. It also involves safeguarding the networks, programs, and systems from online threats. Attackers typically use ransomware to demand money from individuals or disrupt regular corporate operations in an attempt to obtain, alter, and damage confidential information. As there are currently more technology devices than people and attackers are becoming more skilled, implementing effective C-S measures is highly challenging.

Multiple layers of protection are used over the computer, networks, application, or data that one wishes to keep safe in a successful C-S method. To effectively defend against cyber-attacks, an organization's people, procedures, and technology must work in collaboration with one another. Using both positive and negative strategies with two-dimensional data is essential in real-world scenarios. To address such types of information, BCF is an advanced and applicable model. The BCF model is more authentic and generalized in comparison with other structures such as FS, BFS, and CFS. This model provides a more expressive structure for experts to model uncertainty and vagueness in the data. On the other hand, the rough set RS structure consists of lower and upper approximations that provide more space in DM dilemmas. The RS model is more advanced and reliable in handling uncertainty and vagueness in data more accurately. To combine the benefits of BCF, BCFS, and RS, we introduced a novel structure known as BCF rough set (BCFRS) by employing the BCF relation. The established novel notion is more advanced and generalized compared to FS, FRS, BFS, BFRS, CFS, CFRS, BCFS, and BCFRS. We have interpreted operations such as addition, product, scalar product, scalar power, union, intersection, and complement for the theory of BCFRS. Additionally, we introduced the AOs BCFRWAA, BCFROWAA, BCFRWGA, and BCFROWGA for this model. We applied the MADM technique using these novel AOs in the BCFRS model and solved a real-life application of C-S for the security of important data. Finally, we compared our established work with different existing theories to demonstrate the superiority, authenticity, and generalization of our novel established theories. To expand our investigated work on other theories, we will explore several other theories in the future, including the bipolar fuzzy graphs [2], complex Pythagorean fuzzy set [4], complex Fermatean fuzzy N-Soft set [3], intuitionistic FS [16], interval-valued Picture FS [17], BCF soft set [18], q-rung orthopair FS [23], IF bipolar metric spaces [30], TOPSIS Technique [40], BF soft topology [41], and AHP technique [43].

Moreover, the important abbreviations which are used throughout the article are shown in Table 10.

TABLE 10. Important abbreviations are used throughout the article.

Full Name	**Abbreviation**
Fuzzy set	FS
Intuitionistic fuzzy set	IFS
Bipolar fuzzy set	BFS
Complex fuzzy set	CFS
Bipolar complex fuzzy sets	BCFSs
Decision making	DM
Multi-attribute decision making	MADM
Multi-Criteria decision making	MCDM
Multi-criteria group decision making	MCGDM
Cyber Security	C-S
Rough Set	RS
Fuzzy rough set	FRS
Complex fuzzy rough set	CFRS
Bipolar fuzzy rough set	BFRS
Bipolar complex fuzzy rough set	BCFRS
Aggregation operators	AOs
Bipolar complex fuzzy rough weighted arithmetic averaging	BCFRWAA
Bipolar complex fuzzy rough ordered weighted arithmetic averaging	BCFROWAA
Bipolar complex fuzzy rough weighted geometric averaging	BCFRWGA
Bipolar complex fuzzy rough ordered weighted geometric averaging	BCFROWGA

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