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Two Problems on Narayana Numbers And Repeated Digit Numbers

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Abstract. This work aims to solve two problems in the Diophantine equation of the Narayana sequence. In the first question it's proven that there are only 177 solutions of expressing the product of two Narayana numbers as b repdigits numbers, for base $2 \leq b \leq 50$. It's also proven that the Narayana numbers can not be factored as a product of two repdigits numbers for base $2 \leq b \leq 50$, except in two cases. The proofs use some number-theoretic techniques, including Baker's method of linear forms in logarithms height, and some reduction techniques.

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1. Introduction

Let $(N_n)_{n\geq 0}$ be the Narayana sequence, starting with $N_0 = 0, N_1 = 1,$ and $N_2 = 1.$ For $n \geq 3$, the sequence is defined by the recurrence relation

$$N_n = N_{n-1} + N_{n-3}. \qquad (1.\ 1)$$

The first values of N_k are $0, 1, 1, 1, 2, 3, 4, 6, \ldots$. Narayana cow sequence is similar to the problem of Fibonacci's rabbits, as it counts calves produced every four years. This sequence (OEIS A000930 in [12]) appeared for the first time in the book *Ganita Kaumudi*(1365) by the Indian mathematician Narayana Pandita, who gave this sequence its name, and played a role in mathematical developments such as, finding the approximate value of the square roots, and investigations into the Diophantine equation $ax^2 + 1 = y^2$

(Pell's equation). Narayana cows sequence, also known as the supergolden sequence. The real root corresponding to the solution of the characteristic equation is known as the super golden ratio. In Pascal's triangle, starting from $n \geq 3$, the rows with triplicated diagonals sum to Narayana sequence, while the rows with sloping diagonals of 45 degrees sum to the Fibonacci sequence. In the fields of graph theory, coding theory, and cryptography, this sequence is crucial.

In this work, we ascertain every solution to the Diophantine equation

$$N_n N_m = [a, \ldots, a]_b = a\left(\frac{b^\ell - 1}{b-1}\right), \tag{1. 2}$$

in integers (b, a, m, n, ℓ) with $3 \leq m \leq n, 2 \leq b \leq 50, 0 < a < b$ and $\ell > 1$, and the Diophantine equation's solution

$$N_k = a_1 a_2 \left(\frac{b^{\ell_1} - 1}{b-1}\right)\left(\frac{b^{\ell_2} - 1}{b-1}\right), \tag{1. 3}$$

in integers $(k, b, a_1, a_2, \ell_1, \ell_2)$ with $2 \leq \ell_1 \leq \ell_2$, $1 \leq a_1 \leq a_2 \leq b-1$, $k \geq 3$, and $b \geq 2$. More precisely, the theorems listed below have been proven.

Theorem 1.1. *Let* $3 \leq m \leq n$, $b \in \{2, 3, \ldots, 50\}$, $a \in \{1, \ldots, b-1\}$, *and* $\ell \geq 2$. *The Diophantine equation* (1. 2) *has just the following possible solutions:*

(2,1,3,5,2)	*(3,1,3,6,2)*	*(3,1,3,9,3)*	*(3,1,4,4,2)*
(3,2,4,6,2)	*(3,2,4,9,3)*	*(3,1,9,11,6)*	*(5,1,3,7,2)*
(5,1,4,5,2)	*(5,2,4,7,2)*	*(5,3,4,8,2)*	*(5,2,5,6,2)*
(5,3,5,7,2)	*(5,4,6,7,2)*	*(6,4,3,11,2)*	*(6,3,3,15,3)*
(7,1,4,6,2)	*(7,1,5,10,3)*	*(7,2,6,6,2)*	*(7,3,6,7,2)*
(11,10,4,13,2)	*(11,1,5,6,2)*	*(11,7,5,11,2)*	*(11,2,6,7,2)*
(11,3,6,8,2)	*(11,3,7,7,2)*	*(11,4,10,11,3)*	*(12,1,3,9,2)*
(12,2,4,9,2)	*(12,3,5,9,2)*	*(12,4,6,9,2)*	*(12,6,7,9,2)*
(12,9,8,9,2)	*(13,2,3,11,2)*	*(13,4,4,11,2)*	*(13,6,5,11,2)*
(13,8,6,11,2)	*(13,1,6,19,4)*	*(13,12,7,11,2)*	*(13,7,11,19,4)*
(14,4,3,13,2)	*(14,8,4,13,2)*	*(14,12,5,13,2)*	*(15,11,4,14,2)*
(15,1,6,6,2)	*(15,7,6,11,2)*	*(16,9,9,16,3)*	*(17,1,4,8,2)*
(17,1,5,7,2)	*(17,10,5,13,2)*	*(17,2,6,8,2)*	*(17,2,7,7,2)*
(17,3,7,8,2)	*(17,14,8,11,2)*	*(18,1,3,10,2)*	*(18,2,4,10,2)*
(18,3,5,10,2)	*(18,4,6,10,2)*	*(18,6,7,10,2)*	*(18,9,8,10,2)*
(18,13,9,10,2)	*(19,3,3,13,2)*	*(19,6,4,13,2)*	*(19,9,5,13,2)*

(19,12,6,13,2)	(19,18,7,13,2)	(20,9,3,16,2)	(20,18,4,16,2)
(20,4,5,11,2)	(20,8,7,11,2)	(20,12,8,11,2)	(21,4,3,14,2)
(21,8,4,14,2)	(21,12,5,14,2)	(21,16,6,14,2)	(23,5,4,13,2)
(23,11,5,14,2)	(23,1,6,7,2)	(23,10,6,13,2)	(23,7,7,11,2)
(23,15,7,13,2)	(23,22,7,14,2)	(25,1,4,9,2)	(25,2,6,9,2)
(25,3,7,9,2)	(25,14,9,11,2)	(26,7,3,16,2)	(26,14,4,16,2)
(26,1,5,8,2)	(26,21,5,16,2)	(26,2,7,8,2)	(26,3,8,8,2)
(26,20,8,13,2)	(27,1,3,11,2)	(27,2,4,11,2)	(27,3,5,11,2)
(27,4,6,11,2)	(27,6,7,11,2)	(27,9,8,11,2)	(27,13,9,11,2)
(27,19,10,11,2)	(28,14,3,18,2)	(29,2,3,13,2)	(29,4,4,13,2)
(29,6,5,13,2)	(29,8,6,13,2)	(29,12,7,13,2)	(29,18,8,13,2)
(29,26,9,13,2)	(31,11,6,14,2)	(32,8,5,14,2)	(32,16,7,14,2)
(32,24,8,14,2)	(33,14,19,20,4)	(34,17,3,19,2)	(35,5,5,13,2)
(35,1,6,8,2)	(35,21,6,16,2)	(35,1,7,7,2)	(35,10,7,13,2)
(35,7,8,11,2)	(35,15,8,13,2)	(35,22,8,14,2)	(37,1,4,10,2)
(37,2,6,10,2)	(37,3,7,10,2)	(37,14,10,11,2)	(37,30,10,13,2)
(38,1,5,9,2)	(38,2,7,9,2)	(38,3,8,9,2)	(38,20,9,13,2)
(39,3,4,13,2)	(39,6,6,13,2)	(39,9,7,13,2)	(40,1,3,12,2)
(40,2,4,12,2)	(40,3,5,12,2)	(40,4,6,12,2)	(40,6,7,12,2)
(40,9,8,12,2)	(40,13,9,12,2)	(40,19,10,12,2)	(40,28,11,12,2)
(41,9,4,16,2)	(41,2,5,11,2)	(41,29,5,18,2)	(41,18,6,16,2)
(41,4,7,11,2)	(41,27,7,16,2)	(41,6,8,11,2)	(41,40,11,13,2)
(42,3,3,15,2)	(42,6,4,15,2)	(42,9,5,15,2)	(42,12,6,15,2)
(42,18,7,15,2)	(42,27,8,15,2)	(42,39,9,15,2)	(43,2,3,14,2)
(43,4,4,14,2)	(43,6,5,14,2)	(43,8,6,14,2)	(43,12,7,14,2)
(43,18,8,14,2)	(43,26,9,14,2)	(43,38,10,14,2)	(44,4,5,13,2)
(44,8,7,13,2)	(44,12,8,13,2)	(45,8,10,20,3)	(47,5,6,13,2)
(47,11,7,14,2)	(47,35,11,13,2)	(48,16,11,11,2)	(49,1,10,15,3)
(50,35,5,19,2)			

TABLE 1. Continuation of Solutions of equation (1. 2) with $\ell \geq 2$

Theorem 1.2. *The equation* (1. 3) *yields exclusively these solutions*

$$N_8 = \frac{2^2-1}{2-1}\frac{2^2-1}{2-1} = [11]_2[11]_2$$

and

$$N_{16} = \frac{2^2-1}{2-1}\frac{2^6-1}{2-1} = [11]_2[111111]_2.$$

Many authors have studied such Diophantine equations. For example, the authors in [7] showed that, as a product of two repdigits, the biggest Fibonacci number is $F_{10} = 55$, whereas the largest Lucas number is $L_6 = 18$. The author in [4, 5] studied the sum of three Padovan numbers as repdigits in base 10. They demonstrated that the only Tribonacci numbers which can be articulated as concatenations of two repdigits in base 10 are

$T_n \in \{13, 24, 33, 81\}$. According to the authors in [13], the only balanced number that is the concatenation of two repdigits base 10 is 35. The authors in [8] studied all repdigits base $2 \leq b \leq 9$ which can be articulated as a products of two Fibonacci numbers. The author in [3] studied all repdigits which can be articulated as the product of a Fibonacci number and a Pell number. The author in [9] studied Pell-Lucas and Pell numbers which can be articulated as sums of two Jacobsthal numbers. The authors in [16] studied all repdigits which can be articulated as products of consecutive Padovan or Perrin numbers . Researchers in [14] demonstrated that $N_{14} = 88$ and $N_{17} = 277$ are the only Narayana numbers which can be articulated as sums of two repdigits. The authors in [2] studied the sum of two Narayana numbers which can be articulated as b repdigits for the bases $2 \leq b \leq 100$. For further studies on Diophantine equations, you can found in [1, 10, 15, 18].

2. Preliminary

2.1. **Narayana sequence.** The characteristic equation corresponding to the third-order linear recurrence relation (1. 1) is $x^3 - x^2 - 1$, three roots of this polynomial are β, α and $\gamma = \bar{\alpha}$ where

$$\beta = \tfrac{2+r_1+r_2}{6}, \quad \alpha = \tfrac{4-(1+\sqrt{-3})r_1-(1-\sqrt{-3})r_2}{12}$$

and

$$r_1 = \sqrt[3]{116 - 12\sqrt{93}}, \quad r_2 = \sqrt[3]{116 + 12\sqrt{93}}.$$

Furthermore, the Binet formula is

$$N_n = a_1\beta^n + a_2\alpha^n + a_3\gamma^n \quad \text{for all } n \geq 0. \tag{2. 4}$$

The initial values $N_0 = 0$, $N_1 = 1$ and $N_2 = 1$ imply that $a_1 = \frac{\beta}{(\beta-\alpha)(\beta-\gamma)}$, $a_2 = \frac{\alpha}{(\alpha-\gamma)(\alpha-\beta)}$ and $a_3 = \frac{\gamma}{(\gamma-\beta)(\gamma-\alpha)}$. The Binet formula for equation (2. 4) can instead be expressed as:

$$N_n = c_\beta\beta^{n+2} + c_\alpha\alpha^{n+2} + c_\gamma\gamma^{n+2}, \tag{2. 5}$$

where

$$c_t = \tfrac{1}{t^3+2} \quad , t \in \{\beta, \alpha, \gamma\}.$$

It's evident that

$$\begin{aligned} 1.45 <&\beta < 1.5 \\ 0.82 <&|\gamma| = |\alpha| < 0.83 \\ 5 <&c_\beta^{-1} < 5.15 \\ |c_\alpha| \simeq& 0.4075 \\ |\xi(n)| <&\frac{1}{2} \quad \text{where } \xi(n) = c_\alpha\alpha^{n+2} + c_\gamma\gamma^{n+2}. \end{aligned} \tag{2. 6}$$

By induction over n, the following is easily proven

$$\beta^{n-2} \leq N_n \leq \beta^{n-1} \quad \text{for all} \quad n \geq 0. \tag{2. 7}$$

We have

$$2^{\ell-1} \leq b^{\ell-1} \leq a\tfrac{b^\ell-1}{b-1} = N_nN_m \leq \beta^{n+m-2} \leq \beta^{2n-2} \leq (1.5)^{2n-2}$$

$$\ell \leq (2n-2)\tfrac{\log 1.5}{\log 2} + 1 < 2n - 1,$$

and

$$(1.45)^{n-2} < \beta^{n-2} < N_n < N_n N_m = a\tfrac{b^{\ell}-1}{b-1} < b^{\ell} < (50)^{\ell}$$
$$n < \ell \frac{\log 50}{\log 1.45} + 2 < 11\,\ell + 2.$$

Similarly, we have

$$2^{\ell_1 - 1} < b^{\ell_1 - 1} < \frac{b^{\ell_1}-1}{b-1} < a_1 a_2 \frac{(b^{\ell_1}-1)(b^{\ell_2}-1)}{(b-1)^2} = N_k < \beta^{k-1}$$
$$\ell_1 < (k-1)\frac{\log \beta}{\log 2} + 1 < k,$$

and

$$\beta^{k-2} < N_k = a_1 a_2 \frac{(b^{\ell_1}-1)(b^{\ell_2}-1)}{(b-1)^2} < (b^{\ell_2}-1)^2 < b^{2\ell_2} < 50^{2\ell_2}$$
$$\begin{aligned} k &< 2\ell_2 \frac{\log 50}{\log \beta} + 2 \\ &< 22\ell_2 + 2. \end{aligned} \tag{2. 8}$$

2.2. **Logarithmic linear forms of real algebraic numbers.** Let $f(x) \in \mathbb{Z}[x]$ be a polynomial of minimal degree d which has ψ as a root

$$f(X) = a_0 \prod_{i=1}^{d} (X - \psi^{(i)}),$$

where $\psi^{(i)}$'s are the conjugates of ψ. The logarithmic height of ψ is defined by

$$h(\psi) = \frac{1}{d}(\log a_0 + \sum_{i=1}^{d} \log \max\{|\psi^{(i)}|, 1\}).$$

The properties listed below are valid for the algebraic numbers ψ and γ.

$$\begin{aligned} h(\psi \pm \gamma) &\leq h(\psi) + h(\gamma) + \log(2) \\ h(\psi\gamma^{\pm 1}) &\leq h(\psi) + h(\gamma) \\ h(\psi^m) &= |m| h(\psi) \quad (m \in \mathbb{Z}). \end{aligned} \tag{2. 9}$$

To prove our main theorem, we employ lower bounds of the Baker type for nonzero linear forms in the logarithms of real algebraic numbers, according to Matveev's theorem [11].

Theorem 2.3. *(Matveev) Let L be a finite extension field of degree D over $\mathbb{Q}$, $\psi_1, \ldots, \psi_t$ be a positive real algebraic numbers in L, and $r_1, \ldots, r_t$ integers. Put*

$$\Lambda = \psi_1^{r_1} \times \cdots \times \psi_t^{r_t} - 1$$

and $B \geq \max\{|r_1|, \ldots, |r_t|\}$. Let $A_j \geq \max\{D\, h(\psi_j), |\log \psi_j|, 0.16\}$ be real numbers. If $\Lambda \neq 0$, then

$$\log |\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2 (1 + \log D)(1 + \log B) A_1 \times \cdots \times A_t.$$

We obtain some high upper bounds for the problem-related variables from our computations. To reduce those higher boundaries, we need some reduction techniques. We employ a few results from the theory of continued fractions. The following Lemma due to Dujella-Pethő ([6],Lemma 5a).

We define $\|Y\| = \min\{|Y - n| : n \in \mathbb{Z}\}$ as the distance between a real number Y and the nearest integer.

Lemma 2.4. *(Dujella- Pethő) Let M be a positive integer such that $q > 6M$, where $\frac{p}{q}$ is a continued fraction expansion of one of the convergent of the irrational number τ, let A, B, and $\mu \in \mathbb{R}$ with $A > 0,\ B > 1$ and $\epsilon = \|\mu q\| - M\|\tau q\|$. The inequality*

$$0 < |u\tau - v + \mu| < AB^{-w}$$

has no solution if $\epsilon > 0$, in positive integers u, v and w with

$$u \leq M \quad and \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

This lemma analytically demonstrates an upper bound for x with respect to T. Guzmán and Luca proved the following result ([17], Lemma 7).

Lemma 2.5. *If $m \geq 10$, $T > (4m^2)^m$ and $T > \frac{x}{\log^m x}$, then $x < 2^m T \log^m T$.*

3. Verification of Theorem 1.1

3.1. **Bounding on m.** From equation (1. 2), we obtain

$$c_\beta^2 \beta^{n+m+4} - \frac{ab^\ell}{b-1} = -\xi(m)c_\beta\beta^{n+2} - \xi(n)c_\beta\beta^{m+2} - \xi(n)\xi(m) - \frac{a}{b-1}.$$

Using inequalities (2. 6) and dividing both sides by $c_\beta^2\beta^{n+m+4}$, one gets

$$\begin{aligned}
\left|c_\beta^2 \beta^{n+m+4} - \frac{ab^\ell}{b-1}\right| &< \frac{c_\beta\beta^{n+2}}{2} + \frac{c_\beta\beta^{m+2}}{2} + \frac{5}{4} \\
\left|1 - \frac{ab^\ell}{c_\beta^2\beta^{n+m+4}(b-1)}\right| &< \frac{1}{2c_\beta\beta^{m+2}} + \frac{1}{2c_\beta\beta^{n+2}} + \frac{5}{4c_\beta^2\beta^{n+m+4}} \\
&< \frac{1}{c_\beta\beta^{m+2}} + \frac{5}{4c_\beta^2\beta^{m+2}} \\
&< \left(5.15 \times \frac{1}{1.45^2} + \frac{5 \times 5.15^2}{4 \times 1.45^2}\right)\frac{1}{\beta^m} \\
&< \frac{19}{\beta^m}.
\end{aligned}$$

Put

$$\Lambda_1 := \frac{ab^\ell}{c_\beta^2\beta^{n+m+4}(b-1)} - 1.$$

We have

$$\log|\Lambda_1| < \log 19 - m\log\beta. \tag{3. 10}$$

Now, we apply Theorem (2.3), where

$$\psi_1 := \beta, \qquad \psi_2 := b, \qquad \psi_3 := \tfrac{a}{c_\beta^2(b-1)}$$
$$r_1 := -(n+m+4), \qquad r_2 := \ell, \qquad r_3 := 1.$$

First, we show that $\Lambda_1 \neq 0$. If $\Lambda_1 = 0$, then $\frac{ab^\ell}{b-1} = c_\beta^2 \beta^{n+m+4}$. Consider the isomorphism $\sigma : \mathbb{Q}(\beta) \to \mathbb{Q}(\alpha)$, defined by $\sigma(\beta) = \alpha$. Then $|c_\alpha^2 \alpha^{n+m+4}| < |c_\alpha^2| < 1$, while the left-hand side is greater than 4 which is a contradiction. Using properties (2. 9), we obtain:

$$\begin{aligned}
h(\psi_1) &= \frac{\log \beta}{3};\\
h(\psi_2) &= \log b;\\
h(\psi_3) &< h\left(\frac{a}{b-1}\right) + h(c_\beta^2)\\
&< \log(b-1) + \frac{2\log 31}{3}\\
&< \log b + 3.4 \log b\\
&\leq 4.4 \log b,
\end{aligned}$$

since the minimal polynomial of c_β is given by $31x^3 - 31x^2 + 10x - 1$. We take $B := 2n+4$, $L := \mathbb{Q}(\beta)$, thus $D := [L : Q] = 3$. Also, we take

$$A_1 : = \log \beta, \qquad A_2 : = 3 \log b, \qquad A_3 : = 13.2 \log b.$$

Now from theorem 2.3, we get the following

$$\log|\Lambda_1| > -1.4 \times 30^6 \times 3^{4.5} \times 3^3 \times 13.2\,(1 + \log 3)(1 + \log(2n+4))\ \log \beta \log^2 b. \quad (3.\ 11)$$

Using equation (3. 10) and (3. 11), we have the following

$$1 + \log(2n+4) < 3.1 \log n \quad \text{for all } n \geq 3,$$

$$m \log \beta < \log 19 + 1.4 \times 30^6 \times 3^{4.5} \times 3^3 \times 13.2 \times 3.1\,(1 + \log 3)\ \log \beta\ \log n \log^2 b.$$

Using Mathematica we have

$$m < 3.7 \times 10^{14} \log n \log^2 b. \quad (3.\ 12)$$

3.2. **Bounding on n.** Equation (1. 2) implies

$$N_n = \frac{a}{N_m} \frac{b^\ell - 1}{b-1}$$

$$c_\beta \beta^{n+2} - \frac{ab^\ell}{N_m(b-1)} = -\xi(n) - \frac{a}{N_m(b-1)}.$$

Using inequalities (2. 7) and (2. 6) , we get

$$\begin{aligned}
\left| c_\beta \beta^{n+2} - \frac{ab^\ell}{N_m(b-1)} \right| &< |\xi(n)| + |\frac{a}{N_m(b-1)}|\\
&< \frac{1}{2} + \frac{1}{\beta^{m-2}},
\end{aligned}$$

and dividing both sides by $c_\beta \beta^{n+2}$, we get

$$\begin{aligned} \left|1-\frac{ab^\ell}{N_m c_\beta \beta^{n+2}(b-1)}\right| &< \frac{1}{2c_\beta \beta^{n+2}} + \frac{1}{c_\beta \beta^{n+m}} \\ &< \frac{1}{2c_\beta \beta^{n}} + \frac{1}{c_\beta \beta^{n}} \\ &< \frac{8}{\beta^n}. \end{aligned} \tag{3. 13}$$

Put

$$\Lambda_2 := \frac{ab^\ell}{N_m c_\beta \beta^{n+2}(b-1)} - 1,$$

we have

$$|\Lambda_2| < \frac{8}{\beta^n}, \tag{3. 14}$$

and

$$\log|\Lambda_2| < \log 8 - n\log\beta. \tag{3. 15}$$

Now, we apply Theorem 2.3, where

$$\begin{array}{ccc} \psi_1 := \beta, & \psi_2 := b, & \psi_3 := \frac{a}{N_m c_\beta (b-1)} \\ r_1 := -(n+2), & r_2 := \ell, & r_3 := 1. \end{array}$$

The same way as before, we can prove that $\Lambda_2 \neq 0$, moreover, using properties (2. 9), we obtain

$$\begin{aligned} h(\psi_3) &< h(\frac{a}{b-1}) + h(c_\beta) + h(N_m) \\ &< \log(b-1) + \frac{\log 31}{3} + m\log\beta \\ &< \log b + 1.7\log b + m\log\beta \\ &\leq 2.7\log b + m\log\beta. \end{aligned}$$

We take $B := 2n-1$, $L := \mathbb{Q}(\beta)$ thus $D := 3$,

$$A_1 := \log\beta, \qquad A_2 := 3\log b, \qquad A_3 := 3(2.7\log b + m\log\beta).$$

Using Theorem 2.3 we get

$$\log|\Lambda_2| > C\,\log\beta\,\log b\,(1+\log 3)\,(1+\log(2n-1))\,(2.7\,\log b + m\,\log\beta). \tag{3. 16}$$

where $C = -1.4 \times 30^6 \times 3^{4.5} \times 3^4$, using equation (3. 15) and (3. 16), we have the following, and using (3. 12) in addition to using inequality $1 + \log(2n-1) < 2.4\log n$ for all $n \geq 3$, using Mathematica we have

$$\frac{n}{\log^2 n} < 9.6 \times 10^{27}\log^3 b.$$

Now we apply Lemma 2.5, since $5.2 \times 10^{27} \log^3 b > (16)^2$, we obtain

$$\begin{aligned} n &< 2^2 \cdot 9.6 \cdot 10^{27} \log^3 b \log^2(9.6 \times 10^{27} \log^3 b) \\ &< 3.85 \times 10^{28} \log^3 b (64.5 + 3 \log\log b)^2 \\ &< 3.85 \times 10^{28} \log^3 b (93.1 \log b + 3 \log b)^2 \\ &< 3.6 \times 10^{32} \log^5 b, \end{aligned} \tag{3. 17}$$

since $\log\log b < \log b$ for every $b \geq 2$ and $\frac{1}{\log 2} \simeq 1.4427$.

3.3. **Reducing the maximum bound on m.** Let

$$z_1 := \ell \log(b) - (n+m+4)\log\beta + \log(\frac{a}{c_\beta^2(b-1)}),$$

if $z_1 > 0$ then $z_1 < |e^{z_1} - 1|$, if $z_1 < 0$ and $|e^{z_1} - 1| < \frac{1}{2}$ then $|z_1| < 2|e^{z_1} - 1|$. Since $|e^{z_1} - 1| < \frac{1}{2}$ for all $m > 9$ and for all numbers less than 9, we found m positive since $n \geq m$ and $\ell \leq 2n - 1$. Thus we have,

$$|z_1| < 2|e^{z_1} - 1|.$$

By substituting into the inequality (3. 10), we have

$$\begin{aligned} \left| \ell \log b - (n+m+4)\log\beta + \log\left(\frac{a}{c_\beta^2(b-1)}\right) \right| &< \frac{38}{\beta^m}; \\ \left| \ell \frac{\log b}{\log\beta} - (n+m+4) + \frac{\log\left(\frac{a}{c_\beta^2(b-1)}\right)}{\log\beta} \right| &< \frac{103}{\beta^m}. \end{aligned} \tag{3. 18}$$

Let

$$A := 103,\ B := \beta,\ \tau := \frac{\log b}{\log\beta},\ \mu := \log\big(\frac{a}{c_\beta^2(b-1)}\big)/\log\beta \text{ and} M := 7.2 \times 10^{32} \log^5 b.$$

Using Mathematica, for all $b \in \{2, 3, \ldots, 50\}$ and $a \in \{1, 2, \ldots, b-1\}$, we calculate a convergent $\frac{p_k}{q_k}$ such that $q_k > 6M$, furthermore computing $\varepsilon(b) := \|\mu q_k\| - M\|\tau q_k\|$, we find that $\varepsilon(b)$ is positive for all b, so we can conclude from Lemma 2.4. that if there is a solution to the inequality (3. 18) then $m \leq \max\left(\frac{\log(Aq_k/\varepsilon(b))}{\log B}\right) \leq 258$, since $q_{76} :=$ 803432248489035937201196190726584681055 and $\epsilon = 0.00110029$.

3.4. **Reducing the maximum bound on n.** Let

$$z_2 := \ell \log b - (n+2)\log\beta + \log(\frac{a}{N_m c_\beta(b-1)}),$$

as before and by substituting into the inequality (3. 14), we have

$$\left| \ell \frac{\log b}{\log\beta} - (n+2) + \frac{\log\left(\frac{a}{N_m c_\beta(b-1)}\right)}{\log\beta} \right| < \frac{44}{\beta^n}. \tag{3. 19}$$

Let

$$A := 44,\ B := \beta,\ \tau := \frac{\log b}{\log \beta},\ \mu := \log\left(\frac{a}{N_m c_\beta (b-1)}\right) / \log \beta \text{and} M := 7.2{\times}10^{32} \log^5 b,$$

for all $b \in \{2, 3, \ldots, 50\}$, $a \in \{1, 2, \ldots, b-1\}$ and $m \in \{3, \ldots, 258\}$, using Mathematica, we find that $\varepsilon(b) > 0$, so using Lemma 2.4, we can say that if the inequality (3. 19) has a solution then

$$n \leq \max(\frac{\log(Aq_k/\varepsilon(b))}{\log B}) \leq 280,$$

since $q_{83} := 707966339852442030238159382452950843308$2 and $\epsilon := 7.54491 \times 10^{-6}$.

We conclude all solutions (b, a, n, m, ℓ) of Equation (1. 2), where $3 \leq m \leq n, 2 \leq b \leq 50, 0 < a < b$ and $l \geq 2$, reduce to the rang $3 \leq n \leq 280$, with the help of Mathematica, we compute all solution in specified range, and conclude Theorem 1.1.

4. Verification of Theorem 1.2

4.1. **Bounding on ℓ_1.** From equation (1. 3), we obtain

$$c_\beta \beta^{k+2} - \frac{a_1 a_2 b^{\ell_1+\ell_2}}{(b-1)^2} = -\xi(k) - \frac{a_1 a_2 b^{\ell_1}}{(b-1)^2} - \frac{a_1 a_2 b^{l_2}}{(b-1)^2} + \frac{a_1 a_2}{(b-1)^2}.$$

Hence, using inequalities (2. 6), (2. 7)

$$\begin{aligned} \left| c_\beta \beta^{k+2} - \frac{a_1 a_2 b^{\ell_1+\ell_2}}{(b-1)^2} \right| &< \frac{1}{2} + b^{\ell_1} + b^{\ell_2} + 1 \\ &< \frac{3}{2} + 2b^{\ell_2}, \end{aligned}$$

and dividing both sides by $\frac{a_1 a_2 b^{\ell_1+\ell_2}}{(b-1)^2}$, we get

$$\begin{aligned} \left| \frac{c_\beta \beta^{k+2}(b-1)^2}{a_1 a_2 b^{\ell_1+\ell_2}} - 1 \right| &< \frac{3(b-1)^2}{2a_1 a_2 b^{\ell_1+\ell_2}} + \frac{2(b-1)^2}{a_1 a_2 b^{\ell_1}} \\ &< \frac{3(b-1)^2}{b^{\ell_1}} + \frac{2(b-1)^2}{b^{\ell_1}} \\ &< \frac{3b^2}{b^{\ell_1}} + \frac{2b^2}{b^{\ell_1}} \\ &\leq \frac{5}{b^{\ell_1-2}}. \end{aligned}$$

Put

$$\Lambda_3 := \frac{c_\beta \beta^{k+2}(b-1)^2}{a_1 a_2 b^{\ell_1+\ell_2}} - 1,$$

we have

$$|\Lambda_3| \leq \frac{5}{b^{\ell_1-2}} \tag{4. 20}$$

and

$$\log|\Lambda_3| \leq \log 5 - (\ell_1 - 2)\log b. \tag{4. 21}$$

Now, we apply Theorem 2.3, where

$$\psi_1 := \beta \qquad \psi_2 := b \qquad \psi_3 := \tfrac{c_\beta (b-1)^2}{a_1 a_2}$$
$$r_1 := k+2 \qquad r_2 := -(\ell_1+\ell_2) \qquad r_3 := 1.$$

The same way as before we can prove $\Lambda_3 \neq 0$, moreover using properties (2. 9), we obtain:

$$\begin{aligned} h(\psi_3) &< h(c_\beta) + h\left(\frac{b-1}{a_1}\right) + h\left(\frac{b-1}{a_2}\right) \\ &< \frac{\log 31}{3} + 2\log(b-1) \\ &< 3.7 \log b. \end{aligned}$$

We can take $L := \mathbb{Q}(\beta)$ thus $D := 3$, $B := 22\ell_2 + 4$ since $k < 22\ell_2 + 2$

$$A_1 : = \log \beta, \qquad A_2 : = 3\log b, \qquad A_3 : = 11.1 \log b,$$

and then from Theorem 2.3 we get

$$\log \Lambda_3 > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^3 \cdot 11.7 \cdot (1+\log 3)(1+\log(22\ell_2+4)) \log\beta \log^2 b. \quad (4.\ 22)$$

Using equation (4. 21) and (4. 22) , we have the following,

$$(1+\log(22\ell_2+4)) < 7.1 \log \ell_2 \text{ for all } \quad \ell_2 \geq 2,$$

using Mathematica, we have gives us

$$\ell_1 < 2.7 \times 10^{14} \log \ell_2 \log b. \quad (4.\ 23)$$

4.2. **Bounding on ℓ_2.** Let

$$\frac{N_k}{\frac{a_1(b^{\ell_1}-1)}{b-1}} = \frac{a_2(b^{\ell_2}-1)}{b-1}$$
$$\frac{c_\beta \beta^{k+2}(b-1)}{a_1(b^{\ell_1}-1)} - \frac{a_2 b^{\ell_2}}{b-1} = \frac{-\xi(k)(b-1)}{a_1(b^{\ell_1}-1)} - \frac{a_2}{b-1}.$$

Hence,

$$\left|\frac{c_\beta \beta^{k+2}(b-1)}{a_1(b^{\ell_1}-1)} - \frac{a_2 b^{\ell_2}}{b-1}\right| < \frac{(b-1)}{2a_1(b^{\ell_1}-1)} + 1,$$

and dividing both sides by $\frac{a_2 b^{\ell_2}}{b-1}$, we get

$$\begin{aligned} \left|\frac{\beta^{k+2} c_\beta b^{-\ell_2}(b-1)^2}{a_1 a_2 (b^{\ell_1}-1)} - 1\right| &< \frac{(b-1)^2}{a_1 a_2 b^{\ell_2}(b^{\ell_1}-1)} + \frac{b-1}{a_2 b^{\ell_2}} \\ &< \frac{(b-1)^2}{b^{\ell_2}} + \frac{b-1}{b^{\ell_2}} \\ &< \frac{b^2}{b^{\ell_2}} + \frac{b}{b^{\ell_2}} \end{aligned}$$

$$\left|\frac{\beta^{k+2} c_\beta b^{-\ell_2}(b-1)^2}{a_1 a_2 (b^{\ell_1}-1)} - 1)\right| \leq \frac{2}{b^{\ell_2 - 2}}. \quad (4.\ 24)$$

Put $\Lambda_4 := \frac{\beta^{k+2}c_\beta b^{-\ell_2}(b-1)^2}{a_1a_2(b^{\ell_1}-1)} - 1$, we have

$$\log|\Lambda_4| \leq \log 2 - (\ell_2 - 2)\log b. \quad (4.\ 25)$$

Now, we apply Theorem 2.3, where

$$\psi_1 := \beta, \qquad \psi_2 := b, \qquad \psi_3 := \frac{c_\beta(b-1)^2}{a_1a_2(b^{\ell_1}-1)}$$
$$r_1 := k+2, \qquad r_2 := -\ell_2, \qquad r_3 := 1.$$

The same way as before we can prove that $|\Lambda_4| \neq 0$, moreover, using properties (2. 9)

$$\begin{aligned} h(\psi_3) &< h(c_\beta) + h\left(\frac{b-1}{a_1}\right) + h\left(\frac{b-1}{a_2}\right) + h(b^{\ell_1} - 1) \\ &< \frac{\log 31}{3} + 2\log(b-1) + \ell_1 \log b \\ &< 3.7\log b + \ell_1 \log b, \end{aligned}$$

thus, choosing

$$A_1 : = \log\beta, \quad A_2 : = 3\log b, \quad A_3 : = 3(3.7\log b + \ell_1 \log b), \quad B : = 22l_2 + 4,$$

$$\log\Lambda_4 > -1.4\cdot 30^6 \cdot 3^{4.5} \cdot 3^4 \log\beta \log b(1+\log 3)(1+\log(22\ell_2+4))(3.7\log b+\ell_1 \log b), \quad (4.\ 26)$$

from (4. 23), (4. 25) and (4. 26) we deduce

$$\ell_2 < 1.9 \times 10^{28} \log^2 b \log^2 \ell_2.$$

Now we apply Lemma 2.5, since $2 \times 10^{28} \log^2 \ell_2 \log b > (16)^2$, we obtain

$$\begin{aligned} \frac{\ell_2}{\log^2 \ell_2} &< 1.9 \times 10^{28} \log^2 b \\ \ell_2 &< 2^2 \cdot 1.9 \cdot 10^{28} \log^2 b \log^2(1.9 \times 10^{28} \log^2 b) \\ &< 7.7 \times 10^{28} \log^2 b(65.2 + 2\log\log b)^2 \\ &< 7.1 \times 10^{32} \log^4 b. \end{aligned}$$

From (2. 8), we find that $k < 1.562 \times 10^{34} \log^4 b$.

4.3. **Reducing the maximum bound on ℓ_1.** Let

$$z_3 := (k+2)\log\beta - (\ell_1 + \ell_2)\log b + \log\frac{(b-1)^2 c_\beta}{a_1a_2},$$

if $z_3 > 0$ then $z_3 < |e^{z_3} - 1|$, if $z_3 < 0$ and $|e^{z_3} - 1| < \frac{1}{2}$, then $|z_3| < 2|e^{z_3} - 1|$ Since $|e^{z_3} - 1| < \frac{1}{2}$ for all $\ell_1 > 3$ and for all numbers less than 3, we found ℓ_1 positive since $\ell_2 \geq \ell_1$ and $k < 22\ell_2 + 2$. Thus we have,

$$|z_3| < 2|e^{z_3} - 1|.$$

By substituting into the inequality (4. 20)

$$\left|(k+2)\log\beta - (\ell_1 + \ell_2)\log b + \log\left(\frac{(b-1)^2 c_\beta}{a_1a_2}\right)\right| < \frac{10}{b^{\ell_1 - 2}},$$

and dividing both by $\log b$, we have

$$\left|(k+2)\frac{\log\beta}{\log b}-(\ell_1+\ell_2)+\frac{\log\left(\frac{(b-1)^2c_\beta}{a_1a_2}\right)}{\log b}\right|<\frac{10}{\log(b)b^{\ell_1-2}}$$

$$\left|(k+2)\frac{\log\beta}{\log b}-(\ell_1+\ell_2)+\frac{\log\left(\frac{(b-1)^2c_\beta}{a_1a_2}\right)}{\log b}\right|<\frac{14.5}{b^{\ell_1-2}}. \qquad (4.\ 27)$$

Since $\frac{1}{\log 2}=1.4427$. Let

$$A:=14.5, \qquad B:=b, \qquad \tau:=\frac{\log\beta}{\log b},$$

$$\mu:=\log\left(\frac{(b-1)^2c_\beta}{a_1a_2}\right)/\log b, \qquad M:=6.77\times10^{34}\log^4 b \qquad ,$$

for all $b\in\{2,3,\ldots,50\}$ and $a_1,a_2\in\{1,\ldots,b-1\}$, using Mathematica we find that $\varepsilon(b)>0$, so we can conclude from Lemma 2.4 that if there is a solution to the inequality (4. 27) then

$$\ell_1-2\le\max\left(\frac{\log\left(Aq_k/\varepsilon(b)\right)}{\log B}\right)\le 122,$$

since $q_{84}=107652262008477484336487139106538810$ and $\epsilon:=0.231494$. Hence $\ell_1\le 124$.

4.4. **Reducing the maximum bound on ℓ_2.** Let

$$z_4:=(k+2)\log\beta-\ell_2\log b+\log\frac{c_\beta(b-1)^2}{a_1a_2(b^{\ell_1}-1)},$$

if $z_4>0$ then $z_4<|e^{z_4}-1|$ and $|z_4|<2|e^{z_4}-1|\, if z_4<0$ and $|e^{z_4}-1|<\frac{1}{2}$. As previously, hence we have, $|z_4|<2|e^{z_4}-1|$. By substituting into the inequality (4. 24) and dividing both by $\log b$, we have

$$\left|(k+2)\frac{\log\beta}{\log b}-\ell_2+\frac{\log\left(\frac{c_\beta(b-1)^2}{a_1a_2(b^{\ell_1}-1)}\right)}{\log b}\right|<\frac{4}{\log b\, b^{\ell_2-2}}$$
$$<\frac{5.8}{b^{\ell_2-2}}.$$

Let

$$A:=5.8,\ B:=b,\ \tau:=\frac{\log\beta}{\log b},$$

$$\mu:=\log\left(\frac{c_\beta(b-1)^2}{a_1a_2(b^{\ell_1}-1)}\right)/\log b \text{ and } M:=6.77\times10^{34}\log^4 b,$$

for all $b\in\{2,3,\ldots,50\}$, $a_1,a_2\in\{1,\cdots,b-1\}$,and $\ell_1\in\{1,\ldots,124\}$, using Mathematica we find that $\varepsilon(b)>0$, so we can conclude from Lemma 2.4 that if there is a solution to the inequality (4. 27) then

$$\ell_2 - 2 \leq \max\left(\frac{\log\left(Aq_k/\varepsilon(b)\right)}{\log B}\right) \leq 130,$$

since $q_{87} := 233591243711519495005099129781165085 9$ and $\epsilon := 0.00815537$. Hence $\ell_2 \leq 132$ and $k < 2906$.

5. Conclusion

In this manuscript two problems related to Narayana numbers were solved. We found all possible product of two Narayana numbers which give a repdigits in base b from 2 to 50. Also we found Narayana numbers which can be articulated as a product of two repdigit in base b from 2 to 50. The techniques used in this work can be applied to a wide range of similar problems.

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