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Refinements and applications of Hermite-Hadamard type inequalities for fractional integrals based on harmonic convexity

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Abstract. In this paper, several applications of the Hermite-Hadamard inequality for fractional integrals using harmonic convexity are discussed, including some new refinements and similar extensions, as well as several applications in the Beta function.

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1. Introduction

We recall that an interval $\mathbb{I} \subset \mathbb{R}$ is convex if for all $u, v \in \mathbb{I}$, we have $\varkappa u + (1-\varkappa) v \in \mathbb{I}$, where $\varkappa \in [0,1]$ and a function $\kappa : \mathbb{I} \to \mathbb{R}$ is convex if for all $u, v \in \mathbb{I}$, the inequality

$$\kappa\left(\varkappa u + (1-\varkappa) v\right) \leq \varkappa \kappa(u) + (1-\varkappa)\kappa(v) \tag{1. 1}$$

holds. A function $\kappa : \mathbb{I} \to \mathbb{R}$ is concave if the inequality (1. 1) holds in opposite direction.

Due to characteristics and definition convexity, it is significant for understanding and solving problems involving fractional integral inequalities. As demonstrated in [7, 10, 13, 14, 22, 33, 34], convex functions have resulted in the discovery of various novel integral inequalities. Hermite–Hadamard's integral inequalities are usually mentioned when looking for comprehensive inequalities:

$$\kappa\left(\frac{\sigma+\varepsilon}{2}\right) \leq \frac{1}{\varepsilon-\sigma}\int_{\sigma}^{\varepsilon} \kappa(u)\, du \leq \frac{\kappa(\sigma)+\kappa(\varepsilon)}{2}, \tag{1. 2}$$

where the function $\kappa : \mathbb{I} \to \mathbb{R}$ is convex on $\mathbb{I}$ and $\kappa \in L^1([\sigma, \varepsilon])$.

Convexity and convex functions have various generalizations. One of these generalizations is harmonic convexity, which is defined as follows.

Definition 1.1. [19] *Define* $\mathbb{I} \subseteq \mathbb{R}\backslash\{0\}$ *as an interval of real numbers. A function* κ *from* $\mathbb{I}$ *to the real numbers is considered to be harmonically convex, if*

$$\kappa\left(\frac{uv}{\varkappa u+(1-\varkappa)v}\right) \leq \varkappa\kappa(v)+(1-\varkappa)\kappa(u) \tag{1. 3}$$

for all $u, v \in \mathbb{J}$ *and* $\varkappa \in [0,1]$. *Harmonically concave* κ *is defined as the inequality in (1. 3) reversed.*

Using harmonic-convexity, the Hermite–Hadamard type yields the following result.

Theorem 1.2. [19]*Let* $\kappa : \mathbb{I} \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *be a harmonically convex function and* $\sigma, \varepsilon \in \mathbb{I}$ *with* $\sigma < \varepsilon$. *If* $\kappa \in L^1([\sigma, \varepsilon])$ *then the following inequalities hold:*

$$\kappa\left(\frac{2\sigma\varepsilon}{\sigma+\varepsilon}\right) \leq \frac{\sigma\varepsilon}{\varepsilon-\sigma}\int_{\varepsilon}^{\sigma}\frac{\kappa(u)}{u^2}du \leq \frac{\kappa(\sigma)+\kappa(\varepsilon)}{2}. \tag{1. 4}$$

The left-sided and right-sided Riemann–Liouville fractional integrals $\mathbb{J}_{\sigma+}^{\lambda}\kappa$ and $\mathbb{J}_{\varepsilon-}^{\lambda}\kappa$ of order $\lambda > 0$ in (1. 5), are defined respectively as (see [4, 23]):

$$\mathbb{J}_{\sigma+}^{\lambda}\kappa(u) := \frac{1}{\Gamma(\lambda)}\int_{\sigma}^{u}(u-\varkappa)^{\lambda-1}\kappa(\varkappa)\,d\varkappa, 0 \leq \sigma < u < \varepsilon$$

and

$$\mathbb{J}_{\varepsilon-}^{\lambda}\kappa(u) := \frac{1}{\Gamma(\lambda)}\int_{u}^{\varepsilon}(\varkappa-u)^{\lambda-1}\kappa(\varkappa)\,d\varkappa,\ 0 \leq \sigma < u < \varepsilon,$$

where $\Gamma(\lambda)$ is the Gamma function defined by

$$\Gamma(\lambda) = \int_{0}^{\infty} e^{-\varkappa}\varkappa^{\lambda-1}d\varkappa \text{ and } \mathbb{J}_{\sigma+}^{0}\kappa(u) = \mathbb{J}_{\varepsilon-}^{0}\kappa(u) = \kappa(u).$$

The following fractional integral forms can be used to express Hermite–Hadamard type inequalities (1. 4) for harmonically convex functions:

İşcan [18], established the following trapezoidal type inequalities of Hermite–Hadamard type using fractional calculus and harmonic convex functions.

Theorem 1.3. [18] *Let* $\kappa : \mathbb{I} \subseteq (0,\infty) \to \mathbb{R}$ *be a function such that* $\kappa \in L^1([\sigma, \varepsilon])$, *where* $\sigma, \varepsilon \in \mathbb{I}$ *with* $\sigma < \varepsilon$. *If* κ *is a harmonically convex function on* $[\sigma, \varepsilon]$, *then the following inequalities for fractional integrals hold:*

$$\kappa\left(\frac{2\sigma\varepsilon}{\sigma+\varepsilon}\right) \leq \frac{\Gamma(\lambda+1)}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\left[\mathbb{J}_{\frac{1}{\sigma}-}^{\lambda}(\kappa\circ\vartheta)\left(\frac{1}{\varepsilon}\right) + \mathbb{J}_{\frac{1}{\varepsilon}+}^{\lambda}(\kappa\circ\vartheta)\left(\frac{1}{\sigma}\right)\right] \leq \frac{\kappa(\sigma)+\kappa(\varepsilon)}{2} \tag{1. 5}$$

with $\lambda > 0$ *and* $\vartheta(u) = \frac{1}{u}$, $u \in \left[\frac{1}{\varepsilon}, \frac{1}{\sigma}\right]$.

Kunt [21], obtained the following midpoint type inequalities of Hermite–Hadamard type using fractional calculus and the harmonic convexity.

Theorem 1.4. [21] *Let $\kappa : \mathbb{I} \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $\kappa \in L^1([\sigma,\varepsilon])$, where $\sigma,\varepsilon \in \mathbb{I}$ with $\sigma < \varepsilon$. If κ is a harmonically convex function on $[\sigma,\varepsilon]$, then the following inequalities for fractional integrals holds:*

$$\kappa\left(\frac{2\sigma\varepsilon}{\sigma+\varepsilon}\right) \leq \frac{\Gamma(\lambda+1)}{2^{1-\lambda}}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\left[\mathbb{J}^{\lambda}_{\frac{\sigma+b}{\sigma\varepsilon}+}(\kappa\circ\vartheta)\left(\frac{1}{\sigma}\right)\right.$$
$$\left.+\mathbb{J}^{\lambda}_{\frac{\sigma+\varepsilon}{\sigma\varepsilon}-}(\kappa\circ\vartheta)\left(\frac{1}{\varepsilon}\right)\right] \leq \frac{\kappa(\sigma)+\kappa(\varepsilon)}{2} \quad (1.\ 6)$$

with $\lambda > 0$ and $\vartheta(u) = \frac{1}{u}$, $u \in \left[\frac{1}{\varepsilon},\frac{1}{\sigma}\right]$.

İşcan [18], obtained the estimate between the middle term and the rightmost term in (1. 5).

Theorem 1.5. [18] *Let $\kappa : \mathbb{I} \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on $\mathbb{I}^\circ$ (the interior $\mathbb{I}$) such that $\kappa' \in L^1([\sigma,\varepsilon])$, where $\sigma,\varepsilon \in \mathbb{I}^\circ$ with $\sigma < \varepsilon$. If $\left|\kappa'\right|^q$ is harmonically convex on $[\sigma,\varepsilon]$ for some fixed $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\left|\frac{\kappa(\sigma)+\kappa(\varepsilon)}{2} - \frac{\Gamma(\lambda+1)\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}}{2}\left[\mathbb{J}^{\lambda}_{\frac{1}{\sigma}-}(\kappa\circ\vartheta)\left(\frac{1}{\varepsilon}\right)\right.\right.$$
$$\left.\left.+\mathbb{J}^{\lambda}_{\frac{1}{\varepsilon}+}(\kappa\circ\vartheta)\left(\frac{1}{\sigma}\right)\right]\right| \leq \frac{\sigma\varepsilon(\varepsilon-\sigma)}{2}\left(C_1(\lambda;\sigma,\varepsilon)\right)^{1-\frac{1}{q}}$$
$$\times\left[C_3(\lambda;\sigma,\varepsilon)\left|\kappa'(\sigma)\right|^q + C_2(\lambda;\sigma,\varepsilon)\left|\kappa'(\varepsilon)\right|^q\right]^{\frac{1}{q}}, \quad (1.\ 7)$$

where

$$C_1(\lambda;\sigma,\varepsilon) = \frac{\varepsilon^{-2}}{1+\lambda}\left[{}_2F_1\left(2,1;\lambda+2;1-\frac{\sigma}{\varepsilon}\right) +{}_2 F_1\left(2,\lambda+1;\lambda+2;1-\frac{\sigma}{\varepsilon}\right)\right],$$

$$C_2(\lambda;\sigma,\varepsilon) = \frac{\varepsilon^{-2}}{2+\lambda}\left[\frac{1}{1+\lambda}\,{}_2F_1\left(2,2;\lambda+3;1-\frac{\sigma}{\varepsilon}\right)\right.$$
$$\left.+{}_2F_1\left(2,\lambda+2;\lambda+3;1-\frac{\sigma}{\varepsilon}\right)\right],$$

and

$$C_2(\lambda;\sigma,\varepsilon) = \frac{\varepsilon^{-2}}{1+\lambda}\left[\frac{1}{1+\lambda}\,{}_2F_1\left(2,1;\lambda+3;1-\frac{\sigma}{\varepsilon}\right)\right.$$
$$\left.+\frac{1}{1+\lambda}\,{}_2F_1\left(2,\lambda+1;\lambda+3;1-\frac{\sigma}{\varepsilon}\right)\right].$$

Kunt [21], proved the estimate between the middle term and the leftmost term in (1. 5).

Theorem 1.6. [21] *Under the assumptions of Theorem 1.5, the following inequality holds:*

$$\left|\kappa\left(\frac{2\sigma\varepsilon}{\sigma+\varepsilon}\right)-\frac{\Gamma(\lambda+1)\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}}{2^{1-\lambda}}\left[\mathbb{J}^{\lambda}_{\frac{\sigma+\varepsilon}{\sigma\varepsilon}-}(\kappa\circ\vartheta)\left(\frac{1}{\varepsilon}\right)+\mathbb{J}^{\lambda}_{\frac{\sigma+\varepsilon}{\sigma\varepsilon}+}(\kappa\circ\vartheta)\left(\frac{1}{\sigma}\right)\right]\right|$$
$$\leq\frac{\sigma\varepsilon(\varepsilon-\sigma)}{2^{1-\lambda}}\left\{(C_4(\lambda,\sigma,\varepsilon))^{1-\frac{1}{q}}\left[C_5(\lambda,\sigma,\varepsilon)\left|\kappa'(\sigma)\right|^q+C_6(\lambda,\sigma,\varepsilon)\left|\kappa'(\varepsilon)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+C_7(\lambda;\sigma,\varepsilon)\left[C_8(\lambda,\sigma,\varepsilon)\left|\kappa'(\sigma)\right|^q+C_9(\lambda,\sigma,\varepsilon)\left|\kappa'(\varepsilon)\right|^q\right]\right\}^{\frac{1}{q}}, \quad (1.\ 8)$$

where

$$C_4(\lambda;\sigma,\varepsilon)=\frac{(\sigma+\varepsilon)^{-2}}{2^{1-\lambda}(1+\lambda)}\,{}_2F_1\left(2,1;\lambda+2;\frac{\varepsilon-\sigma}{\varepsilon+\varepsilon}\right),$$

$$C_5(\lambda;\sigma,\varepsilon)=\frac{\varepsilon^{-2}}{2^{\lambda}(2+\lambda)}\,{}_2F_1\left(2,1;\lambda+3;\frac{\varepsilon-\sigma}{\varepsilon+\varepsilon}\right),$$

$$C_6(\lambda;\sigma,\varepsilon)=C_4(\lambda;\sigma,\varepsilon)-C_5(\lambda;\sigma,\varepsilon)$$

$$C_7(\lambda;\sigma,\varepsilon)=\frac{\varepsilon^{-2}}{2^{1-\lambda}(1+\lambda)}\,{}_2F_1\left(2,\lambda+1;\lambda+2;\frac{1}{2}\left(1-\frac{\sigma}{\varepsilon}\right)\right),$$

$$C_8(\lambda;\sigma,\varepsilon)=\frac{\varepsilon^{-2}}{2^{\lambda+1}(1+\lambda)}\,{}_2F_1\left(2,\lambda+1;\lambda+2;\frac{1}{2}\left(1-\frac{\sigma}{\varepsilon}\right)\right)$$
$$-\frac{1}{2^{\lambda+2}(2+\lambda)}\,{}_2F_1\left(2,\lambda+2;\lambda+3;\frac{1}{2}\left(1-\frac{\sigma}{\varepsilon}\right)\right)$$

and

$$C_9(\lambda;\sigma,\varepsilon)=C_7(\lambda;\sigma,\varepsilon)-C_8(\lambda;\sigma,\varepsilon)$$

with $\lambda>1$, $\vartheta(u)=\frac{1}{u}$, $u\in\left[\frac{1}{\varepsilon},\frac{1}{\sigma}\right]$.

We establish several new inequalities in this study that refine Hermite–Hadamard type inequalities (1. 4) and Hermite–Hadamard type inequality (1. 5), as well as some comparable extensions of Theorems 1.5, 1.6. There are a few examples of how the Beta function can be used.

2. Main Results

Theorem 2.1. *Let $\kappa : [\sigma, \varepsilon] \to \mathbb{R}$ be a harmonically convex function with $\sigma < \varepsilon$. Then we have the inequality*

$$\begin{aligned}\kappa\left(\frac{2\sigma\varepsilon}{\sigma+\varepsilon}\right) &\leq \frac{3^{\lambda}-1}{4^{\lambda}}\kappa\left(\frac{2\sigma\varepsilon}{\sigma+\varepsilon}\right) \\ &\quad + \frac{4^{\lambda}-3^{\lambda}+1}{2\times 4^{\lambda}}\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right] \\ &\leq \frac{\Gamma(\lambda+1)}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\left\{\mathbb{J}^{\lambda}_{\frac{1}{\sigma}-}(\kappa\circ\vartheta)\left(\frac{1}{\varepsilon}\right)+\mathbb{J}^{\lambda}_{\frac{1}{\varepsilon}+}(\kappa\circ\vartheta)\left(\frac{1}{\sigma}\right)\right\} \\ &\leq \frac{3^{\lambda}-1}{2\times 4^{\lambda}}\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right] \\ &\quad + \frac{4^{\lambda}-3^{\lambda}+1}{4^{\lambda}}\left[\frac{\kappa(\sigma)+\kappa(\varepsilon)}{2}\right] \leq \frac{\kappa(\sigma)+\kappa(\varepsilon)}{2} \qquad (2.\ 9)\end{aligned}$$

for $\lambda > 0$.

Proof. We can derive the following identities from simple computation:

$$\begin{aligned}&\frac{\lambda\Gamma(\lambda)}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\left\{\mathbb{J}^{\lambda}_{\frac{1}{\sigma}-}(\kappa\circ\vartheta)\left(\frac{1}{\varepsilon}\right)+\mathbb{J}^{\lambda}_{\frac{1}{\varepsilon}+}(\kappa\circ\vartheta)\left(\frac{1}{\sigma}\right)\right\} \\ &= \frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right]\kappa\left(\frac{1}{u}\right)du \\ &= \frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right] \\ &\quad\times\left[\kappa\left(\frac{1}{u}\right)+\kappa\left(\frac{1}{\sigma}+\frac{1}{\varepsilon}-\frac{1}{u}\right)\right]du+\frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda} \\ &\times\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right]\left[\kappa\left(\frac{1}{u}\right)+\kappa\left(\frac{1}{\sigma}+\frac{1}{\varepsilon}-\frac{1}{u}\right)\right]du. \qquad (2.\ 10)\end{aligned}$$

It is easy to observe that

$$\frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right]du=\frac{4^{\lambda}-3^{\lambda}+1}{2\times 4^{\lambda}}$$

and

$$\frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right]du=\frac{3^{\lambda}-1}{2\times 4^{\lambda}}.$$

Since

$$\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}=\left(\frac{\sigma u-4\sigma\varepsilon+3\varepsilon u}{4\sigma u-8\sigma\varepsilon+4\varepsilon u}\right)\frac{1}{u}+\left(\frac{3\sigma u-4\sigma\varepsilon+\varepsilon u}{4\sigma u-8\sigma\varepsilon+4\varepsilon u}\right)\left(\frac{1}{\sigma}+\frac{1}{\varepsilon}-\frac{1}{u}\right) \qquad (2.\ 11)$$

and

$$\frac{\sigma+3\varepsilon}{4\sigma\varepsilon}=\left(\frac{3\sigma u-4\sigma\varepsilon+\varepsilon u}{4\sigma u-8\sigma\varepsilon+4\varepsilon u}\right)\frac{1}{u}+\left(\frac{\sigma u-4\sigma\varepsilon+3\varepsilon u}{4\sigma u-8\sigma\varepsilon+4\varepsilon u}\right)\left(\frac{1}{\sigma}+\frac{1}{\varepsilon}-\frac{1}{u}\right), \quad (2.\ 12)$$

where

$$\frac{1}{u}\in\left[\frac{1}{\varepsilon},\frac{\sigma+\varepsilon}{2\sigma\varepsilon}\right] \text{ and } 0\leq\frac{\sigma u-4\sigma\varepsilon+3\varepsilon u}{4\sigma u-8\sigma\varepsilon+4\varepsilon u},\frac{3\sigma u-4\sigma\varepsilon+\varepsilon u}{4\sigma u-8\sigma\varepsilon+4\varepsilon u}\leq 1.$$

We can also write the following identities:

$$\frac{\sigma+\varepsilon}{2\sigma\varepsilon}=\frac{1}{2}\left[\frac{1}{u}+\left(\frac{1}{\sigma}+\frac{1}{\varepsilon}-\frac{1}{u}\right)\right], \quad (2.\ 13)$$

where $\frac{1}{u}\in\left[\frac{3\sigma+\varepsilon}{4\sigma\varepsilon},\frac{\sigma+\varepsilon}{2\sigma\varepsilon}\right]$.
Moreover, we also get that

$$\frac{1}{u}=\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\frac{1}{\sigma}+\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\frac{1}{\varepsilon} \quad (2.\ 14)$$

and

$$\frac{1}{\sigma}+\frac{1}{\varepsilon}-\frac{1}{u}=\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\frac{1}{\sigma}+\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\frac{1}{\varepsilon}, \quad (2.\ 15)$$

where $\frac{1}{u}\in\left[\frac{1}{\varepsilon},\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}\right]$.
Lastly, we get the following identities:

$$\frac{1}{u}=\left(\frac{4\sigma\varepsilon-\sigma u-3\varepsilon u}{2\sigma u-2\varepsilon u}\right)\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}+\left(\frac{3\sigma u-4\sigma\varepsilon+\varepsilon u}{2\sigma u-2\varepsilon u}\right)\frac{\sigma+3\varepsilon}{4\sigma\varepsilon} \quad (2.\ 16)$$

and

$$\frac{1}{\sigma}+\frac{1}{\varepsilon}-\frac{1}{u}=\left(\frac{3\sigma u-4\sigma\varepsilon+\varepsilon u}{2\sigma u-2\varepsilon u}\right)\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}+\left(\frac{4\sigma\varepsilon-\sigma u-3\varepsilon u}{2\sigma u-2\varepsilon u}\right)\frac{\sigma+3\varepsilon}{4\sigma\varepsilon}, \quad (2.\ 17)$$

where $\frac{1}{u}\in\left[\frac{3\sigma+\varepsilon}{4\sigma\varepsilon},\frac{\sigma+\varepsilon}{2\sigma\varepsilon}\right]$ with $0\leq\frac{3\sigma u-4\sigma\varepsilon+\varepsilon u}{2\sigma u-2\varepsilon u},\frac{4\sigma\varepsilon-\sigma u-3\varepsilon u}{2\sigma u-2\varepsilon u}\leq 1$.
Using the harmonic convexity of $\kappa:[\sigma,\varepsilon]\rightarrow\mathbb{R}$, we get

$$\frac{4^{\lambda}-3^{\lambda}+1}{2\times 4^{\lambda}}\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right]\geq\frac{4^{\lambda}-3^{\lambda}+1}{4^{\lambda}}\kappa\left(\frac{2\sigma\varepsilon}{\sigma+\varepsilon}\right) \quad (2.\ 18)$$

and

$$\frac{3^{\lambda}-1}{2\times 4^{\lambda}}\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right]\leq\frac{3^{\lambda}-1}{2\times 4^{\lambda}}\left[\kappa\left(\sigma\right)+\kappa\left(\varepsilon\right)\right]. \quad (2.\ 19)$$

Using the aforementioned identities (2. 11)-(2. 13) and the harmonic convexity of $\kappa : [\sigma, \varepsilon] \to \mathbb{R}$, we obtain the following inequalities:

$$\begin{aligned}
&\frac{4^{\lambda}-3^{\lambda}+1}{2\times 4^{\lambda}}\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right]=\frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\\
&\quad\times\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right]\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right]du\\
&\qquad\leq\frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right]\\
&\quad\times\left[\left(\frac{\sigma u-4\sigma\varepsilon+3\varepsilon u}{4\sigma u-8\sigma\varepsilon+4\varepsilon u}\right)\kappa\left(\frac{1}{u}\right)+\left(\frac{3\sigma u-4\sigma\varepsilon+\varepsilon u}{4\sigma u-8\sigma\varepsilon+4\varepsilon u}\right)\kappa\left(\frac{1}{\sigma}+\frac{1}{\varepsilon}-\frac{1}{u}\right)\right.\\
&\quad\left.+\left(\frac{3\sigma u-4\sigma\varepsilon+\varepsilon u}{4\sigma u-8\sigma\varepsilon+4\varepsilon u}\right)\kappa\left(\frac{1}{u}\right)+\left(\frac{\sigma u-4\sigma\varepsilon+3\varepsilon u}{4\sigma u-8\sigma\varepsilon+4\varepsilon u}\right)\kappa\left(\frac{1}{\sigma}+\frac{1}{\varepsilon}-\frac{1}{u}\right)\right]du\\
&\qquad=\frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right]\\
&\qquad\qquad\times\left[\kappa\left(\frac{1}{u}\right)+\kappa\left(\frac{1}{\sigma}+\frac{1}{\varepsilon}-\frac{1}{u}\right)\right]du \qquad (2.\ 20)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{3^{\lambda}-1}{4^{\lambda}}\kappa\left(\frac{2\sigma\varepsilon}{\sigma+\varepsilon}\right)=\frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\\
&\quad\times\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right]2\kappa\left(\frac{2\sigma\varepsilon}{\sigma+\varepsilon}\right)du\leq\frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\\
&\times\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right]\left[\kappa\left(\frac{1}{u}\right)+\kappa\left(\frac{1}{\sigma}+\frac{1}{\varepsilon}-\frac{1}{u}\right)\right]du. \qquad (2.\ 21)
\end{aligned}$$

Adding (2. 20) and (2. 21), we obtain

$$\begin{aligned}
&\frac{3^{\lambda}-1}{4^{\lambda}}\kappa\left(\frac{2\sigma\varepsilon}{\sigma+\varepsilon}\right)+\frac{4^{\lambda}-3^{\lambda}+1}{2\times 4^{\lambda}}\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right]\\
&\qquad\leq\frac{\lambda\Gamma(\lambda)}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\left\{\mathbb{J}^{\lambda}_{\frac{1}{\sigma}-}(\kappa\circ\vartheta)\left(\frac{1}{\varepsilon}\right)+\mathbb{J}^{\lambda}_{\frac{1}{\varepsilon}+}(\kappa\circ\vartheta)\left(\frac{1}{\sigma}\right)\right\}. \qquad (2.\ 22)
\end{aligned}$$

Applying (2. 14) and (2. 15) and using the convexity of $\kappa:\left[\frac{1}{\varepsilon},\frac{1}{\sigma}\right]\to\mathbb{R}$ give

$$\frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right]$$
$$\times\left[\kappa\left(\frac{1}{u}\right)+\kappa\left(\frac{1}{\sigma}+\frac{1}{\varepsilon}-\frac{1}{u}\right)\right]du\le\frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}$$
$$\times\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right]\left[\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\kappa(\sigma)+\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\kappa(\varepsilon)\right.$$
$$\left.+\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\kappa(\sigma)+\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\kappa(\varepsilon)\right]du\le\frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\left[\kappa(\sigma)+\kappa(\varepsilon)\right]$$
$$\times\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right]=\frac{4^{\lambda}-3^{\lambda}+1}{2\times4^{\lambda}}\left[\kappa(\sigma)+\kappa(\varepsilon)\right].\quad(2.\ 23)$$

Applying (2. 16) and (2. 17) together with the convexity of $\kappa:\left[\frac{1}{\varepsilon},\frac{1}{\sigma}\right]\to\mathbb{R}$ yield

$$\frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right]$$
$$\times\left[\kappa\left(\frac{1}{u}\right)+\kappa\left(\frac{1}{\sigma}+\frac{1}{\varepsilon}-\frac{1}{u}\right)\right]du\le\frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}$$
$$\times\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right]+\left(\frac{3\sigma u-4\sigma\varepsilon+\varepsilon u}{2\sigma u-2\varepsilon u}\right)\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)$$
$$\times\left[\left(\frac{4\sigma\varepsilon-\sigma u-3\varepsilon u}{2\sigma u-2\varepsilon u}\right)\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\left(\frac{3\sigma u-4\sigma\varepsilon+\varepsilon u}{2\sigma u-2\varepsilon u}\right)\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)\right.$$
$$\left.+\left(\frac{4\sigma\varepsilon-\sigma u-3\varepsilon u}{2\sigma u-2\varepsilon u}\right)\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right]du=\frac{\lambda}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}$$
$$\times\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda-1}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda-1}\right]\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right]du$$
$$\le\frac{3^{\lambda}-1}{2\times4^{\lambda}}\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right].\quad(2.\ 24)$$

Adding (2. 23) and (2. 24), we obtain

$$\frac{\lambda\Gamma(\lambda)}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\left\{\mathbb{J}^{\lambda}_{\frac{1}{\sigma}-}(\kappa\circ\vartheta)\left(\frac{1}{\varepsilon}\right)+\mathbb{J}^{\lambda}_{\frac{1}{\varepsilon}+}(\kappa\circ\vartheta)\left(\frac{1}{\sigma}\right)\right\}$$
$$\le\frac{3^{\lambda}-1}{2\times4^{\lambda}}\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right]+\frac{4^{\lambda}-3^{\lambda}+1}{2\times4^{\lambda}}\left[\kappa(\sigma)+\kappa(\varepsilon)\right].\quad(2.\ 25)$$

Combining (2. 22) and (2. 25), we get what is desired. □

Remark 2.2. *In Theorem 2.1, the inequality (2. 9) refines Hermite–Hadamard-type inequality (1. 4).*

Corollary 2.3. *In Theorem 2.1, Let $\lambda = 1$. Then we have the inequality*

$$\kappa\left(\frac{2\sigma\varepsilon}{\sigma+\varepsilon}\right) \leq \frac{1}{2}\kappa\left(\frac{2\sigma\varepsilon}{\sigma+\varepsilon}\right) + \frac{1}{4}\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right) + \kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right]$$
$$\leq \frac{\sigma\varepsilon}{\varepsilon-\sigma}\int_a^b \frac{\kappa(u)}{u^2}du \leq \frac{1}{4}\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right) + \kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right]$$
$$+ \frac{1}{2}\left[\frac{\kappa(\sigma)+\kappa(\varepsilon)}{2}\right] \leq \frac{\kappa(\sigma)+\kappa(\varepsilon)}{2}. \quad (2.\ 26)$$

The inequality provides a refinement of the inequality (1. 4).

3. Extended inequalities for fractional integrals using harmonic convexity

In this section, we establish some results which are offers extensions of some existing results.

Theorem 3.1. *Let $\kappa : \mathbb{I} \subseteq (0,\infty) \to \mathbb{R}$ be an $L^1([\sigma,\varepsilon])$ function with $\kappa' \in L^1([\sigma,\varepsilon])$ for σ, $\varepsilon \in \mathbb{J}^\circ$ (the interior $\mathbb{I}$). If $\kappa : \mathbb{I} \to \mathbb{R}$ is differentiable on $\mathbb{I}^\circ$ and $\left|\kappa'\right|$ is harmonically convex on $[\sigma,\varepsilon]$, then the following inequality holds for $\lambda > 0$:*

$$\left|\frac{\Gamma(\lambda+1)}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^\lambda \left\{\mathbb{J}^\lambda_{\frac{1}{\sigma}-}(\kappa\circ\vartheta)\left(\frac{1}{\varepsilon}\right) + \mathbb{J}^\lambda_{\frac{1}{\varepsilon}+}(\kappa\circ\vartheta)\left(\frac{1}{\sigma}\right)\right\} - \left[\frac{3^\lambda-1}{2\times 4^\lambda}\right.\right.$$
$$\left.\left.\times\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right] + \frac{4^\lambda-3^\lambda+1}{2\times 4^\lambda}\left[\kappa(\sigma)+\kappa(\varepsilon)\right]\right]\right| \leq \frac{1}{2}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)$$
$$\times\left[\frac{4\times\left(4^\lambda+2^\lambda\right)+(\lambda+1)\left(1-3^\lambda\right)}{4^{\lambda+1}(\lambda+1)}\right]\left[\left|\kappa'(\sigma)\right| + \left|\kappa'(\varepsilon)\right|\right]. \quad (3.\ 27)$$

Proof. Let $h : \left[\frac{1}{\varepsilon},\frac{1}{\sigma}\right] \to \mathbb{R}$ be defined as

$$h(u) = \begin{cases} \left(\frac{1}{\sigma}-u\right)^\lambda - \left(u-\frac{1}{\varepsilon}\right)^\lambda - \frac{3^\lambda-1}{4^\lambda}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^\lambda, & u \in \left[\frac{1}{\varepsilon},\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}\right), \\ \left(\frac{1}{\sigma}-u\right)^\lambda - \left(u-\frac{1}{\varepsilon}\right)^\lambda, & u \in \left[\frac{3\sigma+\varepsilon}{4\sigma\varepsilon},\frac{\sigma+3\varepsilon}{4\sigma\varepsilon}\right), \\ \left(\frac{1}{\sigma}-u\right)^\lambda - \left(u-\frac{1}{\varepsilon}\right)^\lambda + \frac{3^\lambda-1}{4^\lambda}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^\lambda, & u \in \left[\frac{\sigma+3\varepsilon}{4\sigma\varepsilon},\frac{1}{\sigma}\right]. \end{cases}$$

We obtain the following identities using the integration by parts:

$$\frac{1}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}} h(u)\,\kappa^{'}\left(\frac{1}{u}\right)du$$
$$=\frac{1}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda}\right]\kappa\left(\frac{1}{u}\right)du$$
$$-\left[\frac{3^{\lambda}-1}{2\times 4^{\lambda}}\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right]+\frac{4^{\lambda}-3^{\lambda}+1}{2\times 4^{\lambda}}\left[\kappa(\sigma)+\kappa(\varepsilon)\right]\right]$$
$$=\frac{\lambda\Gamma(\lambda)}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\left\{\mathbb{J}^{\lambda}_{\frac{1}{\sigma}-}(\kappa\circ\vartheta)\left(\frac{1}{\varepsilon}\right)+\mathbb{J}^{\lambda}_{\frac{1}{\varepsilon}+}(\kappa\circ\vartheta)\left(\frac{1}{\sigma}\right)\right\}$$
$$-\left[\frac{3^{\lambda}-1}{2\times 4^{\lambda}}\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right]+\frac{4^{\lambda}-3^{\lambda}+1}{2\times 4^{\lambda}}\left[\kappa(\sigma)+\kappa(\varepsilon)\right]\right]. \quad (3.\ 28)$$

We observe that

$$\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\varepsilon)\right|du$$
$$+\int_{\frac{\sigma+3\varepsilon}{4\sigma\varepsilon}}^{\frac{1}{\sigma}}\left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{1}{\sigma}-u\right)^{\lambda}-\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\varepsilon)\right|du$$
$$=\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]$$
$$\times\left[\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}+\frac{\sigma u-\sigma u}{\sigma u-\varepsilon u}\right]\left|\kappa^{'}(\varepsilon)\right|du$$
$$=\left|\kappa^{'}(\varepsilon)\right|\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda+1}\left[\frac{4\left(4^{\lambda}-3^{\lambda}\right)+\left(1-3^{\lambda}\right)\lambda}{4^{\lambda+1}(\lambda+1)}\right], \quad (3.\ 29)$$

$$\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\sigma)\right|du$$
$$+\int_{\frac{\sigma+3\varepsilon}{4\sigma\varepsilon}}^{\frac{1}{\sigma}}\left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{1}{\sigma}-u\right)^{\lambda}-\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\sigma)\right|du$$
$$=\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]$$
$$\times\left[\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}+\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right]\left|\kappa^{'}(\sigma)\right|du$$
$$=\left|\kappa^{'}(\sigma)\right|\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda+1}\left[\frac{4\left(4^{\lambda}-3^{\lambda}\right)+\left(1-3^{\lambda}\right)\lambda}{4^{\lambda+1}(\lambda+1)}\right], \quad (3.\ 30)$$

$$\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\varepsilon)\right|du$$
$$+\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+3\varepsilon}{4\sigma\varepsilon}}\left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{1}{\sigma}-u\right)^{\lambda}\right]\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\varepsilon)\right|du$$
$$=\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}\right]\left[\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}+\frac{\sigma u-\sigma u}{\sigma u-\varepsilon u}\right]\left|\kappa^{'}(\varepsilon)\right|du$$
$$=\left|\kappa^{'}(\varepsilon)\right|\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}\right]du$$
$$=\left|\kappa^{'}(\varepsilon)\right|\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda+1}\left[\frac{3^{\lambda+1}-2^{\lambda+2}+1}{4^{\lambda+1}(\lambda+1)}\right], \quad (3.\ 31)$$

and

$$\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\sigma)\right|du$$
$$+\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+3\varepsilon}{4\sigma\varepsilon}}\left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{1}{\sigma}-u\right)^{\lambda}\right]\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\sigma)\right|du$$
$$=\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}\right]\left[\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}+\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right]\left|\kappa^{'}(\sigma)\right|du$$
$$=\left|\kappa^{'}(\sigma)\right|\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda+1}\left[\frac{3^{\lambda+1}-2^{\lambda+2}+1}{4^{\lambda+1}(\lambda+1)}\right]. \quad (3.\ 32)$$

Using the fact that $\left|\kappa^{'}\right|$ is harmonic convex on $[\sigma,\varepsilon]$, thus the mapping $\varphi(u)=\left|\kappa^{'}\left(\frac{1}{u}\right)\right|$ is convex on $\left[\frac{1}{\varepsilon},\frac{1}{\sigma}\right]$, and the identities (3. 29)-(3. 32), we get

$$\left|\frac{1}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}}h(u)\,\kappa^{'}\left(\frac{1}{u}\right)du\right|$$
$$\leq\frac{1}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}}|h(u)|\left|\kappa^{'}\left(\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\frac{1}{\sigma}+\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\frac{1}{\varepsilon}\right)\right|du$$
$$\leq\frac{1}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}}|h(u)|\left[\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\sigma)\right|+\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\varepsilon)\right|\right]du$$
$$=\frac{1}{2}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)\left[\frac{3^{\lambda+1}+1-2^{\lambda+2}}{(\lambda+1)4^{\lambda+1}}\right]\left[\left|\kappa^{'}(\sigma)\right|+\left|\kappa^{'}(\varepsilon)\right|\right]$$
$$+\frac{1}{2}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)\left[\frac{4\left(3^{\lambda}-4^{\lambda}\right)+\left(3^{\lambda}-1\right)\lambda}{(\lambda+1)4^{\lambda+1}}\right]\left[\left|\kappa^{'}(\sigma)\right|+\left|\kappa^{'}(\varepsilon)\right|\right]. \quad (3.\ 33)$$

The inequality (3. 33) together with (3. 28) gives the desired result. □

Theorem 3.2. *Let $\kappa : \mathbb{I} \subseteq (0,\infty) \to \mathbb{R}$ be an $L^1([\sigma,\varepsilon])$ function with $\kappa^{'} \in L^1([\sigma,\varepsilon])$ for σ, $\varepsilon \in \mathbb{I}^{\circ}$ (the interior $\mathbb{I}$). If $\kappa : \mathbb{I} \to \mathbb{R}$ is differentiable on $\mathbb{I}^{\circ}$ and $\left|\kappa^{'}\right|$ is harmonically convex on $[\sigma,\varepsilon]$, then the following inequality holds for $\lambda > 0$:*

$$\begin{aligned}&\left|\frac{\Gamma(\lambda+1)}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\left\{\mathbb{J}^{\lambda}_{\frac{1}{\sigma}-}(\kappa\circ\vartheta)\left(\frac{1}{\varepsilon}\right)+\mathbb{J}^{\lambda}_{\frac{1}{\varepsilon}+}(\kappa\circ\vartheta)\left(\frac{1}{\sigma}\right)\right\}-\left[\frac{4^{\lambda}-3^{\lambda}+1}{2\times 4^{\lambda}}\right.\right.\\&\quad\times\left.\left.\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right]+\frac{3^{\lambda}-1}{4^{\lambda}}\kappa\left(\frac{2\sigma\varepsilon}{\sigma+\varepsilon}\right)\right]\right|\leq\frac{1}{2}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)\\&\quad\times\left[\frac{3^{\lambda}-\lambda-3\times 4^{\lambda}+2^{2\lambda}\lambda+2^{\lambda+2}+3^{\lambda}\lambda-1}{4^{\lambda+1}(\lambda+1)}\right]\left[\left|\kappa^{'}(\sigma)\right|+\left|\kappa^{'}(\varepsilon)\right|\right].\end{aligned}\tag{3. 34}$$

Proof. Let $H : \left[\frac{1}{\varepsilon},\frac{1}{\sigma}\right] \to \mathbb{R}$ be defined as

$$H(u)=\begin{cases}\left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}, & u\in\left[\frac{1}{\varepsilon},\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}\right),\\ \left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}, & u\in\left[\frac{3\sigma+\varepsilon}{4\sigma\varepsilon},\frac{\sigma+\varepsilon}{2\sigma\varepsilon}\right),\\ \left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}+\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}, & u\in\left[\frac{3\sigma+\varepsilon}{4\sigma\varepsilon},\frac{\sigma+3\varepsilon}{4\sigma\varepsilon}\right)\\ \left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}, & u\in\left[\frac{\sigma+3\varepsilon}{4\sigma\varepsilon},\frac{1}{\sigma}\right].\end{cases}$$

We obtain the following identities using the integration by parts:

$$\begin{aligned}&\frac{1}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}}H(u)\kappa^{'}\left(\frac{1}{u}\right)du\\&\qquad=\frac{1}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda}+\left(u-\frac{1}{\varepsilon}\right)^{\lambda}\right]\kappa\left(\frac{1}{u}\right)du\\&\qquad-\left[\frac{4^{\lambda}-3^{\lambda}+1}{2\times 4^{\lambda}}\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right]+\frac{3^{\lambda}-1}{4^{\lambda}}\kappa\left(\frac{\sigma+\varepsilon}{2}\right)\right]\\&\qquad=\frac{\lambda\Gamma(\lambda)}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\left\{\mathbb{J}^{\lambda}_{\frac{1}{\sigma}-}(\kappa\circ\vartheta)\left(\frac{1}{\varepsilon}\right)+\mathbb{J}^{\lambda}_{\frac{1}{\varepsilon}+}(\kappa\circ\vartheta)\left(\frac{1}{\sigma}\right)\right\}\\&\qquad-\left[\frac{4^{\lambda}-3^{\lambda}+1}{2\times 4^{\lambda}}\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)\right]+\frac{3^{\lambda}-1}{4^{\lambda}}\kappa\left(\frac{\sigma+\varepsilon}{2}\right)\right].\end{aligned}\tag{3. 35}$$

We observe that

$$\begin{aligned}
&\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}} \left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{1}{\sigma}-u\right)^{\lambda}+\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}\left(\varepsilon\right)\right|du \\
&\quad+\int_{\frac{\sigma+3\varepsilon}{4\sigma\varepsilon}}^{\frac{1}{\sigma}} \left[\left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}+\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}\left(\varepsilon\right)\right|du \\
&\quad=\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}} \left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{1}{\sigma}-u\right)^{\lambda}+\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}\left(\varepsilon\right)\right|du \\
&\quad+\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}} \left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{1}{\sigma}-u\right)^{\lambda}+\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma u-\sigma u}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}\left(\varepsilon\right)\right|du \\
&\qquad\qquad=\left|\kappa^{'}\left(\varepsilon\right)\right|\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda+1}\left[\frac{4^{\lambda}\times(\lambda-3)+3^{\lambda+1}+1}{4^{\lambda+1}(\lambda+1)}\right], \quad (3.\ 36)
\end{aligned}$$

$$\begin{aligned}
&\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}} \left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(u-\frac{1}{\sigma}\right)^{\lambda}+\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}\left(\sigma\right)\right|du \\
&\quad+\int_{\frac{\sigma+3\varepsilon}{4\sigma\varepsilon}}^{\frac{1}{\sigma}} \left[\left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}+\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}\left(\sigma\right)\right|du \\
&\quad=\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}} \left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{1}{\sigma}-u\right)^{\lambda}+\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}\left(\sigma\right)\right|du \\
&\quad+\int_{\frac{1}{\varepsilon}}^{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}} \left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{1}{\sigma}-u\right)^{\lambda}+\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}\left(\sigma\right)\right|du \\
&\qquad\qquad=\left|\kappa^{'}\left(\sigma\right)\right|\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda+1}\left[\frac{4^{\lambda}\times(\lambda-3)+3^{\lambda+1}+1}{4^{\lambda+1}(\lambda+1)}\right], \quad (3.\ 37)
\end{aligned}$$

$$\begin{aligned}
&\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}} \left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{1}{\sigma}-u\right)^{\lambda}+\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}\left(\varepsilon\right)\right|du \\
&+\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+3\varepsilon}{4\sigma\varepsilon}} \left[\left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}+\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}\left(\varepsilon\right)\right|du \\
&=\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}} \left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{1}{\sigma}-u\right)^{\lambda}+\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}\left(\varepsilon\right)\right|du \\
&+\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}} \left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{1}{\sigma}-u\right)^{\lambda}+\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}\left(\varepsilon\right)\right|du \\
&\qquad=\left|\kappa^{'}\left(\varepsilon\right)\right|\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda+1}\left[\frac{4\times 2^{\lambda}+\left(3^{\lambda}-1\right)\lambda-2\times\left(3^{\lambda}+1\right)}{4^{\lambda+1}(\lambda+1)}\right], \quad (3.\ 38)
\end{aligned}$$

and

$$\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}}\left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{1}{\sigma}-u\right)^{\lambda}+\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\sigma)\right|du$$
$$+\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+3\varepsilon}{4\sigma\varepsilon}}\left[\left(\frac{1}{\sigma}-u\right)^{\lambda}-\left(u-\frac{1}{\varepsilon}\right)^{\lambda}+\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\sigma)\right|du$$
$$=\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}}\left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{1}{\sigma}-u\right)^{\lambda}+\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\sigma)\right|du$$
$$+\int_{\frac{3\sigma+\varepsilon}{4\sigma\varepsilon}}^{\frac{\sigma+\varepsilon}{2\sigma\varepsilon}}\left[\left(u-\frac{1}{\varepsilon}\right)^{\lambda}-\left(\frac{1}{\sigma}-u\right)^{\lambda}+\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda}\right]\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\sigma)\right|du$$
$$=\left|\kappa^{'}(\sigma)\right|\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda+1}\left[\frac{4\times 2^{\lambda}+\left(3^{\lambda}-1\right)\lambda-2\times\left(3^{\lambda}+1\right)}{4^{\lambda+1}(\lambda+1)}\right]. \quad (3.\ 39)$$

Using the fact that $\left|\kappa^{'}\right|$ is harmonic convex on $[\sigma,\varepsilon]$, thus the mapping $\varphi(u)=\left|\kappa^{'}\left(\frac{1}{u}\right)\right|$ is convex on $\left[\frac{1}{\varepsilon},\frac{1}{\sigma}\right]$, and the identities (3. 36)-(3. 39), we get

$$\left|\frac{1}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}}H(u)\,\kappa^{'}\left(\frac{1}{u}\right)du\right|$$
$$\leq\frac{1}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}}|h(u)|\left|\kappa^{'}\left(\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\frac{1}{\sigma}+\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\frac{1}{\varepsilon}\right)\right|du$$
$$\leq\frac{1}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}}|h(u)|\left[\left(\frac{\sigma u-\sigma\varepsilon}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\sigma)\right|+\left(\frac{\sigma\varepsilon-\varepsilon u}{\sigma u-\varepsilon u}\right)\left|\kappa^{'}(\varepsilon)\right|\right]du$$
$$=\frac{1}{2}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda+1}\left[\frac{4^{\lambda}\times(\lambda-3)+3^{\lambda+1}+1}{4^{\lambda+1}(\lambda+1)}\right]\left[\left|\kappa^{'}(\sigma)\right|+\left|\kappa^{'}(\varepsilon)\right|\right]$$
$$+\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda+1}\left[\frac{2\times\left(3^{\lambda}+1\right)-4\times 2^{\lambda}-\left(3^{\lambda}-1\right)\lambda}{4^{\lambda+1}(\lambda+1)}\right]\left[\left|\kappa^{'}(\sigma)\right|+\left|\kappa^{'}(\varepsilon)\right|\right]. \quad (3.\ 40)$$

The inequality (3. 40) when combined with (3. 35) yields the desired result. □

Remark 3.3. *Theorem 3.1 and Theorem 3.2 are similar extensions of Theorems 1.5 and 1.6.*

Remark 3.4. *Let $\lambda=1$ in Theorem 3.1 and Theorem 3.2. The following inequalities are obtained:*

$$\left|\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)\int_{\sigma}^{\varepsilon}\frac{\kappa(u)}{u^{2}}du-\frac{1}{4}\left[\kappa(\sigma)+\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)+\kappa(\varepsilon)\right]\right|$$
$$\leq\frac{1}{16}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)\left[\left|\kappa^{'}(\sigma)\right|+\left|\kappa^{'}(\varepsilon)\right|\right]. \quad (3.\ 41)$$

and

$$\left|\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)\int_{\sigma}^{\varepsilon}\frac{\kappa(u)}{u^{2}}du-\frac{1}{4}\left[\kappa\left(\frac{4\sigma\varepsilon}{3\sigma+\varepsilon}\right)+\kappa\left(\frac{4\sigma\varepsilon}{\sigma+3\varepsilon}\right)+2\kappa\left(\frac{2\sigma\varepsilon}{\sigma+\varepsilon}\right)\right]\right|$$
$$\leq\frac{1}{16}\left(\frac{\varepsilon-\sigma}{\sigma\varepsilon}\right)^{\lambda+1}\left[\left|\kappa^{'}(\sigma)\right|+\left|\kappa^{'}(\varepsilon)\right|\right]. \quad (3.\ 42)$$

4. Applications of the results

Perhaps it would be better to return to the Beta incomplete functions

$$B(r,s)=\int_{0}^{1}u^{r-1}(1-u)^{s-1}du$$

and

$$B_{z}(r,s)=\int_{0}^{z}u^{r-1}(1-u)^{s-1}du.$$

Consider the function $\kappa(u)=u^{-p}$, $u\in[1,2]$ for $p>1$. It is obvious that this function is a harmonically convex function. We observe that

$$\frac{\Gamma(\lambda+1)}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\mathbb{J}^{\lambda}_{\frac{1}{\sigma}-}(\kappa\circ\vartheta)\left(\frac{1}{\varepsilon}\right)=\frac{\lambda}{2}\int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}}\left(\frac{1}{\sigma}-u\right)^{\lambda-1}\kappa\left(\frac{1}{u}\right)du$$
$$=\frac{\lambda}{2}\int_{\frac{1}{2}}^{1}(1-u)^{\lambda-1}u^{p}du=\frac{1}{2}\lambda\left[B(p+1,\lambda)-B_{\frac{1}{2}}(p+1,\lambda)\right]$$

and

$$\frac{\Gamma(\lambda+1)}{2}\left(\frac{\sigma\varepsilon}{\varepsilon-\sigma}\right)^{\lambda}\int_{\frac{1}{\varepsilon}}^{\frac{1}{\sigma}}\left(u-\frac{1}{\varepsilon}\right)^{\lambda}\kappa\left(\frac{1}{u}\right)du=\frac{\lambda}{2}\int_{\frac{1}{2}}^{1}\left(u-\frac{1}{2}\right)^{\lambda-1}\kappa\left(\frac{1}{u}\right)du$$
$$=\frac{\lambda}{2}\int_{\frac{1}{2}}^{1}\left(u-\frac{1}{2}\right)^{\lambda-1}u^{-p-2}du=\frac{\lambda\left[\Gamma(\lambda)\Gamma(-p-\lambda)-\Gamma(-p)B_{\frac{1}{2}}(-p-\lambda,\lambda)\right]}{2^{\lambda+p+1}\Gamma(-p)}.$$

We can now have the following estimates regarding the incomplete beta and beta functions.

Proposition 4.1. *Let $p>2$ and $\lambda>0$ be in Theorem 2.1. Then the following inequality holds:*

$$\left(\frac{3}{4}\right)^{p}\leq\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{3}{4}\right)^{p}+\frac{4^{\lambda}-3^{\lambda}+1}{2\times4^{\lambda}}\left[\left(\frac{5}{8}\right)^{p}+\left(\frac{7}{8}\right)^{p}\right]$$
$$\leq2^{\lambda-1}\lambda\left[B(p+1,\lambda)-B_{\frac{1}{2}}(p+1,\lambda)\right]+\frac{\lambda}{2^{p+1}}\left[B(-p-\lambda,\lambda)-B_{\frac{1}{2}}(-p-\lambda,\lambda)\right]$$
$$\leq\frac{3^{\lambda}-1}{2\times4^{\lambda}}\left[\left(\frac{5}{8}\right)^{p}+\left(\frac{7}{8}\right)^{p}\right]+\frac{4^{\lambda}-3^{\lambda}+1}{2\times4^{\lambda}}\left[\frac{2^{p}+1}{2^{p}}\right]\leq\frac{2^{p}+1}{2^{p+1}} \quad (4.\ 43)$$

Proof. The proof is obvious from Theorem 2.1 for the function $\kappa(u)=u^{-p}$, $u\in[1,2]$ for $p>1$. The graph below also support the validity of the inequality (4. 43). □

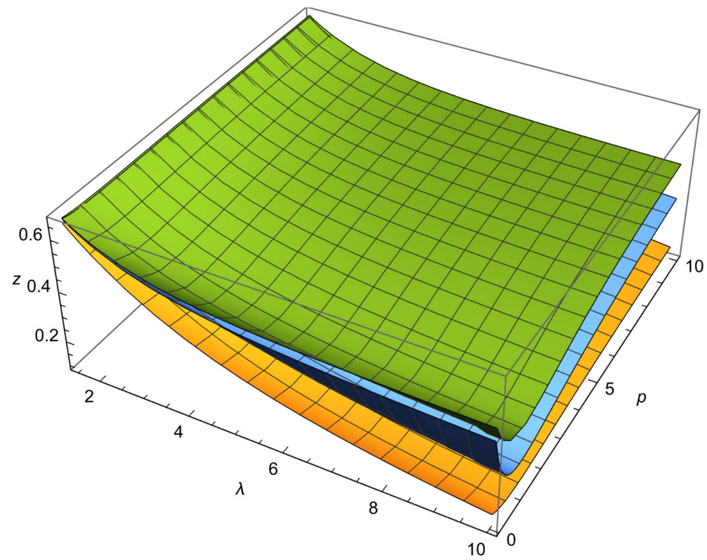

FIGURE 1. The graph of inequality 4. 43 for $p > 2$ and $\lambda > 0$

Proposition 4.2. *Let $p > 2$ and $\lambda > 0$ be in Theorem 3.1. Then the following inequality holds:*

$$\begin{aligned} 0 \leq\ & 2^{\lambda-1}\lambda\left[B(p+1,\lambda)-B_{\frac{1}{2}}(p+1,\lambda)\right]+\frac{\lambda}{2^{p+1}}\left[B\left(-p-\lambda,\lambda\right)-B_{\frac{1}{2}}(-p-\lambda,\lambda)\right] \\ & -\left[\frac{3^{\lambda}-1}{2\times 4^{\lambda}}\left[\left(\frac{5}{8}\right)^{p}+\left(\frac{7}{8}\right)^{p}\right]+\frac{4^{\lambda}-3^{\lambda}+1}{2\times 4^{\lambda}}\left[1+2^{-p}\right]\right] \\ & +\frac{p\left[2^{-p-1}+1\right]}{4}\left[\frac{4\times\left(4^{\lambda}+2^{\lambda}\right)+(\lambda+1)\left(1-3^{\lambda}\right)}{4^{\lambda+1}\left(\lambda+1\right)}\right] \\ & \leq \frac{p\left[2^{p-1}+1\right]}{2}\left[\frac{(\lambda+1)\left(3^{\lambda}-1\right)-4\times\left(4^{\lambda}+2^{\lambda}\right)}{4^{\lambda+1}\left(\lambda+1\right)}\right]. \qquad (4.\ 44) \end{aligned}$$

Proof. The proof follows directly from Theorem 3.1 by using the harmonic convex function $\kappa(u) = u^{-p}$, $u \in [1, 2]$ for $p > 2$. The figure below confirms the validity of the above inequality. □

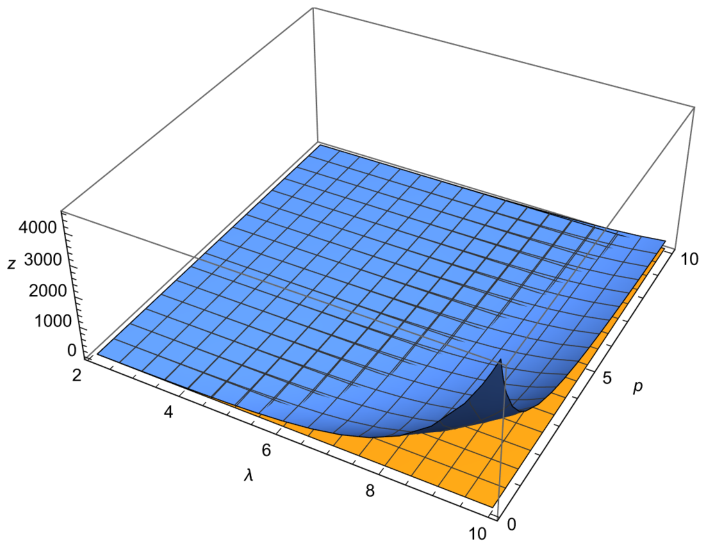

FIGURE 2. The graph of inequality (4. 44) for $p > 2$ and $\lambda > 0$

Proposition 4.3. *Let $p > 2$ $p > 2$ and $\lambda > 0$ be in Theorem 3.2. Then the following inequality holds:*

$$\begin{aligned} 0 &\leq 2^{\lambda-1}\lambda\left[B(p+1,\lambda)-B_{\frac{1}{2}}(p+1,\lambda)\right]+\frac{\lambda}{2^{p+1}}\left[B\left(-p-\lambda,\lambda\right)-B_{\frac{1}{2}}(-p-\lambda,\lambda)\right] \\ &\quad -\left[\frac{4^{\lambda}-3^{\lambda}+1}{2\times 4^{\lambda}}\left[\left(\frac{5}{8}\right)^{p}+\left(\frac{7}{8}\right)^{p}\right]+\frac{3^{\lambda}-1}{4^{\lambda}}\left(\frac{3}{4}\right)^{p}\right] \\ &\quad +\frac{p\left[2^{-p-1}+1\right]}{4}\left[\frac{3^{\lambda}-\lambda-3\times 4^{\lambda}+2^{2\lambda}\lambda+2^{\lambda+2}+3^{\lambda}\lambda-1}{4^{\lambda+1}\left(\lambda+1\right)}\right] \\ &\leq \frac{p\left[2^{-p-1}+1\right]}{2}\left[\frac{3^{\lambda}-\lambda-3\times 4^{\lambda}+2^{2\lambda}\lambda+2^{\lambda+2}+3^{\lambda}\lambda-1}{4^{\lambda+1}\left(\lambda+1\right)}\right]. \end{aligned} \tag{4. 45}$$

Proof. The proof follows directly from Theorem 3.2 by using the harmonic convex function $\kappa(u) = u^{-p}$, $u \in [1, 2]$ for $p > 2$. The figure below confirms the validity of the inequality 4. 45 . □

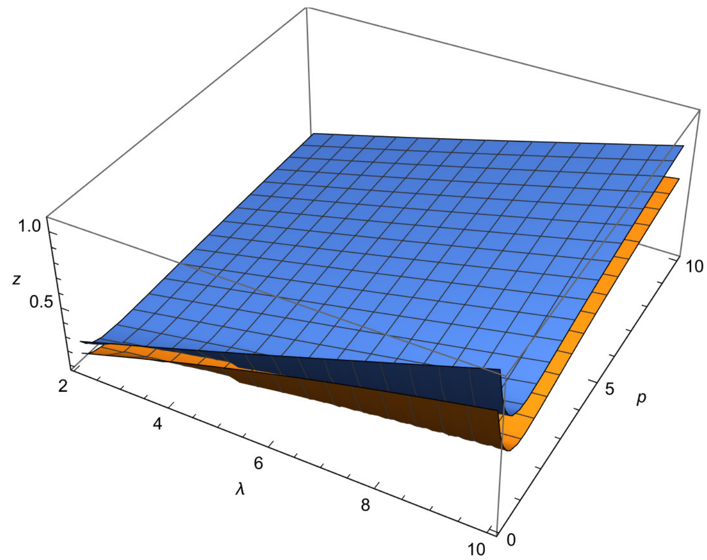

FIGURE 3. The graph of inequality 4. 45 for $p > 2$ and $\lambda > 0$

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REFERENCES

[1] T. Abdeljawad, P. O. Mohammed, A. Kashuri, New modified conformable fractional integral inequalities of Hermite–Hadamard type with applications, J. Funct. Space, 2020 (2020) 4352357.

[2] R. Almeida, A Caputo fractional derivative of a function with respect to another function, Commun. Nonlinear Sci. Numer. Simul., 44 (2017), 460-481.

[3] D. Baleanu, P.O. Mohammed, M. Vivas-Cortez, Y. Rangel-Oliveros, Some modifications in conformable fractional integral inequalities, Adv. Differ. Equ., 2020, 2020, 374.

[4] C. Bardaro, P. L. Butzer, I. Mantellini, The foundations of fractional calculus in the Mellin transform setting with applications, J. Fourier Anal. Appl., 21 (2015), 961-1017.

[5] D. Baleanu, O. O. Mohammed, S. Zeng, Inequalities of trapezoidal type involving generalized fractional integrals, Alex. Eng. J., 59 (5) (2020), 2975-2984.

[6] F. Chen and S. Wu, Fej´er and Hermite–Hadamard type inqequalities for harmonically convex functions, Jurnal of applied Mathematics, volume 2014, article ID:386806.

[7] S. S. Dragomir, C. E. M. Pearce, Selected Topics on Hermite–Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University: Footscray, Australia, 2000.

[8] S. S. Dragomir, R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett., 11 (1998), 91-95.

[9] M. R. Delavar, M. Aslani, M. De La Sen, Hermite–Hadamard-Fejér inequality related to generalized convex functions via fractional integrals, Journal of Mathematics, 2018 (2018) 5864091.

[10] L. Fejér, Uberdie Fourierreihen, II, Math. Naturwise Anz Ung. Akad. Wiss., 24 (1906), 369-390.

[11] A. Fernandez, P. O. Mohammed, Hermite–Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels, Math. Methods Appl. Sci., 2020, 1-18.

[12] B. Gavrea, I. Gavrea, On some Ostrowski type inequalities, Gen. Math., 18 (2010), 33-44.

[13] H. Gunawan, Eridani, Fractional integrals and generalized Olsen inequalities, Kyungpook Math. J., 49 (2009), 31-39.
[14] J. Hadamard, Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann. J. Math. Pures Appl., 1893, 58, 171-215.
[15] J. Han, P. O. Mohammed, H. Zeng, Generalized fractional integral inequalities of Hermite–Hadamard-type for a convex function, Open Math., 18 (2020), 794-806.
[16] S. -R. Hwang, K. -L. Tseng, K. -C. Hsu, New inequalities for fractional integrals and their applications, Turkish Journal of Mathematics, 40 (2016), 471-486.
[17] İ. İşcan, Hermite–Hadamard-Fejér type inequalities for convex functions via fractional integrals, Stud. Univ. Babes Bolyai Math., 60 (2015), 355-366.
[18] İ. İşcan, S. Wu, Hermite–Hadamard type inequalities for harmonically convex functions via fractional integrals, Appl. Math. Comput., 238 (2014), 237-244.
[19] İ. İşcan, M. Kunt and N. Yazici, Hermite–Hadamard-Fejér type inequalities for harmonically convex functions via fractional integrals, New Trends in Mathematical Sciences, 4 (3) (2016), 239-253
[20] F. Jarad, T. Abdeljawad, K. Shah, On the weighted fractional operators of a function with respect to another function, Fractals, 28 (8) (2020), 2040011, 12 pages.
[21] M. Kunt, İ. İşcan, N. Yazıcı and U. Gözütok, On new inequalities of Hermite–Hadamard-Fejér type for harmonically convex functions via fractional integrals, SpringerPlus 5, 635 (2016).
[22] S. Kaijser, L. Nikolova, L .-E. Persson, Wedestig, A Hardy type inequalities via convexity, Math. Inequal. Appl., 8 (2005), 403-417.
[23] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations; North-Holland Mathematics Studies; Elsevier Sci. B.V.: Amsterdam, The Netherlands, 2006; Volume 204.
[24] M. Kunt, I. Iscan, On new Hermite–Hadamard-Fejér type inequalities for p-convex functions via fractional integrals, CMMA 2 (2017), 1-15.
[25] P. O. Mohammed, I. Brevik, A new version of the Hermite–Hadamard inequality for Riemann–Liouville fractional integrals, Symmetry, 12 (2020) 610.
[26] P. O. Mohammed, M. Z. Sarikaya, On generalized fractional integral inequalities for twice differentiable convex functions, J. Comput. Appl. Math., 372 (2020), 112740.
[27] P. O. Mohammed, Hermite–Hadamard inequalities for Riemann–Liouville fractional integrals of a convex function with respect to a monotone function, Math. Methods Appl. Sci., (2019), 1-11.
[28] S. Mehmood, F. Zafar, N. Asmin, New Hermite–Hadamard-Fejér type inequalities for (h_1,h_2)-convex functions via fractional calculus, ScienceAsia, 46 (2020), 102-108.
[29] F. Qi, O. P. Mohammed, J. C. Yao, Y. H. Yao, Generalized fractional integral inequalities of Hermite–Hadamard type for (λ,m)-convex functions, J. Inequal. Appl., 2019 (2019), 135.
[30] M.Z. Sarikaya, E. Set, H. Yaldiz, N. Bas ak, Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities, Math. Comput. Model., 57 (2013), 2403-2407.
[31] M. Z. Sarikaya, H. Yaldiz, On generalization integral inequalities for fractional integrals, Nihonkai Math. J., 25 (2014), 93-104.
[32] M. Z. Sarikaya, C. C. Bilisik, P. O. Mohammed, Some generalizations of Opial type inequalities, Appl. Math. Inf. Sci. 14 (2020), 809-816.
[33] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities, Math. Comput. Model., 57 (2013), 2403-2407.
[34] D. -P. Shi, B. -Y Xi, F. Qi, Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals of (λ,m)-convex functions, Fract. Differ. Calc., 4 (2014), 31-43.
[35] C. -J. Zhao, W. -S. Cheung, On improvements of the Rozanova's inequality, J. Inequal. Appl., 2011 (2011) 33.