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Parametric Equations of Geodesic and Magnetic Surfaces in three Dimensional Heisenberg Group

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Abstract. In this paper, we extend the concept of a magnetic curve, which is viewed as a one-dimensional manifold representing the trajectory of a charged particle moving under the action of a magnetic field, to a magnetic surface considered as a two-dimensional manifold. For this purpose, we investigate the contact geometry of Heisenberg three-group denoted by $\mathbb{H}_3$. Subsequently, we determine the parametric equations of geodesic surfaces, which can be interpreted as magnetic surfaces in the absence of a magnetic field, and of magnetic surfaces. Lastly, we conclude with illustrative examples of such surfaces in $\mathbb{H}_3$ with graphical presentation in $\mathbb{R}^3$.

AMS (MOS) Subject Classification Codes: 53A05; 53A35; 53Z05
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1. Introduction

The study of magnetic structures represents a significant research fields where geometry and physics, particularly magnetism, intersect. In geometry, a magnetic curve, seen as a one-dimensional manifold, is a curve on a Riemannian manifold that follows the trajectory of a charged particle moving under the influence of a magnetic field. It generalizes the concept of a geodesic, which is the trajectory of a particle moving in the absence of a magnetic field "free fall". Several research works on magnetic curves have appeared in Riemannian, Lorentzian and generally in pseudo-Riemannian contexts. (See [6–8, 11, 18])

Thereafter, a magnetic surface, that defines a flux surface [9], is two-dimensional magnetic structure in which the magnetic field lines lie. This signifies that the magnetic vector fields at any point on a magnetic surface are tangent to the surface itself. In the absence of magnetism, this surface is called a geodesic surface. The study of such surfaces in Euclidean space is given in [16].

The importance of such surfaces has long been recognized in magnetic fusion research that we can cite [2–4, 15, 16]. We present two concrete examples of magnetic surface that we dissect in the following:

Plasma can serve as an excellent example to elucidate the concept of magnetic surfaces. Considered as fourth state of matter, it is a hot ionized gas made up of approximately equal numbers of positively charged ions and negatively charged electrons that it makes it a good electrical conductor. The electrical conductivity creates currents flowing in a plasma that interact with magnetic fields to produce the forces necessary for containment. Ordinary matter ionizes and forms a plasma at temperatures above about 5000 K, and most of the visible matter in the universe is in the plasma state. Plasma particles can be confined and shaped by magnetic field lines that combine to act like an invisible bottle. By fixing magnetic field lines toroidally around the interior of the tokamak, the ions and electrons in the plasma are forced to move slightly around these field lines, preventing them from escaping from the container. The creation of the first limited laboratory positron-electron plasma presented an important applications of the magnetic surfaces. (See [3], [14] and [15])

Earth's magnetic field, characterized by its magnetic lines and magnetic surfaces, serves as another example of magnetic surfaces as depicted in Figure 1.

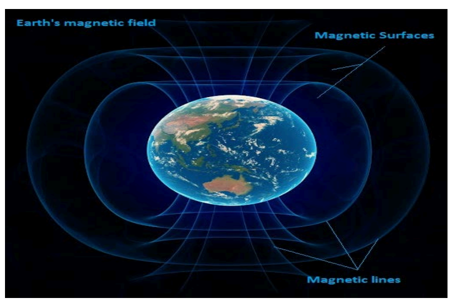

FIGURE 1. Earth's magnetic field

Consequently, in this paper, we focus our geometric study of a magnetic surfaces in three-dimensional Heisenberg group, denoted by $\mathbb{H}_3$, by defining and determining the parametric equations of geodesic and magnetic surfaces.

The paper is organized as follow;

After an introduction in Section 1, we give in Section 2 an overview of contact geometry of three-dimensional Heisenberg group $\mathbb{H}_3$ as a set definition, metric, orthonormal basis, connection and contact structure. The Section 3 is devoted to the definitions of magnetic and geodesic curves, and we define the magnetic and geodesic surfaces of a given manifold. In the fourth Section, we determine the parametric equations that define the geodesic surfaces in $\mathbb{H}_3$. Subsequently, in the final Section 5, we determine the parametric equations of magnetic surfaces in $\mathbb{H}_3$. The Sections (4 and 5) are concluded by an illustrative examples with graphical representations in $\mathbb{R}^3$.

Note that we have used the computer software, Wolfram Mathematica and Scientific WorkPlace to solve PDEs and ODEs (in Eq. 5. 21 , Eq. 4. 15), as well as graphical presentation.

2. Geometry of three-dimensional Heisenberg group

The three-dimensional Heisenberg group, $\mathbb{H}_3$, is a group endowed with a multiplication given as

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 - \tfrac{1}{2}(x_1 y_2 + x_2 y_1)).$$

and seen as the Riemannian real space $\mathbb{R}^3$ in which the invariant Riemannian metric is

$$g_\lambda = \frac{1}{\lambda^2} dx_1^2 + dx_2^2 + (x_1 dx_2 + dx_3)^2 \tag{2. 1}$$

where λ is a strictly positive real number. This gives $\mathbb{H}_3$ the structure of a Riemannian manifold. Note that all left-invariant Riemannian metrics on the $\mathbb{H}_3$ are isometric to the metric g_λ.

Next, we define an orthonormal basis in $(\mathbb{H}_3, g_\lambda)$ and its associated dual basis as

$$e_1 = \partial x_2 - x_1 \partial x_3, \;\; e_2 = \lambda \partial x_1, \;\; e_3 = \partial x_3, \tag{2. 2}$$

and

$$e^1 = dx_2, \;\; e^2 = \tfrac{1}{\lambda} dx_1, \;\; e^3 = x_1 dx_2 + dx_3.$$

Let ∇ be the Levi-Civita connection of g, then the non vanishing components of ∇ with respect to the left-invariant orthonormal basis are

$$\begin{array}{l} \nabla_{e_1} e_2 = \frac{\lambda}{2} e_3, \;\; \nabla_{e_2} e_1 = -\frac{\lambda}{2} e_3, \;\; \nabla_{e_3} e_1 = -\frac{\lambda}{2} e_2 \\ \nabla_{e_1} e_3 = -\frac{\lambda}{2} e_2, \;\; \nabla_{e_2} e_3 = \frac{\lambda}{2} e_1, \;\; \nabla_{e_3} e_2 = \frac{\lambda}{2} e_1 \end{array} \tag{2. 3}$$

Moreover, Heisenberg group $(\mathbb{H}_3, \phi, \xi, \eta, g)$ is an almost contact manifold where the contact form is

$$\eta = x_1 dx_2 + dx_3 = e^3, \tag{2. 4}$$

$\xi = e_3$ and the $(1,1)$-tensor φ is

$$\phi(e_1) = e_2; \; \phi(e_2) = -e_1, \; \phi(e_3) = 0, \tag{2. 5}$$

which satisfy

$$\begin{array}{rcl} \eta(e_3) & = & 1; \; \phi^2(X) = -X + \eta(X) e_3 \text{ and} \\ g_\lambda(\phi(X), \phi(Y)) & = & g(X,Y) + \eta(X)\eta(Y), \end{array}$$

for any vector fields X, Y on $(\mathbb{H}_3, g_\lambda)$. In addition, if

$$d\eta(X,Y) = g_\lambda(X, \phi(Y)) \tag{2. 6}$$

$(\mathbb{H}_3, \phi, \xi, \eta, g_\lambda)$ become a contact manifold and the fundamental 2-form $d\eta$ is closed and it defines a magnetic field. (See [5, 12, 13])

3. Magnetic curves and surfaces

In this section, we give an overview of the magnetic curves and we define the magnetic surface in the Riemannian manifold. A magnetic field F is a closed 2-form on n-dimensional Riemannian manifold (M, g) and the LORENTZ force of F on (M, g) is a $(1, 1)$-tensor field given by

$$F(X, Y) = g(\Phi(X), Y).$$

The magnetic trajectories, of F, are curves on (M, g) which satisfies the LORENTZ equation

$$\nabla_{\mathbf{t}}\mathbf{t} = \Phi(\mathbf{t}),$$

where $\mathbf{t}$ is speed vector of magnetic curve (or trajectory) and ∇ is the Levi-Civita connection associated to g. Hence, the LORENTZ equation generalizes the geodesic equation, namely

$$\nabla_{\mathbf{t}}\mathbf{t} = 0.$$

Subsequently, we introduce a new definition of 'magnetic surfaces,' extending the concept of magnetic curves (one-dimensional magnetic manifolds) to two-dimensional magnetic manifolds. This generalizes geodesic surfaces in the absence of a magnetic field, as detailed in the following definitions.

Definition 3.1. *Let $\mathbb{M}$ be a regular surface and F be a magnetic tensor fields in (M, g). $\mathbb{M}$ is a magnetic surface if the integral curves of the tangent vectors X_u and X_v are magnetic curves i.e.*

$$\begin{cases} \nabla_{X_u}X_u = \Phi(X_u) \\ \nabla_{X_v}X_v = \Phi(X_v) \end{cases}.$$

We can find also another equivalent definition of magnetic surfaces in [16].

Definition 3.2. *Let $\mathbb{M}$ be a regular surface in n-dimensional Riemannian manifold (M, g) parameterized with $X(u, v) = (x(u, v)_i)_{i=\overline{1,n}}$. $\mathbb{M}$ is a geodesic surface if*

$$\nabla_{X_u}X_u = \nabla_{X_v}X_v = 0,$$

where $X_u = \frac{\partial X}{\partial u}$ and $X_v = \frac{\partial X}{\partial v}$.

Remark 3.3. *Note that the definition of magnetic surfaces, in the absence of magnetism F (i.e. $F \equiv 0$), generalizes the definition of geodesic surfaces seen as a surface in which each point has a geodesic as a curve which locally minimizes the distance.*

Therefore, the determination of the parametric equation of such surfaces requires the resolution of a second-order partial differential system, with two equations.

4. Geodesic surfaces in $\mathbb{H}_3$

We consider in this section, the determination of geodesic surfaces in $(\mathbb{H}_3, g_\lambda)$ according to the Definition 3.2.

Let $\mathbb{M}$ be a regular surface in $\mathbb{H}_3$ parameterized by

$$\begin{array}{ccc} X & : & I \times J \subset \mathbb{R}^2 \to (\mathbb{H}_3, g) \\ (u, v) & \longmapsto & X(u, v) = (x(u, v), y(u, v), z(u, v)). \end{array}$$

(Here, the functions (x_1, x_2, x_3) are replaced by (x, y, z)). Its tangent vectors are

$$\left\{ \begin{array}{l} X_u = x_u \partial x + y_u \partial y + z_u \partial z \\ X_v = x_v \partial x + y_v \partial y + z_v \partial z \end{array} \right. . \tag{4. 7}$$

From the Eq.(2. 2), the vectors X_u and X_v are expressed, in the basis $(e_i)_{i=\overline{1,3}}$, as

$$\left\{ \begin{array}{l} X_u = y_u e_1 + \frac{x_u}{\lambda} e_2 + (xy_u + z_u)\, e_3 \\ X_v = y_v e_1 + \frac{x_v}{\lambda} e_2 + (xy_v + z_v)\, e_3 \end{array} \right. , \tag{4. 8}$$

where the partial derivative is denoted by $\frac{\partial}{\partial x} = \partial x$.

Using the connection formulas given in the Eq.(2. 3), the covariant derivative of the basis $(e_i)_{i=\overline{1,3}}$ with respect to the tangent vectors are

$$\left\{ \begin{array}{l} \nabla_{X_u} e_1 = -\frac{x_u}{2} e_3 - \frac{\lambda}{2}\left(xy_u + z_u\right) e_2 \\ \nabla_{X_u} e_2 = \frac{\lambda}{2} y_u e_3 + \frac{x_u}{\lambda} e_2 + \frac{\lambda}{2}\left(xy_u + z_u\right) e_1 \\ \nabla_{X_u} e_3 = -\frac{\lambda}{2} y_u e_2 + \frac{x_u}{2} e_1 \end{array} \right. \quad \text{and} \tag{4. 9}$$

$$\left\{ \begin{array}{l} \nabla_{X_v} e_1 = -\frac{x_v}{2} e_3 - \frac{\lambda}{2}\left(xy_v + z_v\right) e_2 \\ \nabla_{X_v} e_2 = \frac{\lambda}{2} y_v e_3 + \frac{x_v}{\lambda} e_2 + \frac{\lambda}{2}\left(xy_v + z_v\right) e_1 \\ \nabla_{X_v} e_3 = -\frac{\lambda}{2} y_v e_2 + \frac{x_v}{2} e_1 \end{array} \right. .$$

Next, the covariant derivative of the tangent vectors in the direction of X_u and X_v are

$$\left\{ \begin{array}{l} \nabla_{X_u} X_u = \left(y_{uu} + x_u\left(z_u + xy_u\right)\right) e_1 + \left(\frac{x_{uu}}{\lambda} - \lambda y_u\left(z_u + xy_u\right)\right) e_2 + \left(z_u + xy_u\right)_u e_3 \\ \nabla_{X_v} X_v = \left(y_{vv} + x_v\left(z_v + xy_v\right)\right) e_1 + \left(\frac{x_{vv}}{\lambda} - \lambda y_v\left(z_v + xy_v\right)\right) e_2 + \left(z_v + xy_v\right)_v e_3 \end{array} \right. . \tag{4. 10}$$

Now, using the Definition 3.2, we have the system (S)

$$S : \left\{ \begin{array}{l} \nabla_{X_u} X_u = 0 \\ \nabla_{X_v} X_v = 0 \end{array} \right. ,$$

by the Eq. (4. 10) and taking into account that $(e_i)_{i=\overline{1,3}}$ is an orthonormal basis, we get two systems

$$SG : \left\{ \begin{array}{l} x_{uu} - \lambda^2 y_u\left(z_u + xy_u\right) = 0 \\ y_{uu} + x_u\left(z_u + xy_u\right) = 0 \\ \left(z_u + xy_u\right)_u = 0 \end{array} \right. \quad \text{and } \overline{SG} : \left\{ \begin{array}{l} x_{vv} - \lambda^2 y_v\left(z_v + xy_v\right) = 0 \\ y_{vv} + x_v\left(z_v + xy_v\right) = 0 \\ \left(z_v + xy_v\right)_v = 0 \end{array} \right.$$

which, after an integration of equations (SG_3 and $\overline{SG}_3$), turns to

$$SG : \left\{ \begin{array}{l} x_{uu} - \lambda^2 y_u \varphi(v) = 0 \\ y_{uu} + x_u \varphi(v) = 0 \\ z_u + xy_u = \varphi(v) \end{array} \right. \quad \text{and } \overline{SG} : \left\{ \begin{array}{l} x_{vv} - \lambda^2 y_v \overline{\varphi}(u) = 0 \\ y_{vv} + x_v \overline{\varphi}(u) = 0 \\ z_v + xy_v = \overline{\varphi}(u) \end{array} \right. , \tag{4. 11}$$

then we have the following theorem.

Theorem 4.1. *Let $\mathbb{M}$ be a regular surface in $\mathbb{H}_3$ parameterized with $X(u, v) = (x(u, v), y(u, v), z(u, v))$, then $\mathbb{M}$ is geodesic surface if and only if the systems (SG) and $\left(\overline{SG}\right)$ holds.*

In order to determine the parametric equations of geodesic surfaces, it is necessary to solve the systems (SG) and $\left(\overline{SG}\right)$. Due to the complexity of the general case, we assume that the functions φ and $\overline{\varphi}$ are zero or constants.

Case 1. If $\varphi=\overline{\varphi}\equiv 0$, the equations $(SG_{1,2})$ becomes

$$x_{uu}=0 \text{ and } y_{uu}=0. \tag{4. 12}$$

From Eqs.(4. 11 $_3$ and 4. 12), the general solution of (SG) is

$$\begin{cases} x(u,v)=\varphi_1 u+\varphi_2 \\ y(u,v)=\varphi_3 u+\varphi_4 \\ z(u,v)=-\frac{\varphi_1\varphi_3}{2}u^2-\varphi_2\varphi_3 u+\varphi_5 \end{cases}, \tag{4. 13}$$

where $\varphi_{\overline{1,5}}$ are arbitrary smooth functions in v. Substituting the Eq.(4. 13) in equation $\left(\overline{SG}_{1,2}\right)$, we get

$$\begin{cases} x_{vv}=\varphi_{1vv}u+\varphi_{2vv}=0 \\ y_{vv}=\varphi_{3vv}u+\varphi_{4vv}=0 \end{cases},$$

by comparing with respect to u, the functions $(\varphi_i)_{i=\overline{1,4}}$ are linear in v (i.e. $\varphi_i=a_i v+b_i \mid a_i,b_i\in\mathbb{R},\ i=\overline{1,4}$).
Using the Eqs.(4. 13 $_3$ and 4. 11 $_6$), we have

$$\begin{aligned} -\frac{(\varphi_1\varphi_3)_v}{2}u^2-(\varphi_2\varphi_3)_v u+\varphi_{5v} &= -(\varphi_1 u+\varphi_2)(\varphi_{3v}u+\varphi_{4v}) \\ &= -\varphi_1\varphi_{3v}u^2-(\varphi_1\varphi_{4v}+\varphi_2\varphi_{3v})u-\varphi_2\varphi_{4v}, \end{aligned}$$

comparing again with respect to u and after with respect to v, we obtain

$$\begin{cases} \varphi_{1v}\varphi_3=\varphi_1\varphi_{3v} \\ \varphi_{2v}\varphi_3=\varphi_1\varphi_{4v} \\ \varphi_{5v}=-\varphi_2\varphi_{4v} \end{cases} \text{ and } \begin{cases} a_1a_4=a_2a_3 \\ a_1b_3=b_1a_3 \\ \varphi_5=-\frac{a_2a_4}{2}v^2-b_2a_4v+a_5 \end{cases},$$

where a_5 is a real constant. Then, the solution of the systems $\left(SG \text{ and } \overline{SG}\right)$ are

$$\begin{cases} x(u,v)=a_1vu+b_1u+a_2v+b_2 \\ y(u,v)=a_3vu+b_3u+a_4v+b_4 \\ z(u,v)=-\frac{1}{2}u^2v^2a_1a_3+uv^2a_2a_3-a_1b_3u^2v-\frac{1}{2}u^2b_1b_3-\frac{1}{2}v^2a_2a_4 \\ \qquad +uv(a_2b_3+a_3b_2)+ub_2b_3-va_4b_2+a_5 \end{cases} \tag{4. 14}$$

with a conditions $a_1a_4=a_2a_3$ and $a_1b_3=b_1a_3$.

Case 2. If $\varphi\equiv 0$ and $\overline{\varphi}$ is a non-zero constant, the general solution of (SG) is given in the Eq.(4. 13), by substituting it in the Eqs.$\left(\overline{SG}_{1,2}\right)$, we obtain

$$\begin{cases} \left(\varphi_{1vv}-\lambda^2\overline{\varphi}\varphi_{3v}\right)u+\left(\varphi_{2vv}-\lambda^2\overline{\varphi}\varphi_{4v}\right)=0 \\ (\varphi_{3vv}+\overline{\varphi}\varphi_{1v})u+(\varphi_{4vv}+\overline{\varphi}\varphi_{2v})=0 \end{cases},$$

which gives the ODEs

$$\begin{aligned} \varphi_{1vv} &= \lambda^2\overline{\varphi}\varphi_{3v};\ \varphi_{3vv}+\overline{\varphi}\varphi_{1v}=0 \text{ and} \\ \varphi_{2vv} &= \lambda^2\overline{\varphi}\varphi_{4v};\ \varphi_{4vv}+\overline{\varphi}\varphi_{2v}=0, \end{aligned}$$

with solutions

$$\begin{cases} \varphi_{1,2}(v)=\frac{a_{1,2}}{\lambda\varphi}\sin\lambda\overline{\varphi}v+b_{1,2} \\ \varphi_{3,4}(v)=\frac{a_{1,2}}{\lambda^2\overline{\varphi}}\cos\lambda\overline{\varphi}v+b_{3,4} \end{cases}, \tag{4. 15}$$

where $a_{1,2}$ and $b_{\overline{1,4}}$ are real constants. Using the Eq.(4. 13 $_3$) and Eq.(4. 11 $_6$), and comparing with respect to u, we have

$$\varphi_1\varphi_{4v} = \varphi_{2v}\varphi_3;\ \ \varphi_{1v}\varphi_3 = \varphi_1\varphi_{3v} \text{ and } \varphi_{5v} = \overline{\varphi} - \varphi_2\varphi_{4v}.$$

Again, using the Eqs.(4. 15) and comparing with respect to v, we find

$$\begin{aligned} a_1 &= 0;\ \ a_2b_3 = a_2b_1 = 0; \\ \varphi_{5v} &= -\frac{a_2b_2}{\lambda^2\overline{\varphi}}\cos v\lambda\varphi - \frac{a_2^2}{4\lambda^3\overline{\varphi}^2}\sin 2v\lambda\varphi + \left(\overline{\varphi} + \frac{a_2^2}{2\lambda^2\overline{\varphi}}\right)v + b_5. \end{aligned}$$

Hence, the solution of systems $\left(SG \text{ and } \overline{SG}\right)$ is

$$\left\{\begin{array}{l} x(u,v) = b_1u + \frac{a_2}{\lambda\varphi}\sin\lambda\overline{\varphi}v + b_2 \\ y(u,v) = b_3u + \frac{a_2}{\lambda^2\overline{\varphi}}\cos\lambda\overline{\varphi}v + b_4 \\ z(u,v) = -\frac{a_2b_2}{\lambda^2\overline{\varphi}}\cos v\lambda\overline{\varphi} - \frac{a_2^2}{4\lambda^3\overline{\varphi}^2}\sin 2v\lambda\overline{\varphi} - \frac{b_1b_3}{2}u^2 + b_2b_3u + \left(\overline{\varphi} + \frac{a_2^2}{2\lambda^2\overline{\varphi}}\right)v + b_5 \end{array}\right., \tag{4. 16}$$

with condition $a_2b_3 = a_2b_1 = 0$

Remark 4.2. *The only surface* $\mathbb{M}$ *with the parametrization given in Eq.(4. 16), in the case* $a_2 = 0$*, is*

$$X(u,v) = \left(b_1u + b_2, b_3u + b_4, -\frac{b_1b_3}{2}u^2 + b_2b_3u + \overline{\varphi}v + b_5\right),$$

where $b_{\overline{1,5}}$ *are real constants.*

Case 3. If φ and $\overline{\varphi}$ are non-zero constants, the general solution of (SG) is

$$\left\{\begin{array}{l} x(u,v) = \frac{\varphi_1}{\lambda\varphi}\sin\lambda\varphi u + \varphi_2, \\ y(u,v) = \frac{\varphi_1}{\lambda^2\varphi}\cos\lambda\varphi u + \varphi_3 \\ z(u,v) = \varphi u - \frac{\varphi_1}{4\lambda^3\varphi^2}\left(\varphi_1\sin 2\lambda\varphi u + 4\lambda\varphi\varphi_2\cos\lambda\varphi u - 2\lambda\varphi\varphi_1 u\right) + \varphi_4 \end{array}\right. . \tag{4. 17}$$

Substituting the last equation in $\left(\overline{SG}_{1,2}\right)$, we have

$$\begin{array}{c} \frac{\varphi_{1vv}}{\lambda\varphi}\sin\lambda\varphi u + \varphi_{2vv} - \frac{\overline{\varphi}}{\varphi}\varphi_{1v}\cos\lambda\varphi u - \lambda^2\overline{\varphi}\varphi_{3v} = 0 \\ \frac{\varphi_{1vv}}{\lambda^2\varphi}\cos\lambda\varphi u + \varphi_{3vv} + \frac{\overline{\varphi}}{\lambda\varphi}\varphi_{1v}\sin\lambda\varphi u + \overline{\varphi}\varphi_{2v} = 0 \end{array},$$

comparing with respect to u, give the ODEs

$$\varphi_{1vv} = \varphi_{2vv} = \varphi_{3vv} = \varphi_{1v} = \varphi_{3v} = \varphi_{2v} = 0,$$

its solutions are

$$\varphi_{\overline{1,3}} = a_{\overline{1,3}} \text{ reals.}$$

From the Eq.(4. 17 $_3$) and equation $\left(\overline{SG}_3\right)$, we have

$$\varphi_4 = \overline{\varphi}v + b_4,$$

which give the solution of the systems $\left(SG \text{ and } \overline{SG}\right)$

$$\left\{\begin{array}{l} x(u,v) = \frac{a_1}{\lambda\varphi}\sin\lambda\varphi u + a_2, \\ y(u,v) = \frac{a_1}{\lambda^2\varphi}\cos\lambda\varphi u + a_3 \\ z(u,v) = -\frac{a_1^2}{4\lambda^3\varphi^2}\sin 2\lambda\varphi u - \frac{a_1a_2}{\lambda^2\varphi}\cos\lambda\varphi u + \left(\varphi + \frac{a_1^2}{2\lambda^2\varphi}\right)u + \overline{\varphi}v + b_4 \end{array}\right. . \tag{4. 18}$$

Now, we can present the following proposition.

Proposition 4.3. *The parametric surfaces $\mathbb{M}_{1,2,3}$ parameterized by*
1.

$$X(u,v)=\begin{pmatrix} x(u,v)=a_1vu+b_1u+a_2v+b_2 \\ y(u,v)=a_3vu+b_3u+a_4v+b_4 \\ z(u,v)=-\frac{1}{2}u^2v^2a_1a_3-a_1b_3u^2v-\frac{1}{2}u^2b_1b_3-\frac{1}{2}v^2a_2a_4 \\ +uv\left(a_2b_3+a_3b_2\right)+ub_2b_3-va_4b_2+a_5 \end{pmatrix},$$

with conditions $a_1a_4=a_2a_3$ and $a_1b_3=b_1a_3$,
2.

$$X(u,v)=\left(b_1u+b_2,b_3u+b_4,-\frac{b_1b_3}{2}u^2+b_2b_3u+\overline{\varphi}v+b_5\right),$$

3.

$$X(u,v)=\begin{pmatrix} x(u,v)=\frac{a_1}{\lambda\varphi}\sin\lambda\varphi u+a_2, \\ y(u,v)=\frac{a_1}{\lambda^2\varphi}\cos\lambda\varphi u+a_3 \\ z(u,v)=-\frac{a_1^2}{4\lambda^3\varphi^2}\sin 2\lambda\varphi u-\frac{a_1a_2}{\lambda^2\varphi}\cos\lambda\varphi u+\left(\varphi+\frac{a_1^2}{2\lambda^2\varphi}\right)u+\overline{\varphi}v+b_4 \end{pmatrix},$$

respectively, are geodesic surfaces in $\mathbb{H}_3$, where $a_{\overline{1,5}}$, $b_{\overline{1,5}}$, φ and $\overline{\varphi}$ are real constants.

Proof. Its direct consequence from the Theorem 4.1, Definition 3.2, Remark 4.2 and the Eqs(4. 14 and 4. 18).

Corollary 4.4. *The curves $\alpha_{1,2}: I\subset R\to\mathbb{H}_3$ parameterized by*

$$\begin{aligned} u &\mapsto X(u,v)\mid v \textit{ is fixed real number and} \\ v &\mapsto X(u,v)\mid u \textit{ is fixed real number} \end{aligned}$$

respectively, are geodesic curves in $\mathbb{H}_3$ where $X(u,v)$ is the parametrization given in Proposition 4.3.

□

Example 4.5. *1. Using the assertion 1,2,3 of the Proposition 4.3 with conditions*

$$\begin{aligned} a_{\overline{2,5}} &= b_4=0; a_1=a_4=\frac{b_1}{2}=b_2=2b_3=1; (u,v)\in[-2,2]^2, \\ b_1 &= 2b_{2,3}=2\overline{\varphi}=2;\ b_{4,5}=0\ ;\ (u,v)\in[-5,3]^2 \\ a_{1,2} &= \overline{\varphi}=\varphi=\frac{\lambda}{2}=1;\ a_3=b_4=0;\ (u,v)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[-4,4] \end{aligned}$$

then the surfaces $\mathbb{M}_{1,2,3}$ (See Figures 2, 4 and 5 presented in $\mathbb{R}^3$) parameterized by

$$\begin{aligned} X_1(u,v) &= \left(vu+u+1,\frac{1}{2}u,-\frac{1}{2}u^2v-\frac{1}{2}u^2+u\frac{1}{2}-v\right), \\ X_2(u,v) &= \left(2u+1,u,-u^2+u+v\right), \\ X_3(u,v) &= \left(\frac{1}{2}\sin 2u+1,\frac{1}{4}\cos 2u,\frac{9}{8}u+v-\frac{1}{32}\sin 4u-\frac{1}{4}\cos 2u\right), \end{aligned}$$

respectively, are geodesic surfaces in $\mathbb{H}_3$. Moreover, we plot a geodesic curves $\alpha_{1,2}$, in blue color, when $u=3$ and $v=3$ on geodesic surface $\mathbb{M}_2$. (See Figure 3)

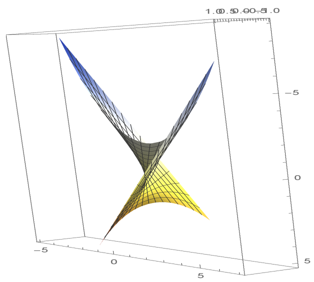

FIGURE 2. Geodesic surface $\mathbb{M}_1$

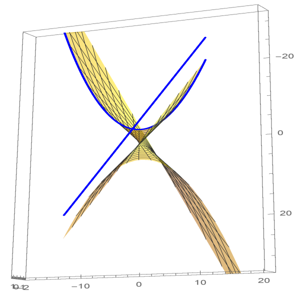

FIGURE 3. Geodesic curves $\alpha_{1,2}$ in $\mathbb{M}_1$.

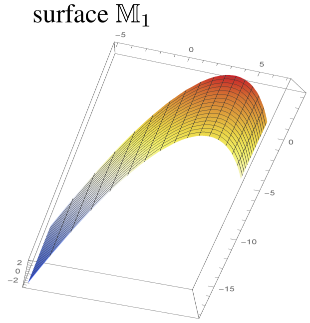

FIGURE 4. Geodesic surface $\mathbb{M}_2$

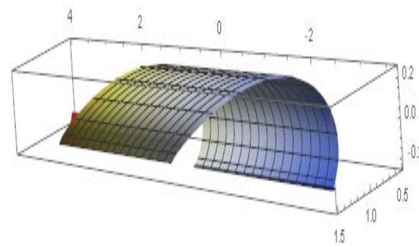

FIGURE 5. Geodesic surface $\mathbb{M}_3$

5. MAGNETIC SURFACES IN $\mathbb{H}_3$

In order to determine magnetic surfaces, we introduce a magnetic field F_q on contact manifold $(\mathbb{H}_3, \phi, \xi, \eta, g_\lambda)$, defined as

$$F_q(X, Y) = g(\Phi(X), Y),$$

where X and Y are vector fields on $\mathbb{H}_3$ and q is a real constant (the value of the magnetic charge). Taking into account the Eq.(2. 6), F_q is called the contact magnetic field with strength q. The $(1,1)$-tensor field Φ that present the LORENTZ force associated to the magnetic fields F_q is given as

$$\Phi = q\phi.$$

Then the Definition 3.1 turns to;

Definition 5.1. *Let $\mathbb{M}$ be a regular surface and F_q be a magnetic tensor fields in $\mathbb{H}_3$. $\mathbb{M}$ is a magnetic surface if the integral curves of the vectors X_u and X_v are magnetic curves i.e.*

$$\begin{cases} \nabla_{X_u} X_u = q\phi(X_u) \\ \nabla_{X_v} X_v = q\phi(X_v) \end{cases}. \tag{5. 19}$$

From the Definition 5.1 and using the Eqs(2. 2 , 2. 5), we have

$$\phi(X_u) = -\frac{x_u}{\lambda}e_1 + y_u e_2 \text{ and } \phi(X_v) = -\frac{x_v}{\lambda}e_1 + y_v e_2,$$

then, we obtain the system

$$\begin{cases} \left(y_{uu}+x_u\left(z_u+xy_u\right)\right)e_1+\left(\frac{x_{uu}}{\lambda}-\lambda y_u\left(z_u+xy_u\right)\right)e_2+\left(z_u+xy_u\right)_u e_3=-q\frac{x_u}{\lambda}e_1+qy_ue_2 \\ \left(y_{vv}+x_v\left(z_v+xy_v\right)\right)e_1+\left(\frac{x_{vv}}{\lambda}-\lambda y_v\left(z_v+xy_v\right)\right)e_2+\left(z_v+xy_v\right)_v e_3=-q\frac{x_v}{\lambda}e_1+qy_ve_2 \end{cases}.$$

After an integration of the component of e_3 and taking into account that $(e_i)_{i=\overline{1,3}}$ is an orthonormal basis, we get two systems

$$S:\begin{cases} x_{uu}-\lambda\left(\lambda\varphi\left(v\right)+q\right)y_u=0 \\ y_{uu}+\frac{1}{\lambda}\left(\lambda\varphi\left(v\right)+q\right)x_u=0 \\ z_u+xy_u=\varphi\left(v\right) \end{cases} \text{and } \overline{S}:\begin{cases} x_{vv}-\lambda\left(\lambda\overline{\varphi}\left(u\right)+q\right)y_v=0 \\ y_{vv}+\frac{1}{\lambda}\left(\lambda\overline{\varphi}\left(u\right)+q\right)x_v=0 \\ z_v+xy_v=\overline{\varphi}\left(u\right) \end{cases}, \tag{5. 20}$$

where φ and $\overline{\varphi}$ are arbitrary smooth functions.

Theorem 5.2. *Let $\mathbb{M}$ be a regular surface in $\mathbb{H}_3$ parameterized by $X(u,v)=(x(u,v),y(u,v),z(u,v))$, then $\mathbb{M}$ is magnetic surface if and only if the systems (S) and $\left(\overline{S}\right)$ holds.*

Similarly as the above section, we will solve the systems $\left(S \text{ and } \overline{S}\right)$ in order to determine the magnetic surfaces in the case when the functions φ and $\overline{\varphi}$ are zero or non-zero constants.

Case 1. If $\varphi=\overline{\varphi}\equiv 0$, the equations $(S_{1,2})$ becomes

$$x_{uu}=q\lambda y_u;\quad y_{uu}=-\frac{q}{\lambda}x_u. \tag{5. 21}$$

Using the Eqs.(5. 20 $_3$), the solution of the system (S) is

$$\begin{cases} x\left(u,v\right)=\frac{\varphi_1}{q}\sin qu+\varphi_2 \\ y\left(u,v\right)=\frac{\varphi_3}{q\lambda}\cos qu+\varphi_4 \\ z(u,v)=-\frac{1}{4q^2\lambda}\varphi_3\left(\varphi_1\sin 2qu-2q\varphi_1u+4q\varphi_2\cos qu\right)+\varphi_5 \end{cases}, \tag{5. 22}$$

where $\varphi_{\overline{1,5}}$ are arbitrary smooth functions in v.
Now, let calculate the unknown functions $\varphi_{\overline{1,5}}$.

Substituting the Eq.(5. 22) in the equation $\left(\overline{S}_{1,2}\right)$, we have

$$\begin{cases} \frac{\varphi_{1vv}}{q}\sin qu+\varphi_{2vv}=\varphi_{3v}\cos qu+q\lambda\varphi_{4v} \\ \frac{\varphi_{3vv}}{q\lambda}\cos qu+\varphi_{4vv}=-\frac{\varphi_{1v}}{\lambda}\sin qu-\frac{q}{\lambda}\varphi_{2v} \end{cases},$$

by a comparing with respect to u, we deduce the ODEs

$$\varphi_{1v}=\varphi_{3v}=0;\ \ \varphi_{2vv}=+q\lambda\varphi_{4v};\ \ \varphi_{4vv}=-\frac{q}{\lambda}\varphi_{2v},$$

its solutions are

$$\varphi_{1,3}=a_{1,3}\quad \varphi_2\left(v\right)=\frac{a_2}{q}\sin qv+b_1;\ \ \varphi_4\left(v\right)=\frac{a_2}{q\lambda}\cos qv+b_2,$$

where $a_{\overline{1,3}}$ and $b_{\overline{1,2}}$ are arbitrary real constants. Using the Eq.(5. 20 $_6$ and 5. 22 $_3$) and comparing with respect to u, we obtain

$$a_3\varphi_{2v}=a_1\varphi_{4v}=0;\ \ \varphi_{5v}=-\varphi_2\varphi_{4v}.$$

Again, comparing with respect to v, we get the conditions

$$a_2a_3=a_2a_1=0, \tag{5. 23}$$

and

$$\varphi_5 = -\frac{a_2^2}{4q^2\lambda}\sin 2qv - \frac{a_2 b_1}{q\lambda}\cos qv + \frac{a_2^2}{2q\lambda}v + b_3, \tag{5. 24}$$

where b_3 is arbitrary real constant.

Then the solution of systems $\left(S \text{ and } \overline{S}\right)$ is

$$X(u,v) = \left(\begin{array}{l} x\,(u,v) = \frac{a_1}{q}\sin qu + \frac{a_2}{q}\sin qv + b_1 \\ y\,(u) = \frac{a_3}{q\lambda}\cos qu + \frac{a_2}{q\lambda}\cos qv + b_2 \\ z(u,v) = -\frac{1}{2q^2\lambda}a_2^2\cos qv\sin qv - \frac{1}{2q^2\lambda}a_1a_3\cos qu\sin qu - \frac{1}{q^2\lambda}a_2a_3\cos qu\sin qv \\ \qquad -\frac{1}{q\lambda}a_3b_1\cos qu - \frac{1}{q\lambda}a_2b_1\cos qv + \frac{a_2^2}{2q\lambda}v + \frac{a_1a_3}{2q\lambda}u + b_3 \end{array} \right) \tag{5. 25}$$

with conditions

$$a_2a_3 = a_2a_1 = 0. \tag{5. 26}$$

Now, we have the proposition.

Proposition 5.3. *There are no magnetic surfaces, given by the Theorem 5.2, parameterized by the Eq.(5. 25) when* $\varphi = \overline{\varphi} \equiv 0$.

Proof. The parametrization given in Eq.(5. 25) is reduced only to curve for possible cases $a_2 = 0$ or $a_3 = a_1 = 0$ given in the condition Eq.(5. 26). □

Case 2. If $\varphi = 0$ and $\overline{\varphi}$ is a non-zero constant, then the equations $(S_{1,2})$ give a same solutions given in Eq.(5. 22). Substituting the Eq.(5. 22) in the equations $\left(\overline{S}_{1,2}\right)$, we have

$$\left\{ \begin{array}{l} \left(-1 - \frac{\lambda}{q}\overline{\varphi}\right)\varphi_{3v}\cos qu + \frac{1}{q}\varphi_{1vv}\sin qu - q\lambda\varphi_{4v} + \varphi_{2vv} - \lambda^2\overline{\varphi}\varphi_{4v} = 0 \\ (q + \lambda\overline{\varphi})\,\varphi_{1v}\sin qu + \varphi_{3vv}\cos qu + q\lambda\overline{\varphi}\varphi_{2v} + q\lambda\varphi_{4vv} + q^2\varphi_{2v} = 0 \end{array} \right. ,$$

by a comparing with respect to u, we get the ODEs

$$\left\{ \begin{array}{c} \varphi_{1vv} = \varphi_{3vv} = 0;\ \ (q + \lambda\overline{\varphi})\,\varphi_{3v} = (q + \lambda\overline{\varphi})\,\varphi_{1v} = 0 \\ -q\lambda\varphi_{4v} + \varphi_{2vv} - \lambda^2\overline{\varphi}\varphi_{4v} = \lambda\overline{\varphi}\varphi_{2v} + \lambda\varphi_{4vv} + q\varphi_{2v} = 0 \end{array} \right. , \tag{5. 27}$$

We observe two cases for above ODEs,

i. if $\overline{\varphi} = \frac{-q}{\lambda}$, then the functions $\varphi_{\overline{1,4}}$ are linear in v, i.e.

$$\varphi_{\overline{1,4}} = a_{\overline{1,4}}v + b_{\overline{1,4}} \mid a_{\overline{1,4}}, b_{\overline{1,4}} \in \mathbb{R}. \tag{5. 28}$$

Using the Eqs.(5. 22 $_3$ and $\left(\overline{S}_3\right)$) and comparing with respect to v, we deduce

$$\begin{array}{c} \varphi_1\varphi_{4v} = \varphi_1\varphi_{3v} = \varphi_3\varphi_{2v} = \varphi_3\varphi_{1v} = 0 \\ \frac{-q}{\lambda} - \varphi_2\varphi_{4v} = \varphi_{5v} \end{array} ,$$

and

$$\begin{array}{rcl} a_1a_4 &=& a_1a_3 = a_2a_3 = b_1a_4 = b_1a_3 = b_3a_2 = b_3a_1 = 0 \\ \varphi_5 &=& \dfrac{-q}{\lambda}v - \dfrac{a_2a_4}{2}v^2 + a_4b_2v + b_5, \end{array} \tag{5. 29}$$

where b_5 is real constant.
ii. if $\overline{\varphi} \neq \frac{-q}{\lambda}$, then, we have

$$\begin{array}{l} \varphi_1 = a_1; \;\; \varphi_3 = a_3 \\ \varphi_2 = -\lambda a_2 \cos \overline{A} v + b_2 \;\; , \\ \varphi_4 = a_2 \sin \overline{A} v + b_4 \end{array} \tag{5. 30}$$

where $\overline{A} = q + \lambda\overline{\varphi}$.

Using the Eqs.(5. 22 $_3$ and $\left(\overline{S}\right)_3$) and comparing with respect to v, we deduce

$$\begin{aligned} a_1 a_2 &= a_2 a_3 = 0, \\ \varphi_5 &= a_2 b_2 \sin \overline{A} v - \frac{1}{4}\lambda a_2^2 \left(\sin 2\overline{A} v + 2\overline{A} v\right) + b_3. \end{aligned} \tag{5. 31}$$

Hence, We can present the following proposition.

Proposition 5.4. *The surfaces* $\mathbb{M}_{1,2}$ *are magnetic surfaces in* $\mathbb{H}_3$ *parameterized by:*
1.

$$X(u,v) = \left(\begin{array}{l} x\,(u,v) = \frac{\varphi_1}{q} \sin qu + \varphi_2 \\ y\,(u) = \frac{\varphi_3}{q\lambda} \cos qu + \varphi_4 \\ z(u,v) = -\frac{1}{4q^2\lambda}\varphi_3 \left(\varphi_1 \sin 2qu - 2q\varphi_1 u + 4q\varphi_2 \cos qu\right) \\ \qquad -\frac{q}{\lambda} v - \frac{a_2 a_4}{2} v^2 + a_4 b_2 v + b_5 \end{array} \right)$$

where the functions $\varphi_{\overline{1,4}} = a_{\overline{1,4}} v + b_{\overline{1,4}}$ *are linear and* $a_1 a_4 = a_1 a_3 = a_2 a_3 = b_1 a_4 = b_1 a_3 = b_3 a_2 = b_3 a_1 = 0$.
2.

$$X(u,v) = \left(\begin{array}{l} x\,(u,v) = \frac{a_1}{q} \sin qu - \lambda a_2 \cos \overline{A} v + b_2 \\ y\,(u) = \frac{a_3}{q\lambda} \cos qu + a_2 \sin \overline{A} v + b_4 \\ z(u,v) = -\frac{1}{4q^2\lambda} \left(a_1 a_3 \sin 2qu - 2q a_1 a_3 u + 4q a_3 \left(-\lambda a_2 \cos \overline{A} v + b_2\right) \cos qu\right) \\ \qquad + \overline{\varphi} v + a_2 \sin \overline{A} v + b_4 \end{array} \right)$$

where $a_{\overline{1,4}}$, $b_{\overline{1,5}}$ *are real constants with the condition* $a_1 a_2 = a_2 a_3 = 0$.

Proof. Its direct consequence from the Theorem 5.2, the Definition 5.1 and the Eqs.(5. 22 , 5. 28 , 5. 29 , 5. 30 and 5. 31). □

Example 5.5. *The surface* $\mathbb{M}$ *parameterized by*

$$X(u,v) = \left(\sin u + 1, \frac{1}{2}\cos u + 1, -\frac{1}{8}\left(\sin 2u - 2u + 4\cos u\right) + v\right)$$

is magnetic surface in $\mathbb{H}_3$. *Using the Proposition 5.4 for the assertion 1 with the condition* $a_{\overline{1,4}} = b_5 = 0;\ b_{\overline{1,4}} = q = \frac{1}{2}\lambda = 1$ *and for the assertion 2 with the condition* $\varphi_2 = b_2 = a_{1,3} = \frac{1}{2}\lambda = q = \overline{\varphi} = 1;\ a_2 = b_4 = 0$.*(See Figure 6 presented in* $\mathbb{R}^3$*)*

Case 3. If φ and $\overline{\varphi}$ are non-zero constants, then the solution of equations $(S_{1,2})$

$$\left\{ \begin{array}{l} x_{uu} - \lambda\left(\lambda\varphi + q\right) y_u = 0 \\ y_{uu} + \frac{1}{\lambda}\left(\lambda\varphi + q\right) x_u = 0 \end{array} \right.$$

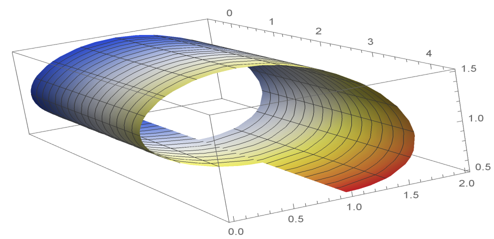

FIGURE 6. Magnetic surface $\mathbb{M}$ in $\mathbb{H}_3$

have two cases

$$\begin{array}{l}\textbf{i.}\left\{\begin{array}{l} x(u,v)=\varphi_1 u+\varphi_2 \\ y(u,v)=\varphi_3 u+\varphi_4 \end{array}\right. \text{if } \varphi=\frac{-q}{\lambda}, \\ \textbf{ii.}\left\{\begin{array}{l} x(u,v)=\frac{\varphi_1}{A}\sin Au+\varphi_2 \\ y(u,v)=\frac{\varphi_1}{A}\cos Au+\varphi_3 \end{array}\right. \text{if } \varphi\neq\frac{-q}{\lambda}, \end{array} \tag{5. 32}$$

where $\varphi_{\overline{1,4}}$ are arbitrary function in v and $A=q+\lambda\varphi$. Using the equation $(S)_3$,, we obtain

$$\left\{\begin{array}{l} \textbf{i.}\ z(u,v)=-\frac{\varphi_1\varphi_3}{2}u^2-\left(\frac{q}{\lambda}+\varphi_2\varphi_3\right)u+\varphi_5 \\ \textbf{ii.}\ z(u,v)=-\varphi_1\left(1+\frac{1}{A}\varphi_2\right)\cos Au+\varphi u+\varphi_5 \end{array}\right., \tag{5. 33}$$

where φ_5 is arbitrary smooth function in v. Substituting the last solutions in the equations $(\overline{S}_{1,2})$, we have

$$\textbf{i.}\left\{\begin{array}{l} \left(\varphi_{1vv}-\lambda\overline{A}\varphi_{3v}\right)u+\varphi_{2vv}-\lambda\overline{A}\varphi_{4v}=0 \\ \left(\varphi_{3vv}+\frac{\overline{A}}{\lambda}\varphi_{1v}\right)u+\varphi_{4vv}+\frac{\overline{A}}{\lambda}\varphi_{2v}=0 \end{array}\right.,$$

$$\textbf{ii.}\left\{\begin{array}{l} \frac{\varphi_{1vv}}{A}\sin Au-\frac{\lambda\overline{A}}{A}\varphi_{1v}\cos Au+\varphi_{2vv}-\lambda\overline{A}\varphi_{3v}=0 \\ \frac{\varphi_{1vv}}{A}\cos Au+\frac{\overline{A}}{\lambda A}\varphi_{1v}\sin Au+\frac{1}{\lambda}\varphi_{2v}+\varphi_{3vv}=0 \end{array}\right.,$$

here $\overline{A}=\lambda\overline{\varphi}+q$. Comparing with respect to u, we get the ODEs

$$\textbf{i.}\left\{\begin{array}{l} \varphi_{1vv}=\lambda\overline{A}\varphi_{3v};\ \ \varphi_{2vv}=\lambda\overline{A}\varphi_{4v} \\ \varphi_{3vv}=-\frac{\overline{A}}{\lambda}\varphi_{1v};\ \ \varphi_{4vv}=-\frac{\overline{A}}{\lambda}\varphi_{2v} \end{array}\right.$$

$$\textbf{ii.}\left\{\begin{array}{l} \varphi_{1vv}=0;\ \ \overline{A}\varphi_{1v}=0 \\ \frac{1}{\lambda}\varphi_{2v}+\varphi_{3vv}=0;\ \ \varphi_{2vv}=\lambda\overline{A}\varphi_{3v} \end{array}\right.$$

according to the value and the sign of $\overline{A}$ (i.e. $\overline{\varphi} =$ or $<$ or $>$ to $-\frac{q}{\lambda}$), its solutions are

$$\textbf{i}.\begin{cases} \textbf{a}.\ \varphi_i = a_i v + b_i \ \text{ if } \overline{A} = 0 \\ \textbf{b}.\ \varphi_{1,2}(v) = \frac{a_{1,2}}{\overline{A}}\sin \overline{A}v + b_{1,2};\ \varphi_{3,4}(v) = \frac{a_{3,4}}{\overline{A}\lambda}\cos \overline{A}v + b_{3,4} \text{ if } \overline{A} \neq 0 \end{cases} \tag{5. 34}$$

$$\textbf{ii}.\begin{cases} \textbf{a}.\ \varphi_{1,2} = a_{1,2}v + b_{1,2};\ \varphi_3 = -\frac{a_2}{\lambda}v + b_3 \ \text{ if } \overline{A} = 0 \\ \textbf{b}.\ \varphi_1 = a_1, \begin{cases} \left.\begin{array}{l} \varphi_2(v) = b_2 + \frac{a_2}{\sqrt{\overline{A}}}\sin\sqrt{\overline{A}}v \\ \varphi_3(v) = b_3 + \frac{a_2}{\overline{A}\lambda}\cos\sqrt{\overline{A}}v, \end{array}\right. \text{if } \overline{A} > 0 \\ \left.\begin{array}{l} \varphi_2(v) = b_2 + \frac{a_2}{\sqrt{a}}e^{\sqrt{-\overline{A}}v} - \frac{a_3}{\sqrt{a}}e^{-\sqrt{-\overline{A}}v} \\ \varphi_3(v) = b_3 - \frac{a_2}{a\lambda}e^{\sqrt{-\overline{A}}v} - \frac{a_3}{\overline{A}\lambda}e^{-\sqrt{-\overline{A}}v}, \end{array}\right. \text{if } \overline{A} < 0 \end{cases} \end{cases}$$

Using the Eqs.(5. 33 and $(\overline{S}_3)$), we obtain

$$\textbf{i}.\ \varphi_{1v}\varphi_3 = 0,\ \varphi_1\varphi_{4v} = \varphi_{2v}\varphi_3,\ \varphi_{5v} = \overline{\varphi} - \varphi_2\varphi_{4v}$$
$$\textbf{ii}.\begin{cases} \overline{A} = 0,\ \varphi_1 = a_1,\ \varphi_1\varphi_{3v} = 0 \\ \varphi_{2v}\varphi_1 = 0;\ \ \varphi_{5v} = \overline{\varphi} - \varphi_2\varphi_{3v} \end{cases}$$

and comparing with respect to v, we deduce

$$\textbf{i}.\begin{cases} \textbf{a}.\ a_1a_3 = a_1b_3 = 0;\ a_1a_4 = a_2a_3; \\ \textbf{b}.\ a_1b_3 = a_1a_3 = a_2a_3 = a_2b_3 = a_1a_4 = b_1a_4 = 0 \end{cases} \tag{5. 35}$$
$$\textbf{ii}.\begin{cases} \textbf{a}.\ \varphi_1 = a_1,\ a_1a_2 = 0 \text{ if } \overline{A} = 0 \\ \textbf{b}.\ \text{non-existent case} \end{cases}$$

and

$$\textbf{i}.\begin{cases} \textbf{a}.\ \ \varphi_5 = -\frac{a_2a_4}{2}v^2 - \left(\frac{q}{\lambda} + b_2a_4\right)v + b_4 \\ \textbf{b}.\ \varphi_5 = \frac{a_2a_4}{4\overline{A}^2\lambda}\sin 2\overline{A}v + \frac{a_4b_2}{\overline{A}\lambda}\cos\overline{A}v + \left(\overline{\varphi} - \frac{a_2a_4}{2\overline{A}\lambda}\right)v + b_5 \end{cases} \tag{5. 36}$$
$$\textbf{ii}.\begin{cases} \textbf{a}.\ \varphi_5 = \frac{a_2a_2}{2\lambda}v^2 - \left(\frac{q}{\lambda} - \frac{a_2b_2}{\lambda}\right)v + b_4 \\ \textbf{b}.\ \text{non-existent case} \end{cases}$$

Finally, we can present the following proposition.

Proposition 5.6. *The surfaces $\mathbb{M}_{1,2,3}$ are magnetic surfaces in $\mathbb{H}_3$ parameterized by*
1.

$$X(u,v) = \begin{pmatrix} x(u,v) = a_1vu + b_1u + a_2v + b_2 \\ y(u,v) = a_3vu + b_3u + a_4v + b_4 \\ z(u,v) = -\frac{1}{2}u^2b_1b_3 - \frac{1}{2}v^2a_2a_4 - uv^2a_2a_3 - \frac{1}{2}u^2va_3b_1 - (a_2b_3 + a_3b_2)uv \\ \qquad -ub_2b_3 - va_4b_2 - \frac{q}{\lambda}(u+v) + b_4 \end{pmatrix}$$

with conditions $a_1a_3 = a_1b_3 = 0;\ a_1a_4 = a_2a_3,$
2.

$$X(u,v) = \begin{pmatrix} x(u,v) = \frac{a_1}{\overline{A}}u\sin\overline{A}v + \frac{a_2}{\overline{A}}\sin\overline{A}v + b_1u + b_2 \\ y(u,v) = \frac{a_3}{\overline{A}}u\sin\overline{A}v + \frac{a_4}{\overline{A}}\sin\overline{A}v + b_3u + b_4 \\ z(u,v) = \frac{a_2a_4}{2\overline{A}^2\lambda}\cos\overline{A}v\sin\overline{A}v - \frac{1}{\overline{A}\lambda}\left(\frac{a_3b_1}{2}u^2 + a_3b_2u - a_4b_2\right)\cos\overline{A}v \\ \qquad -\frac{b_1b_3}{2}u^2 + \left(\overline{\varphi} - \frac{a_2a_4}{2\overline{A}\lambda}\right)v - \left(\frac{q}{\lambda} + b_2b_3\right)u + b_5 \end{pmatrix}$$

with conditions $a_1b_3 = a_1a_3 = a_2a_3 = a_2b_3 = a_1a_4 = b_1a_4 = 0$ *and* $\overline{\varphi} \neq \frac{-q}{\lambda}$*(i.e.* $\overline{A} = \lambda\overline{\varphi} + q \neq 0$*).*
3.

$$X(u,v) = \begin{pmatrix} x(u,v) = \frac{a_1}{A}\sin Au + a_2v + b_2 \\ y(u,v) = \frac{a_1}{A}\cos Au - \frac{a_2}{\lambda}v + b_3 \\ z(u,v) = -\left(a_1 + \frac{a_1b_2}{A}\right)\cos Au + \frac{a_2^2}{2\lambda}v^2 - \left(\frac{q}{\lambda} - \frac{a_2b_2}{\lambda}\right)v + \varphi u + b_4 \end{pmatrix}$$

with conditions $a_1a_2 = 0$ *and* $\varphi \neq \frac{-q}{\lambda}$ *(i.e.* $A = \lambda\varphi + q \neq 0$*) where* $a_{1,4}$*,* $b_{1,5}$ *and* $\overline{\varphi}$ *are a real constants.*

Proof. The proof follows from the Theorem 5.2 , the Definition 5.1 the Eqs.(5. 32 and 5. 33) and the conditions given in Eqs.(5. 35 and 5. 36).

Corollary 5.7. *The curves parameterized by*

$$\begin{aligned} \alpha_1 &: I \subset R \to \mathbb{H}_3,\ u \mapsto X(u,v) \mid v \text{ is fixed real number and} \\ \alpha_2 &: I \subset R \to \mathbb{H}_3,\ v \mapsto X(u,v) \mid u \text{ is fixed real number} \end{aligned}$$

are magnetic curves in $\mathbb{H}_3$*, where* $X(u,v)$ *is the parametrization given in Propositions 5.3,5.4 and 5.6.*

□

Example 5.8. *1. The surface* $\mathbb{M}_1$ *(see Figure 7) parameterized by*

$$X(u,v) = \left(u + 1, vu + 2u + v, -u^2 - \frac{1}{2}u^2v - uv - 3u - 2v\right)$$

is magnetic surface in $\mathbb{H}_3$*. Using the assertion 1 of the Proposition 5.6 with conditions*

$$a_3 = 2a_4 = b_3 = 2b_1 = 2b_2 = q = \lambda = 2;\ a_1 = a_2 = b_4 = 0$$

2. The surface $\mathbb{M}_2$ *(see Figure 8) parameterized by*

$$X(u,v) = \left(2u + 1, \frac{1}{2}u\sin 2v + \frac{1}{2}\sin 2v + u, -\frac{1}{2}\left(u^2 + u\right)\cos 2v - u^2 + v - 2u\right)$$

is magnetic surface in $\mathbb{H}_3$*. Using the assertion 2 of the Proposition 5.6 with conditions*

$$\begin{aligned} a_1 &= a_2 = a_4 = b_4 = b_5 = 0;\ 0.5b_1 = b_2 = b_3 = a_3 = a_4 = 1 \\ q &= \lambda = \overline{\varphi} = 1;\ \overline{A} = \lambda\overline{\varphi} + q = 2 \text{ and } (u,v) \in [-1,1] \times [-\pi,\pi] \end{aligned}$$

In the magnetic surface $\mathbb{M}_2$ *we plot a magnetic curves* $\alpha_{1,2}$ *when* $u = 1.$ *and* $v = 1.$ *(See Figure 9)*
3. The surfaces $\mathbb{M}_{3\cdot 1}$ *and* $\mathbb{M}_{3\cdot 2}$ *parameterized by*

$$\begin{aligned} X(u,v) &= \left(v + 1, -v, \frac{1}{2}v^2 + u\right) \\ X(u,v) &= \left(\frac{1}{2}\sin 2u + 1, \frac{1}{2}\cos 2u, -\frac{3}{2}\cos 2u - v + u\right) \end{aligned}$$

respectively, are magnetic surfaces in $\mathbb{H}_3$. *Using the assertion 3 of the Proposition 5.6 with conditions*

$$
\begin{aligned}
a_1 &= b_3 = b_4 = 0,\ a_2 = b_2 = \lambda = \varphi = q = 1;\ A = \lambda\varphi + q = 2 \text{ and} \\
a_2 &= b_3 = b_4 = 0,\ a_1 = b_2 = \lambda = \varphi = q = 1;\ A = \lambda\varphi + q = 2
\end{aligned}
$$

respectively.(See Figures 10 and 11 presented in $\mathbb{R}^3$*)*

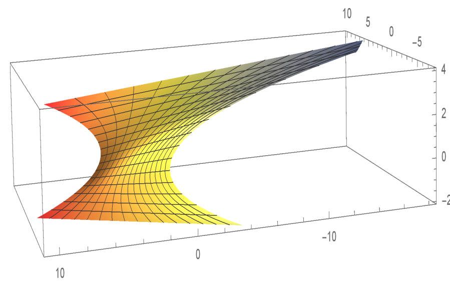

FIGURE 7. Magnetic surface $\mathbb{M}_1$

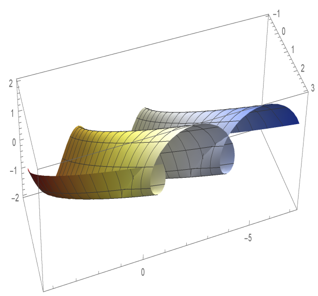

FIGURE 8. Magnetic surface $\mathbb{M}_2$

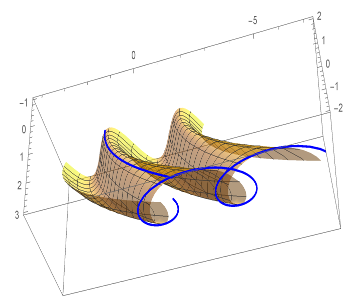

FIGURE 9. Magnetic curves $\alpha_{1,2}$ (blue color) in $\mathbb{M}_2 \subset \mathbb{H}_3$

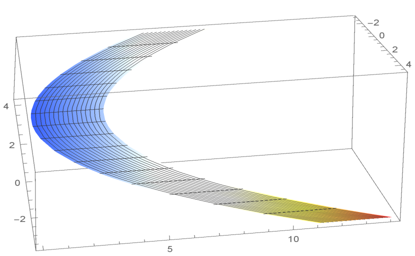

FIGURE 10. Magnetic surface $\mathbb{M}_{3.1}$

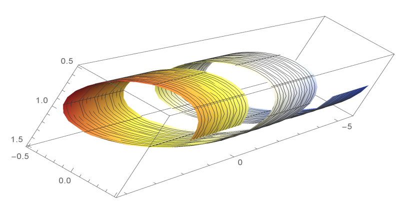

FIGURE 11. Magnetic surface $\mathbb{M}_{3.2}$

Corollary 5.9. *There exist a magnetic geodesic surfaces in* $\mathbb{H}_3$.

Proof. Just compare the geodesic surface equation in the case 3 and the magnetic surface equation in the case 2 for good selection of constants, we get a magnetic geodesic surfaces in $\mathbb{H}_3$. □

CONCLUSION

A magnetic curve, which is the trajectory of a charged particle moving under the influence of a magnetic field, can be viewed as a one-dimensional magnetic manifold. Motivated by real world examples, we have extended this concept to two-dimensional manifold, namely the magnetic surface, where the magnetic field vector at any point on the surface is tangent to the surface itself. As an application, we have presented a method for deriving parametric equations of magnetic surfaces within the Heisenberg three-group, considered as a Riemannian manifold. Subsequently, a concrete examples of such surfaces proving their existence are given. In Euclidean three-space this study was given in [16], We invite readers to apply this study to other three-dimensional contact manifolds such as $Sol3$ and $Sl(2, \mathbb{R})$.

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