





























and

$$\begin{aligned}
& \int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}} \left[ \left( u - \frac{1}{\varepsilon} \right)^\lambda - \left( \frac{1}{\sigma} - u \right)^\lambda + \frac{3^\lambda - 1}{4^\lambda} \left( \frac{\varepsilon - \sigma}{\sigma\varepsilon} \right)^\lambda \right] \left( \frac{\sigma u - \sigma\varepsilon}{\sigma u - \varepsilon u} \right) |\kappa'(\sigma)| du \\
& + \int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+3\varepsilon}{4\sigma\varepsilon}} \left[ \left( \frac{1}{\sigma} - u \right)^\lambda - \left( u - \frac{1}{\varepsilon} \right)^\lambda + \frac{3^\lambda - 1}{4^\lambda} \left( \frac{\varepsilon - \sigma}{\sigma\varepsilon} \right)^\lambda \right] \left( \frac{\sigma u - \sigma\varepsilon}{\sigma u - \varepsilon u} \right) |\kappa'(\sigma)| du \\
& = \int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}} \left[ \left( u - \frac{1}{\varepsilon} \right)^\lambda - \left( \frac{1}{\sigma} - u \right)^\lambda + \frac{3^\lambda - 1}{4^\lambda} \left( \frac{\varepsilon - \sigma}{\sigma\varepsilon} \right)^\lambda \right] \left( \frac{\sigma u - \sigma\varepsilon}{\sigma u - \varepsilon u} \right) |\kappa'(\sigma)| du \\
& + \int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+3\varepsilon}{4\sigma\varepsilon}} \left[ \left( u - \frac{1}{\varepsilon} \right)^\lambda - \left( \frac{1}{\sigma} - u \right)^\lambda + \frac{3^\lambda - 1}{4^\lambda} \left( \frac{\varepsilon - \sigma}{\sigma\varepsilon} \right)^\lambda \right] \left( \frac{\sigma\varepsilon - \varepsilon u}{\sigma u - \varepsilon u} \right) |\kappa'(\sigma)| du \\
& = |\kappa'(\sigma)| \left( \frac{\varepsilon - \sigma}{\sigma\varepsilon} \right)^{\lambda+1} \left[ \frac{4 \times 2^\lambda + (3^\lambda - 1)\lambda - 2 \times (3^\lambda + 1)}{4^{\lambda+1}(\lambda + 1)} \right]. \quad (3.39)
\end{aligned}$$

Using the fact that  $|\kappa'|$  is harmonic convex on  $[\sigma, \varepsilon]$ , thus the mapping  $\varphi(u) = |\kappa'(\frac{1}{u})|$  is convex on  $[\frac{1}{\varepsilon}, \frac{1}{\sigma}]$ , and the identities (3.36)-(3.39), we get

$$\begin{aligned}
& \left| \frac{1}{2} \left( \frac{\sigma\varepsilon}{\varepsilon - \sigma} \right)^\lambda \int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}} H(u) \kappa' \left( \frac{1}{u} \right) du \right| \\
& \leq \frac{1}{2} \left( \frac{\sigma\varepsilon}{\varepsilon - \sigma} \right)^\lambda \int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}} |h(u)| \left| \kappa' \left( \left( \frac{\sigma u - \sigma\varepsilon}{\sigma u - \varepsilon u} \right) \frac{1}{\sigma} + \left( \frac{\sigma\varepsilon - \varepsilon u}{\sigma u - \varepsilon u} \right) \frac{1}{\varepsilon} \right) \right| du \\
& \leq \frac{1}{2} \left( \frac{\sigma\varepsilon}{\varepsilon - \sigma} \right)^\lambda \int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}} |h(u)| \left[ \left( \frac{\sigma u - \sigma\varepsilon}{\sigma u - \varepsilon u} \right) |\kappa'(\sigma)| + \left( \frac{\sigma\varepsilon - \varepsilon u}{\sigma u - \varepsilon u} \right) |\kappa'(\varepsilon)| \right] du \\
& = \frac{1}{2} \left( \frac{\varepsilon - \sigma}{\sigma\varepsilon} \right)^{\lambda+1} \left[ \frac{4^\lambda \times (\lambda - 3) + 3^{\lambda+1} + 1}{4^{\lambda+1}(\lambda + 1)} \right] [|\kappa'(\sigma)| + |\kappa'(\varepsilon)|] \\
& + \left( \frac{\varepsilon - \sigma}{\sigma\varepsilon} \right)^{\lambda+1} \left[ \frac{2 \times (3^\lambda + 1) - 4 \times 2^\lambda - (3^\lambda - 1)\lambda}{4^{\lambda+1}(\lambda + 1)} \right] [|\kappa'(\sigma)| + |\kappa'(\varepsilon)|]. \quad (3.40)
\end{aligned}$$

The inequality (3.40) when combined with (3.35) yields the desired result.  $\square$

**Remark 3.3.** Theorem 3.1 and Theorem 3.2 are similar extensions of Theorems 1.5 and 1.6.

**Remark 3.4.** Let  $\lambda = 1$  in Theorem 3.1 and Theorem 3.2. The following inequalities are obtained:

$$\begin{aligned}
& \left| \left( \frac{\sigma\varepsilon}{\varepsilon - \sigma} \right) \int_{\sigma}^{\varepsilon} \frac{\kappa(u)}{u^2} du - \frac{1}{4} \left[ \kappa(\sigma) + \kappa \left( \frac{4\sigma\varepsilon}{3\sigma + \varepsilon} \right) + \kappa \left( \frac{4\sigma\varepsilon}{\sigma + 3\varepsilon} \right) + \kappa(\varepsilon) \right] \right| \\
& \leq \frac{1}{16} \left( \frac{\varepsilon - \sigma}{\sigma\varepsilon} \right) [|\kappa'(\sigma)| + |\kappa'(\varepsilon)|]. \quad (3.41)
\end{aligned}$$

and

$$\begin{aligned} & \left| \left( \frac{\sigma\varepsilon}{\varepsilon - \sigma} \right) \int_{\sigma}^{\varepsilon} \frac{\kappa(u)}{u^2} du - \frac{1}{4} \left[ \kappa \left( \frac{4\sigma\varepsilon}{3\sigma + \varepsilon} \right) + \kappa \left( \frac{4\sigma\varepsilon}{\sigma + 3\varepsilon} \right) + 2\kappa \left( \frac{2\sigma\varepsilon}{\sigma + \varepsilon} \right) \right] \right| \\ & \leq \frac{1}{16} \left( \frac{\varepsilon - \sigma}{\sigma\varepsilon} \right)^{\lambda+1} \left[ |\kappa'(\sigma)| + |\kappa'(\varepsilon)| \right]. \quad (3.42) \end{aligned}$$

#### 4. APPLICATIONS OF THE RESULTS

Perhaps it would be better to return to the Beta incomplete functions

$$B(r, s) = \int_0^1 u^{r-1} (1-u)^{s-1} du$$

and

$$B_z(r, s) = \int_0^z u^{r-1} (1-u)^{s-1} du.$$

Consider the function  $\kappa(u) = u^{-p}$ ,  $u \in [1, 2]$  for  $p > 1$ . It is obvious that this function is a harmonically convex function. We observe that

$$\begin{aligned} & \frac{\Gamma(\lambda + 1)}{2} \left( \frac{\sigma\varepsilon}{\varepsilon - \sigma} \right)^{\lambda} \mathbb{J}_{\frac{1}{\sigma}-}^{\lambda} (\kappa \circ \vartheta) \left( \frac{1}{\varepsilon} \right) = \frac{\lambda}{2} \int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}} \left( \frac{1}{\sigma} - u \right)^{\lambda-1} \kappa \left( \frac{1}{u} \right) du \\ & = \frac{\lambda}{2} \int_{\frac{1}{2}}^1 (1-u)^{\lambda-1} u^p du = \frac{1}{2} \lambda \left[ B(p+1, \lambda) - B_{\frac{1}{2}}(p+1, \lambda) \right] \end{aligned}$$

and

$$\begin{aligned} & \frac{\Gamma(\lambda + 1)}{2} \left( \frac{\sigma\varepsilon}{\varepsilon - \sigma} \right)^{\lambda} \int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}} \left( u - \frac{1}{\varepsilon} \right)^{\lambda} \kappa \left( \frac{1}{u} \right) du = \frac{\lambda}{2} \int_{\frac{1}{2}}^1 \left( u - \frac{1}{2} \right)^{\lambda-1} \kappa \left( \frac{1}{u} \right) du \\ & = \frac{\lambda}{2} \int_{\frac{1}{2}}^1 \left( u - \frac{1}{2} \right)^{\lambda-1} u^{-p-2} du = \frac{\lambda \left[ \Gamma(\lambda) \Gamma(-p - \lambda) - \Gamma(-p) B_{\frac{1}{2}}(-p - \lambda, \lambda) \right]}{2^{\lambda+p+1} \Gamma(-p)}. \end{aligned}$$

We can now have the following estimates regarding the incomplete beta and beta functions.

**Proposition 4.1.** *Let  $p > 2$  and  $\lambda > 0$  be in Theorem 2.1. Then the following inequality holds:*

$$\begin{aligned} & \left( \frac{3}{4} \right)^p \leq \frac{3^{\lambda} - 1}{4^{\lambda}} \left( \frac{3}{4} \right)^p + \frac{4^{\lambda} - 3^{\lambda} + 1}{2 \times 4^{\lambda}} \left[ \left( \frac{5}{8} \right)^p + \left( \frac{7}{8} \right)^p \right] \\ & \leq 2^{\lambda-1} \lambda \left[ B(p+1, \lambda) - B_{\frac{1}{2}}(p+1, \lambda) \right] + \frac{\lambda}{2^{p+1}} \left[ B(-p - \lambda, \lambda) - B_{\frac{1}{2}}(-p - \lambda, \lambda) \right] \\ & \leq \frac{3^{\lambda} - 1}{2 \times 4^{\lambda}} \left[ \left( \frac{5}{8} \right)^p + \left( \frac{7}{8} \right)^p \right] + \frac{4^{\lambda} - 3^{\lambda} + 1}{2 \times 4^{\lambda}} \left[ \frac{2^p + 1}{2^p} \right] \leq \frac{2^p + 1}{2^{p+1}} \quad (4.43) \end{aligned}$$

*Proof.* The proof is obvious from Theorem 2.1 for the function  $\kappa(u) = u^{-p}$ ,  $u \in [1, 2]$  for  $p > 1$ . The graph below also support the validity of the inequality (4.43). □

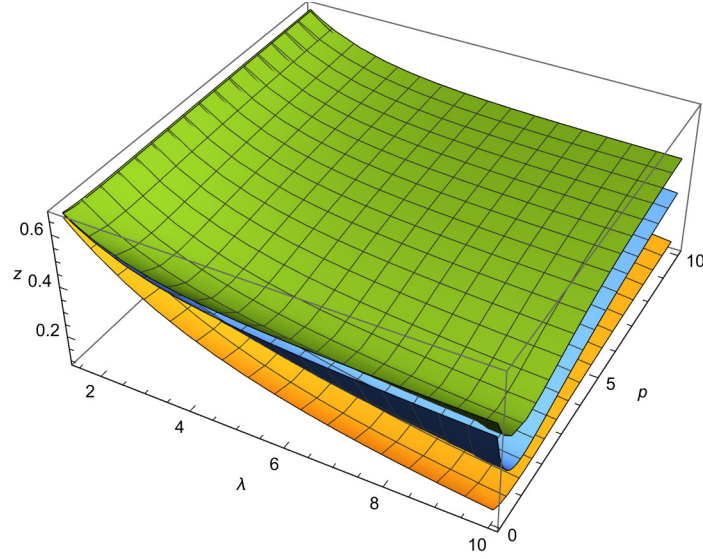


FIGURE 1. The graph of inequality 4.43 for  $p > 2$  and  $\lambda > 0$

**Proposition 4.2.** Let  $p > 2$  and  $\lambda > 0$  be in Theorem 3.1. Then the following inequality holds:

$$\begin{aligned}
 0 &\leq 2^{\lambda-1}\lambda \left[ B(p+1, \lambda) - B_{\frac{1}{2}}(p+1, \lambda) \right] + \frac{\lambda}{2^{p+1}} \left[ B(-p-\lambda, \lambda) - B_{\frac{1}{2}}(-p-\lambda, \lambda) \right] \\
 &\quad - \left[ \frac{3^\lambda - 1}{2 \times 4^\lambda} \left[ \left( \frac{5}{8} \right)^p + \left( \frac{7}{8} \right)^p \right] + \frac{4^\lambda - 3^\lambda + 1}{2 \times 4^\lambda} [1 + 2^{-p}] \right] \\
 &\quad + \frac{p [2^{-p-1} + 1]}{4} \left[ \frac{4 \times (4^\lambda + 2^\lambda) + (\lambda + 1) (1 - 3^\lambda)}{4^{\lambda+1} (\lambda + 1)} \right] \\
 &\leq \frac{p [2^{p-1} + 1]}{2} \left[ \frac{(\lambda + 1) (3^\lambda - 1) - 4 \times (4^\lambda + 2^\lambda)}{4^{\lambda+1} (\lambda + 1)} \right]. \quad (4.44)
 \end{aligned}$$

*Proof.* The proof follows directly from Theorem 3.1 by using the harmonic convex function  $\kappa(u) = u^{-p}$ ,  $u \in [1, 2]$  for  $p > 2$ . The figure below confirms the validity of the above inequality.  $\square$



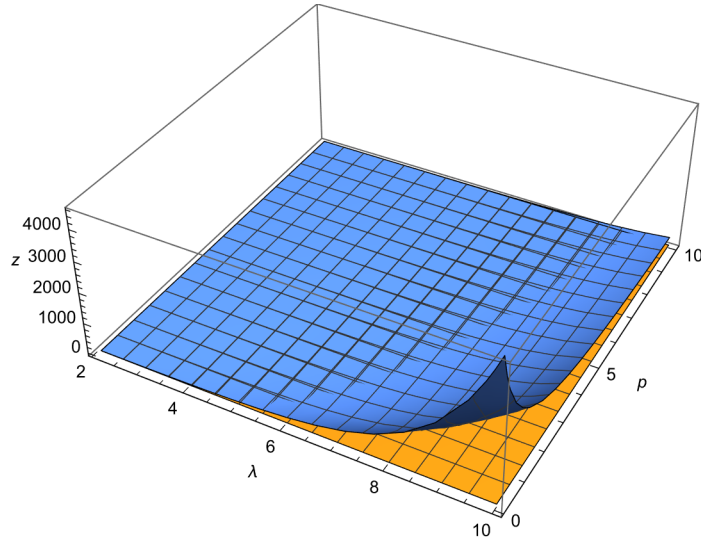


FIGURE 2. The graph of inequality ( 4. 44 ) for  $p > 2$  and  $\lambda > 0$

**Proposition 4.3.** *Let  $p > 2$  and  $\lambda > 0$  be in Theorem 3.2. Then the following inequality holds:*

$$\begin{aligned}
 0 &\leq 2^{\lambda-1} \lambda \left[ B(p+1, \lambda) - B_{\frac{1}{2}}(p+1, \lambda) \right] + \frac{\lambda}{2^{p+1}} \left[ B(-p-\lambda, \lambda) - B_{\frac{1}{2}}(-p-\lambda, \lambda) \right] \\
 &\quad - \left[ \frac{4^\lambda - 3^\lambda + 1}{2 \times 4^\lambda} \left[ \left(\frac{5}{8}\right)^p + \left(\frac{7}{8}\right)^p \right] + \frac{3^\lambda - 1}{4^\lambda} \left(\frac{3}{4}\right)^p \right] \\
 &\quad + \frac{p [2^{-p-1} + 1]}{4} \left[ \frac{3^\lambda - \lambda - 3 \times 4^\lambda + 2^{2\lambda} \lambda + 2^{\lambda+2} + 3^\lambda \lambda - 1}{4^{\lambda+1} (\lambda + 1)} \right] \\
 &\leq \frac{p [2^{-p-1} + 1]}{2} \left[ \frac{3^\lambda - \lambda - 3 \times 4^\lambda + 2^{2\lambda} \lambda + 2^{\lambda+2} + 3^\lambda \lambda - 1}{4^{\lambda+1} (\lambda + 1)} \right]. \quad (4. 45)
 \end{aligned}$$

*Proof.* The proof follows directly from Theorem 3.2 by using the harmonic convex function  $\kappa(u) = u^{-p}$ ,  $u \in [1, 2]$  for  $p > 2$ . The figure below confirms the validity of the inequality 4. 45 . □

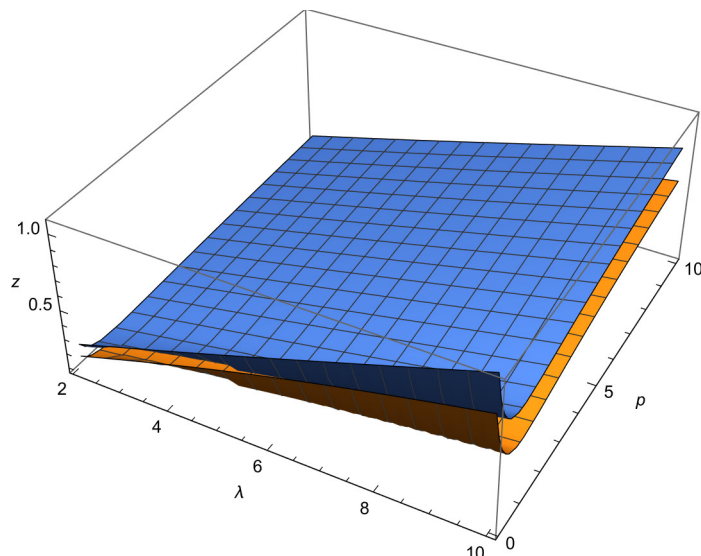


FIGURE 3. The graph of inequality 4.45 for  $p > 2$  and  $\lambda > 0$

## 5. ACKNOWLEDGMENTS

I would like to thank to the referees whose suggestion/remarks have greatly improved the manuscript before publication.

## REFERENCES

- [1] R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Non-linear Sci. Numer. Simul.*, 44 (2017), 460-481.
- [2] T. Abdeljawad, P. O. Mohammed, A. Kashuri, New modified conformable fractional integral inequalities of Hermite–Hadamard type with applications, *J. Funct. Space*, 2020 (2020) 4352357.
- [3] D. Baleanu, P.O. Mohammed, M. Vivas-Cortez, Y. Rangel-Oliveros, Some modifications in conformable fractional integral inequalities, *Adv. Differ. Equ.*, 2020, 2020, 374.
- [4] C. Bardaro, P. L. Butzer, I. Mantellini, The foundations of fractional calculus in the Mellin transform setting with applications, *J. Fourier Anal. Appl.*, 21 (2015), 961-1017.
- [5] D. Baleanu, O. O. Mohammed, S. Zeng, Inequalities of trapezoidal type involving generalized fractional integrals, *Alex. Eng. J.*, 59 (5) (2020), 2975-2984.
- [6] F. Chen and S. Wu, Fej'er and Hermite–Hadamard type inequalities for harmonically convex functions, *Journal of applied Mathematics*, volume 2014, article ID:386806.
- [7] S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite–Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University: Footscray, Australia, 2000.
- [8] S. S. Dragomir, R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.*, 11 (1998), 91-95.
- [9] M. R. Delavar, M. Aslani, M. De La Sen, Hermite–Hadamard-Fejér inequality related to generalized convex functions via fractional integrals, *Journal of Mathematics*, 2018 (2018) 5864091.
- [10] L. Fejér, Über die Fourierreihen, II, *Math. Naturwiss. Anz Ung. Akad. Wiss.*, 24 (1906), 369-390.
- [11] A. Fernandez, P. O. Mohammed, Hermite–Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels, *Math. Methods Appl. Sci.*, 2020, 1-18.
- [12] B. Gavrea, I. Gavrea, On some Ostrowski type inequalities, *Gen. Math.*, 18 (2010), 33-44.

- [13] H. Gunawan, Eridani, Fractional integrals and generalized Olsen inequalities, *Kyungpook Math. J.*, 49 (2009), 31-39.
- [14] J. Hadamard, Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann. *J. Math. Pures Appl.*, 1893, 58, 171-215.
- [15] J. Han, P. O. Mohammed, H. Zeng, Generalized fractional integral inequalities of Hermite–Hadamard-type for a convex function, *Open Math.*, 18 (2020), 794-806.
- [16] S. -R. Hwang, K. -L. Tseng, K. -C. Hsu, New inequalities for fractional integrals and their applications, *Turkish Journal of Mathematics*, 40 (2016), 471-486.
- [17] İ. İşcan, Hermite–Hadamard–Fejér type inequalities for convex functions via fractional integrals, *Stud. Univ. Babeş Bolyai Math.*, 60 (2015), 355-366.
- [18] İ. İşcan, S. Wu, Hermite–Hadamard type inequalities for harmonically convex functions via fractional integrals, *Appl. Math. Comput.*, 238 (2014), 237-244.
- [19] İ. İşcan, M. Kunt and N. Yazici, Hermite–Hadamard–Fejér type inequalities for harmonically convex functions via fractional integrals, *New Trends in Mathematical Sciences*, 4 (3) (2016), 239-253
- [20] F. Jarad, T. Abdeljawad, K. Shah, On the weighted fractional operators of a function with respect to another function, *Fractals*, 28 (8) (2020), 2040011, 12 pages.
- [21] M. Kunt, İ. İşcan, N. Yazıcı and U. Gözütok, On new inequalities of Hermite–Hadamard–Fejér type for harmonically convex functions via fractional integrals, *SpringerPlus* 5, 635 (2016).
- [22] S. Kaijser, L. Nikolova, L. -E. Persson, Wedestig, A Hardy type inequalities via convexity, *Math. Inequal. Appl.*, 8 (2005), 403-417.
- [23] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies; Elsevier Sci. B.V.: Amsterdam, The Netherlands, 2006; Volume 204.
- [24] M. Kunt, I. Iscan, On new Hermite–Hadamard–Fejér type inequalities for  $p$ -convex functions via fractional integrals, *CMMA* 2 (2017), 1-15.
- [25] P. O. Mohammed, I. Brevik, A new version of the Hermite–Hadamard inequality for Riemann–Liouville fractional integrals, *Symmetry*, 12 (2020) 610.
- [26] P. O. Mohammed, M. Z. Sarikaya, On generalized fractional integral inequalities for twice differentiable convex functions, *J. Comput. Appl. Math.*, 372 (2020), 112740.
- [27] P. O. Mohammed, Hermite–Hadamard inequalities for Riemann–Liouville fractional integrals of a convex function with respect to a monotone function, *Math. Methods Appl. Sci.*, (2019), 1-11.
- [28] S. Mehmood, F. Zafar, N. Asmin, New Hermite–Hadamard–Fejér type inequalities for  $(h_1, h_2)$ -convex functions via fractional calculus, *ScienceAsia*, 46 (2020), 102-108.
- [29] F. Qi, O. P. Mohammed, J. C. Yao, Y. H. Yao, Generalized fractional integral inequalities of Hermite–Hadamard type for  $(\lambda, m)$ -convex functions, *J. Inequal. Appl.*, 2019 (2019), 135.
- [30] M.Z. Sarikaya, E. Set, H. Yaldiz, N. Bas ak, Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.*, 57 (2013), 2403-2407.
- [31] M. Z. Sarikaya, H. Yaldiz, On generalization integral inequalities for fractional integrals, *Nihonkai Math. J.*, 25 (2014), 93-104.
- [32] M. Z. Sarikaya, C. C. Bilisik, P. O. Mohammed, Some generalizations of Opial type inequalities, *Appl. Math. Inf. Sci.* 14 (2020), 809-816.
- [33] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.*, 57 (2013), 2403-2407.
- [34] D. -P. Shi, B. -Y Xi, F. Qi, Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals of  $(\lambda, m)$ -convex functions, *Fract. Differ. Calc.*, 4 (2014), 31-43.
- [35] C. -J. Zhao, W. -S. Cheung, On improvements of the Rozanova's inequality, *J. Inequal. Appl.*, 2011 (2011) 33.