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Some Remarks on Results Related to Ostrowski−Grüss Inequality

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Abstract. The article [14] published in 2007 titled "New bounds for the first inequality of Ostrowski−Grüss type and applications in numerical integration" contains very important results in the sense that the main idea of the article is unique, and it helps to generalize many established results. In the article [11], which appeared in the year 2013, we find exactly the same proving techniques used, leading to similar but slightly different results as compared to results stated in [14]. This arises the question: Why do two different authors, using exactly same proving techniques, arrive at different results? Now we compare results of the two articles and give some remarks on it and produce a better version.

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1. Introduction

In [5] Dragomir and Wang introduced a new inequality, termed the Ostrowski −Grüss inequality, by combining two well-known inequality Grüss [6] and Ostrowski [10] inequalities. For the rest of the article J denotes an interval of $\mathbb{R}$, J^0 represents the interior of J and $S = \frac{f(\zeta_2)-f(\zeta_1)}{\zeta_2-\zeta_1}$, provided that $S \neq \gamma$ and $S \neq \Gamma$, where γ and Γ are the lower and upper bounds of f', respectively, with $\gamma \neq \Gamma$. We begin by recalling the Ostrowski−Grüss inequality:

Theorem 1.1. *Let $f : J \to \mathbb{R}$ be a differentiable function on J^0; $\zeta_1, \zeta_2 \in J^0$ and $\zeta_1 < \zeta_2$. If $\exists\, \Gamma, \gamma \in \mathbb{R}$ such that $\Gamma \geq f'(\chi) \geq \gamma\ \forall\, \chi \in [\zeta_1, \zeta_2]$, then the following inequality holds:*

$$\left| f(\chi) - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt - \left(\chi - \frac{\zeta_1+\zeta_2}{2}\right) S \right| \leq \frac{\zeta_2-\zeta_1}{4}(\Gamma-\gamma), \quad (1.\ 1)$$

$\forall\, \chi \in [\zeta_1, \zeta_2]$.

Matić et al. in 2000 [9], gave the following improvement of Ostrowski−Grüss inequality:

Theorem 1.2. *Under the assumptions of Theorem* 1.1*, the following inequality holds:*

$$\left| f(\chi) - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(t)dt - \left(\chi - \frac{\zeta_1 + \zeta_2}{2}\right) S \right| \leq \frac{\zeta_2 - \zeta_1}{4\sqrt{3}} (\Gamma - \gamma), \tag{1. 2}$$

$\forall\, \chi \in [\zeta_1, \zeta_2]$.

In [3] authors improve inequality (1.2) as follows:

Theorem 1.3. *Let* $f : [\zeta_1, \zeta_2] \to \mathbb{R}$ *be an absolutely continuous function such that* $f' \in L_2[\zeta_1, \zeta_2]$*. Then* $\forall \chi \in [\zeta_1, \zeta_2]$*, the following inequalities hold:*

$$\left| f(\chi) - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(t)dt - \left(\chi - \frac{\zeta_1 + \zeta_2}{2}\right) S \right|$$
$$\leq \frac{\zeta_2 - \zeta_1}{2\sqrt{3}} \left[\frac{1}{\zeta_2 - \zeta_1} \|f'\|_2^2 - S^2 \right]^{\frac{1}{2}} \leq \frac{\zeta_2 - \zeta_1}{4\sqrt{3}} (\Gamma - \gamma), \tag{1. 3}$$

where $\gamma \leq f'(\chi) \leq \Gamma$ *a.e.* $\forall \chi \in [\zeta_1, \zeta_2]$.

Ujević in [17], demonstrated the initial inequality of Theorem 1.3 using a different methodology. Additionally, Ujević established the sharpness of this inequality. However, Cheng in [4], provided the sharper bound of first inequality using product $(\zeta_2 - \zeta_1)(\Gamma - \gamma)$. This fact can be summarized as under:

Theorem 1.4. *Under the assumptions of Theorem 1.1, the following inequality holds:*

$$\left| f(\chi) - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(t)dt - \left(\chi - \frac{\zeta_1 + \zeta_2}{2}\right) S \right| \leq \frac{\zeta_2 - \zeta_1}{8} (\Gamma - \gamma), \tag{1. 4}$$

$\forall \chi \in [\zeta_1, \zeta_2]$.

In [16] Ujević presented an estimation of inequality (1.1) in slightly different way:

Theorem 1.5. *Let* $f : J \to \mathbb{R}$*, be a differentiable function on* J^0 *and let* $\zeta_1, \zeta_2 \in J^0$*,* $\zeta_1 < \zeta_2$*. If* $\exists\, \Gamma, \gamma \in \mathbb{R}$ *such that* $\Gamma \geq f'(\chi) \geq \gamma, \forall\, x \in [\zeta_1, \zeta_2]$ *and* $f' \in L_1[\zeta_1, \zeta_2]$*, then*

$$\left| f(\chi) - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(t)dt - \left(\chi - \frac{\zeta_1 + \zeta_2}{2}\right) S \right| \leq \frac{\zeta_2 - \zeta_1}{2} (S - \gamma), \tag{1. 5}$$

and

$$\left| f(\chi) - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(t)dt - \left(\chi - \frac{\zeta_1 + \zeta_2}{2}\right) S \right| \leq \frac{\zeta_2 - \zeta_1}{2} (\Gamma - S), \tag{1. 6}$$

$\forall\, \chi \in [\zeta_1, \zeta_2]$*, where* $S = \frac{f(\zeta_2) - f(\zeta_1)}{\zeta_2 - \zeta_1}$.

In the same paper [16], Ujević also proved that:

Theorem 1.6. *Let* $f : J \to \mathbb{R}$ *be a twice continuously differentiable mapping in* J^0 *with* $f'' \in L_2(\zeta_1, \zeta_2)$ *and let* $\zeta_1, \zeta_2 \in J^0$*,* $\zeta_1 < \zeta_2$*. Then the following inequality holds:*

$$\left| f(\chi) - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(t)dt - \left(\chi - \frac{\zeta_1 + \zeta_2}{2}\right) S \right| \leq \frac{(\zeta_2 - \zeta_1)^{3/2}}{2\pi\sqrt{3}} \|f''\|_2.$$

In 2007, Rafiq and Zafar in [14] by introducing a parameter improved the result stated in Theorem 1.5 as:

Theorem 1.7. *Under the assumptions of Theorem* 1.5*, the following inequalities hold:*

$$\left|(1-h)\left(f(\chi)-\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right)f'(\chi)\right)+h\frac{f(\zeta_1)+f(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right| \le \frac{\zeta_2-\zeta_1}{2}(1-h^2)(S-\gamma), \quad (1.\ 7)$$

and

$$\left|(1-h)\left(f(\chi)-\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right)f'(\chi)\right)+h\frac{f(\zeta_1)+f(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right| \le \frac{\zeta_2-\zeta_1}{2}(1-h^2)(\Gamma-S), \quad (1.\ 8)$$

where $S=\frac{f(\zeta_2)-f(\zeta_1)}{\zeta_2-\zeta_1}$, $\chi\in\left[\zeta_1+h\frac{\zeta_2-\zeta_1}{2},\zeta_2-h\frac{\zeta_2-\zeta_1}{2}\right]$ *and* $h\in[0,1]$.

In [14] authors also stated following result:

Theorem 1.8. *Under the assumptions of Theorem* 1.6*, the following inequality holds:*

$$\left|(1-h)\left(f(\chi)-\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right)f'(\chi)\right)+h\frac{f(\zeta_1)+f(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right| \le \frac{(\zeta_2-\zeta_1)^{1/2}}{\pi}\left[h(1-h)\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right)+\frac{1}{12}(\zeta_2-\zeta_1)^2(h^3+(1-h)^3)\right]^{1/2}\times \times\|f''\|_2, \quad (1.\ 9)$$

$\forall\,\chi\in[\zeta_1,\zeta_2]$.

Park in [11], obtained slightly different results by employing the same proving techniques of those used for proof of Theorems 1.7 and 1.8 which can be seen in the following theorem.

Theorem 1.9. *Under the assumptions of Theorem* 1.7*, the following inequalities hold:*

$$\left|(1-h)\left(f(\chi)-\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right)S\right)+h\frac{f(\zeta_1)+f(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right| \le \frac{\zeta_2-\zeta_1}{2}(S-\gamma), \quad (1.\ 10)$$

and

$$\left|(1-h)\left(f(\chi)-\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right)S\right)+h\frac{f(\zeta_1)+f(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right| \le \frac{\zeta_2-\zeta_1}{2}(\Gamma-S), \quad (1.\ 11)$$

$\forall\,\chi\in[\zeta_1,\zeta_2]$ *and* $h\in[0,1]$.

Theorem 1.10. *Under the assumptions of Theorem* 1.8*, the following inequality holds:*

$$\left|(1-h)\left(f(\chi)-\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right)f'(\chi)\right)+h\frac{f(\zeta_1)+f(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right|$$
$$\leq \frac{1}{\pi}\left[(\zeta_2-\zeta_1)\left\{h(1-h)\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right)^2+\frac{1}{12}(\zeta_2-\zeta_1)^2(h^3+(1-h)^3)\right\}\right]^{1/2}\times$$
$$\times\|f''\|_2, \quad (1.\ 12)$$

$\forall\, \chi \in [\zeta_1, \zeta_2]$ *and* $h \in [0,1]$.

For recent work on Ostrwoski-Grüss type inequalities we refer the reader to [2, 7, 8, 13].

2. Some Remarks

We now present several remarks concerning Theorems 1.7, 1.8, 1.9 and 1.10:

Remark 2.1. (1) *The authors introduced* $f'(\chi)$ *(instead of* $\frac{f(\zeta_2)-f(\zeta_1)}{\zeta_2-\zeta_1}$*) in both the Theorems* 1.7 *and* 1.8 *unnecessarily and mistakenly. In similar manner it should not be present in other remarks and corollaries and applications followed by and directly linked by Theorems* 1.7 *and* 1.8 *(see* [14]*).*

(2) *The authors also claim that setting* $h = 0$ *in inequalities* (1.7) *and* (1.8) *reproduces inequalities* (1.5) *and* (1.6)*, respectively, which is incorrect.*

(3) *The bound on right-hand side of both the inequalities* (1.7) *and* (1.8) *does not depend on variable* χ*. Whereas, it should depend on* χ*. This observation also applies to inequalities* (1.10) *and* (1.11)*.*

(4) *The bound on the right-hand side of both inequalities* (1.7) *and* (1.8) *depends only on the parameter* $h \in [0,1]$*. However, there are calculation mistakes, leading to incorrect results when* $h = 1$ *is substituted into* (1.7) *and* (1.8)*. This error propagates to all corollaries, remarks, and applications in* [14]*. On the other hand, one cannot find any* h *on right-hand side of* (1.10) *and* (1.11) *which should be there.*

(5) *We further see some minor typos and/or calculation mistakes in some of the remarks given in* [14]*, for example, Remarks* $2.5-2.8$ *(of* [14]*). These results cannot be derived by any method based on the given techniques.*

(6) *In Theorem* 1.8 *authors did not define* h *and considered that* $\chi \in [\zeta_1, \zeta_2]$ *in contrast to their first result in which they chose* $\chi \in \left[\zeta_1 + h\frac{\zeta_2-\zeta_1}{2},\ \zeta_2 - h\frac{\zeta_2-\zeta_1}{2}\right]$*. Also in both the results Theorems* 1.9 *and* 1.10*, author assumes that* $\chi \in [\zeta_1, \zeta_2]$*. We believe that in all four results* χ *should be a member of* $[\zeta_1 + h\frac{\zeta_2-\zeta_1}{2}, \zeta_2 - h\frac{\zeta_2-\zeta_1}{2}]$ *for* $h \in [0,1]$*.*

(7) *Throughout article* [14] *there is used a term* $R_n(\chi)$*, where* n *is neither used nor defined explicitly.*

(8) *In Theorem* 1.8 *on right-hand side of inequality* (1.9)*, there should be* $(\chi - \frac{\zeta_1+\zeta_2}{2})^2$ *instead of* $(\chi - \frac{\zeta_1+\zeta_2}{2})$*. Inequality* (1.12) *is* 100% *correct and it justifies our claim.*

(9) *In article* [11] *the author used the same technique as of* [14] *and got similar results but did not cite it in references and bibliography.*

(10) *We can relax the stringent requirement of function differentiability by instead assuming the function to be absolutely continuous* [15, p. 5]*, as demonstrated by Barnett et al. in Theorem* 1.3 *(see also* [3]*). This approach yields two essential properties (see* [1] *for details):*
   (a) *An absolutely continuous function is almost everywhere differentiable.*
   (b) *The fundamental theorem of integral calculus holds true for absolutely continuous functions as well, meaning* $\int_{\zeta_1}^{\zeta_2} f'(t)dt = f(\zeta_2) - f(\zeta_1)$.

3. Rectified Results

Here we state corrected versions of aforementioned results.

Theorem 3.1. *Let* $f : J \to \mathbb{R}$ *be an absolutely continuous function on* J^0 *and let* $\zeta_1, \zeta_2 \in J^0$, $\zeta_1 < \zeta_2$. *If* $\exists\, \gamma, \Gamma \in \mathbb{R}$ *such that* $\gamma \le f'(\chi) \le \Gamma, \forall\, x \in [\zeta_1, \zeta_2]$ *a. e., then we have:*

$$\left|(1-h)\left(f(\chi) - \left(\chi - \frac{\zeta_1+\zeta_2}{2}\right)S\right) + h\frac{f(\zeta_1)+f(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right|$$
$$\le \max\left\{\left|(\chi-\zeta_2)h + \frac{\zeta_2-\zeta_1}{2}\right|, \left|(\chi-\zeta_1)h + \left(\frac{\zeta_1+\zeta_2}{2} - \chi\right)\right|\right\}(S-\gamma), \quad (3.\ 13)$$

and

$$\left|(1-h)\left(f(\chi) - \left(\chi - \frac{\zeta_1+\zeta_2}{2}\right)S\right) + h\frac{f(\zeta_1)+f(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right|$$
$$\le \max\left\{\left|(\chi-\zeta_2)h + \frac{\zeta_2-\zeta_1}{2}\right|, \left|(\chi-\zeta_1)h + \left(\frac{\zeta_1+\zeta_2}{2} - \chi\right)\right|\right\}(\Gamma-S), \quad (3.\ 14)$$

where $S = \frac{f(\zeta_2)-f(\zeta_1)}{\zeta_2-\zeta_1}$, $\chi \in \left[\zeta_1 + h\frac{\zeta_2-\zeta_1}{2}, \zeta_2 - h\frac{\zeta_2-\zeta_1}{2}\right]$ *and* $h \in [0,1]$.

Proof. Let us consider $K : [\zeta_1, \zeta_2]^2 \to \mathbb{R}$ given by

$$K(\chi,t) = \begin{cases} t - \left(\zeta_1 + h\dfrac{\zeta_2-\zeta_1}{2}\right), & \text{if} \quad t \in [\zeta_1, \chi], \\ t - \left(\zeta_2 - h\dfrac{\zeta_2-\zeta_1}{2}\right), & \text{if} \quad t \in (\chi, \zeta_2], \end{cases}$$

where $\chi \in \left[\zeta_1 + h\frac{\zeta_2-\zeta_1}{2}, \zeta_2 - h\frac{\zeta_2-\zeta_1}{2}\right]$ for $h \in [0,1]$.

Using integration by parts we get:

$$\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} K(\chi,t)f'(t)dt = (1-h)f(\chi) + h\frac{f(\zeta_1)+f(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt. \quad (3.\ 15)$$

Moreover,

$$\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} K(\chi,t)dt = (1-h)\left(\chi - \frac{\zeta_1+\zeta_2}{2}\right), \quad (3.\ 16)$$

and

$$\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f'(t)dt = \frac{f(\zeta_2)-f(\zeta_1)}{\zeta_2-\zeta_1} = S \tag{3. 17}$$

By using (3.15), (3.16) and (3.17) we have:

$$(1-h)\left(f(\chi)-\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right)S\right)+h\frac{f(\zeta_1)+f(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt$$

$$=\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} K(\chi,t)f'(t)dt-\left(\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} K(\chi,s)ds\right)\left(\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f'(t)dt\right).$$

It is denoted

$$R(\chi)=\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f'(t)\left(K(\chi,t)-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} K(\chi,s)ds\right)dt.$$

Since

$$\int_{\zeta_1}^{\zeta_2}\left(K(\chi,t)-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} K(\chi,s)ds\right)dt=0,$$

therefore it is followed that, if C is an arbitrary real constant, then

$$R(\chi)=\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} (f'(t)-C)\left(K(\chi,t)-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} K(\chi,s)ds\right)dt. \tag{3. 18}$$

If we choose $C=\gamma$ in (3.18), then

$$R(\chi)=\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} (f'(t)-\gamma)\left(K(\chi,t)-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} K(\chi,s)ds\right)dt$$

which gives us

$$|R(\chi)|\le\frac{1}{\zeta_2-\zeta_1}\sup_{t\in[\zeta_1,\zeta_2]}\left|K(\chi,t)-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} K(\chi,s)ds\right|\int_{\zeta_1}^{\zeta_2}|f'(t)-\gamma|\,dt.$$

If we choose $C=\Gamma$ in (3.18), then

$$R(\chi)=\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} (f'(t)-\Gamma)\left(K(\chi,t)\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} dt-\int_{\zeta_1}^{\zeta_2} K(\chi,s)ds\right)dt$$

which gives us

$$|R(\chi)|\le\frac{1}{\zeta_2-\zeta_1}\sup_{t\in[\zeta_1,\zeta_2]}\left|K(\chi,t)-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} K(\chi,s)ds\right|\int_{\zeta_1}^{\zeta_2}|f'(t)-\Gamma|\,dt,$$

in order to completes the proof we are left only with calculation of

$$\sup_{t\in[\zeta_1,\zeta_2]}\left|K(\chi,t)-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} K(\chi,s)ds\right|$$

for that we proceed as follows:

$$\sup_{t\in[\zeta_1,\zeta_2]}\left|K(\chi,t)-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}K(\chi,s)ds\right|$$
$$=\sup_{t\in[\zeta_1,\zeta_2]}\left|\begin{cases} t-\left(\zeta_1+h\dfrac{\zeta_2-\zeta_1}{2}\right)-(1-h)\left(\chi-\dfrac{\zeta_1+\zeta_2}{2}\right), & t\in[\zeta_1,\chi],\\ t-\left(\zeta_2-h\dfrac{\zeta_2-\zeta_1}{2}\right)-(1-h)\left(\chi-\dfrac{\zeta_1+\zeta_2}{2}\right), & t\in(\chi,\zeta_2], \end{cases}\right|.$$

Now we discuss each case one by one:

Case 1: For $t\in[\zeta_1,\chi]$, (where $\chi\in\left[\zeta_1+h\frac{\zeta_2-\zeta_1}{2},\zeta_2-h\frac{\zeta_2-\zeta_1}{2}\right]$ for $h\in[0,1]$) we have

$$G(t)=t-\left(\zeta_1+h\frac{\zeta_2-\zeta_1}{2}\right)-(1-h)\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right)$$

which is a continuous and increasing function defined on closed interval $[\zeta_1,\chi]$, so it must attain its maximum (or supremum) at χ, $i.\ e.$,

$$\max_{t\in[\zeta_1,\chi]}G(t)=x-\left(\zeta_1+h\frac{\zeta_2-\zeta_1}{2}\right)-(1-h)\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right),$$

after some simplification we finally get

$$\max_{t\in[\zeta_1,\chi]}G(t)=(\chi-\zeta_2)h+\frac{\zeta_2-\zeta_1}{2}.$$

Case 2: For $t\in(\chi,\zeta_2]$, we have

$$H(t)=t-\left(\zeta_2-h\frac{\zeta_2-\zeta_1}{2}\right)-(1-h)\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right)$$

which is again a continuous and increasing function defined on semi-closed interval $(\chi,\zeta_2]$ and it must attain its maximum (or supremum) at b, i. e.,

$$\max_{t\in(\chi,\zeta_2]}H(t)=\zeta_2-\left(\zeta_2-h\frac{\zeta_2-\zeta_1}{2}\right)-(1-h)\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right),$$

after some simplification, we get:

$$\max_{t\in(\chi,\zeta_2]}H(t)=(\chi-\zeta_1)h+\frac{\zeta_1+\zeta_2}{2}-\chi.$$

□

In order to check validity of our proposed result, we would like to verify Theorem 3.1 through an example.

Example 3.2. *Let $f(x)=x^2$ on the interval $[0,2]$. Then $f'(x)=2x$, so $\gamma=0$ and $\Gamma=4$. The slope $S=\frac{f(2)-f(0)}{2-0}=\frac{4}{2}=2$. The integral mean is $\dfrac{1}{2}\displaystyle\int_0^2 f(t)dt=\dfrac{4}{3}$.*

For Theorem 3.1, the left-hand side of (3.13) becomes:

$$E(x,h)=(1-h)(x^2-2x+2)+2h-\frac{4}{3}.$$

where $x \in [h, 2-h]$ *and* $h \in [0,1]$*. The right-hand side bounds are:*

$$\max\{|(x-2)h+1|, |x(h-1)+1|\}(2-0)$$

and

$$\max\{|(x-2)h+1|, |x(h-1)+1|\}(4-2)$$

which are equal. For all tested $h \in \{0, 0.25, 0.5, 0.75, 1.0\}$*, the inequality*

$$|E(x,h)| \leq 2\max\{|(x-2)h+1|, |x(h-1)+1|\}$$

holds (see Figure 1*). This verifies Theorem* 3.1 *for* $f(x) = x^2$*.*

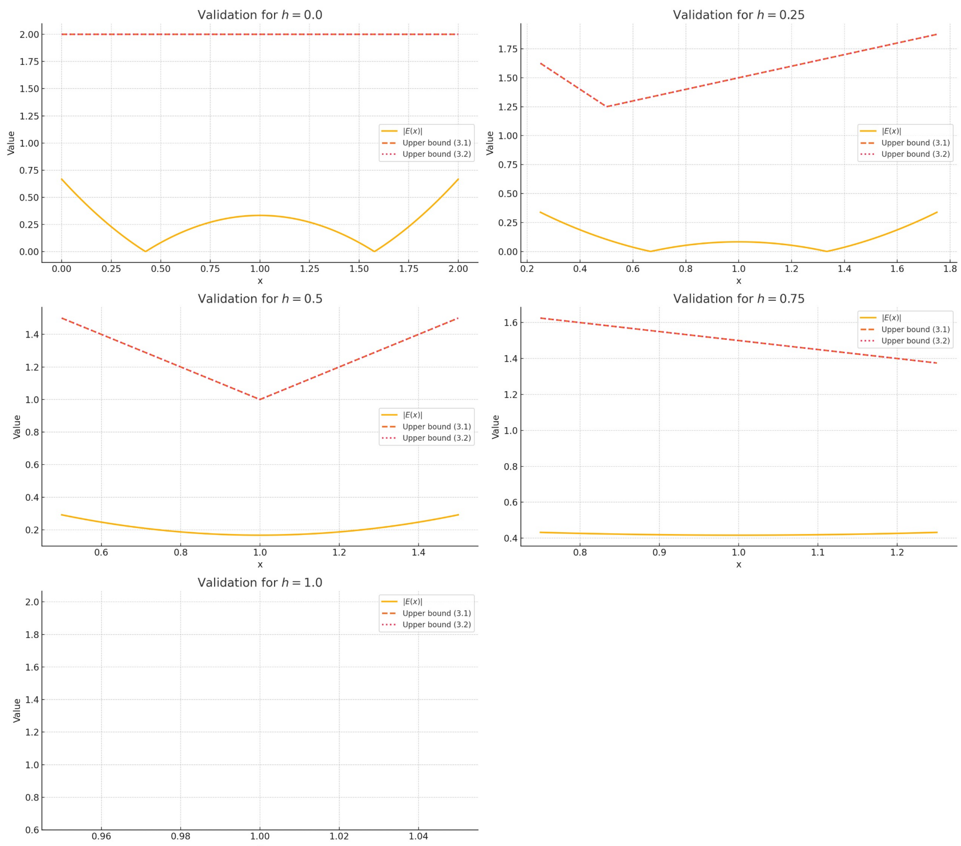

FIGURE 1. Validation plots for different h values for the function $f(x) = x^2$ on the interval $[0, 2]$.

Remark 3.3. *Similar validation holds for* $f(x) = e^x$*,* $\sin(x)$*,* $\ln(x)$*,* x^3 *etc. on suitable intervals (details omitted for brevity).*

Now we state some remarks and corollaries of Theorem 3.1 as under:

Remark 3.4. *By putting $h = 0$ in* (3.13) *and* (3.14)*, we get exactly* (1.5) *and* (1.6) *respectively, $i.$ e.,*

$$\left|f(\chi) - \left(\chi - \frac{\zeta_1 + \zeta_2}{2}\right) S - \frac{1}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right|$$
$$\leq \max\left\{\left|\frac{\zeta_2 - \zeta_1}{2}\right|, \left|\frac{\zeta_1 + \zeta_2}{2} - \chi\right|\right\}(S - \gamma) = \frac{\zeta_2 - \zeta_1}{2}(S - \gamma). \quad (3.\ 19)$$

Because for all values of $\chi \in [\zeta_1, \zeta_2]$, clearly $\left|\frac{\zeta_2-\zeta_1}{2}\right| \geq \left|\frac{\zeta_1+\zeta_2}{2} - \chi\right|$.

Similarly

$$\left|f(\chi) - \left(\chi - \frac{\zeta_1 + \zeta_2}{2}\right) S - \frac{1}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right|$$
$$\leq \max\left\{\left|\frac{\zeta_2 - \zeta_1}{2}\right|, \left|\frac{\zeta_1 + \zeta_2}{2} - \chi\right|\right\}(\Gamma - S) = \frac{\zeta_2 - \zeta_1}{2}(\Gamma - S). \quad (3.\ 20)$$

As a consequences we get all corollaries and remarks of Theorem 1.5 *as our special cases except Remark* 1 *and Theorem* 5 *of* [16]*, as in these results the author used constant* $1/8$ *which cannot be obtained form his main result. The best what we can get from Theorem* 1.5 *is constant* $1/4$*. For our justification we have a simple scheme, if we add the inequalities* (1.5) *and* (1.6) *we get constant* $1/4$ *not* $1/8$*. This can be seen in the following inequality:*

$$\left|f(\chi) - \frac{1}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt - \left(\chi - \frac{\zeta_1 + \zeta_2}{2}\right) S\right| \leq \frac{\zeta_2 - \zeta_1}{4}(\Gamma - \gamma).$$

Remark 3.5. *By putting $h = 1$ in* (3.13) *and* (3.14) *(in this case $\chi = \frac{\zeta_1+\zeta_2}{2}$), we get:*

$$\left|\frac{f(\zeta_1) + f(\zeta_2)}{2} - \frac{1}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right| \leq \frac{\zeta_2 - \zeta_1}{2}(S - \gamma),$$

and

$$\left|\frac{f(\zeta_1) + f(\zeta_2)}{2} - \frac{1}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right| \leq \frac{\zeta_2 - \zeta_1}{2}(\Gamma - S),$$

which are bounds of Hermite−Hadamard right inequality (or trapezoidal inequality)(for details of Hermite−Hadamard dual inequality kindly see [12]*). In this case we also get special case of Theorem* 1.9 *for $h = 0$.*

Corollary 3.6. *Let the assumptions of Theorem* 3.1 *be valid. Then we have:*

$$\left|(1-h)\left(f(\chi) - \left(\chi - \frac{\zeta_1 + \zeta_2}{2}\right) S\right) + h\frac{f(\zeta_1) + f(\zeta_2)}{2} - \frac{1}{\zeta_2 - \zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right|$$
$$\leq \frac{1}{2}\max\left\{\left|(\chi - \zeta_2)h + \frac{\zeta_2 - \zeta_1}{2}\right|, \left|(\chi - \zeta_1)h + \left(\frac{\zeta_1 + \zeta_2}{2} - \chi\right)\right|\right\}(\Gamma - \gamma), \quad (3.\ 21)$$

where $\chi \in \left[\zeta_1 + h\frac{\zeta_2-\zeta_1}{2}, \zeta_2 - h\frac{\zeta_2-\zeta_1}{2}\right]$ and $h \in [0, 1]$.

Proof. We get this result by simply adding inequalities (3.13) and (3.14). □

Remark 3.7. *If we put* $h = 0$ *in Corollary* 3.6*, we get Theorem* 1.1*. Further by putting different values of* h *and* χ *we get various results of high importance particulary at* $h = 1, 1/2, 1/3, \dots$ *and choosing* $\chi \in \{\zeta_1, \zeta_2, \frac{\zeta_1+\zeta_2}{2}, \dots\}$ *etc. Here we highlight one important result of* [16] *as our special case which may be stated as*

$$\left| \frac{f(\zeta_1) + f(\zeta_2)}{2} - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(t)dt \right| \leq \frac{\zeta_2 - \zeta_1}{4} (f'(\zeta_2) - f'(\zeta_1)).$$

The result can easily be obtained by simply putting $\Gamma - \gamma = f'(\zeta_2) - f'(\zeta_1)$ *in* (3.21) *with* $h = 1$ *(in* [16] *author got* $1/8$ *constant mistakenly, here it must be* $1/4$*).*

Remark 3.8. *Under the assumptions of Theorem* 3.1*, if we put* $\chi = \frac{\zeta_1+\zeta_2}{2}$ *in* (3.13) *and* (3.14)*, then we get respectively:*

$$\left| (1-h) f\left(\frac{\zeta_1+\zeta_2}{2}\right) + h \frac{f(\zeta_1) + f(\zeta_2)}{2} - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(t)dt \right| \leq \frac{\zeta_2 - \zeta_1}{2} \max\{h, 1-h\} (S - \gamma), \quad (3.\ 22)$$

and

$$\left| (1-h) f\left(\frac{\zeta_1+\zeta_2}{2}\right) + h \frac{f(\zeta_1) + f(\zeta_2)}{2} - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(t)dt \right| \leq \frac{\zeta_2 - \zeta_1}{2} \max\{h, 1-h\} (\Gamma - S), \quad (3.\ 23)$$

where $h \in [0, 1]$.

Remark 3.9. *If we add inequalities* (3.22) *and* (3.23)*, then we get:*

$$\left| (1-h) f\left(\frac{\zeta_1+\zeta_2}{2}\right) + h \frac{f(\zeta_1) + f(\zeta_2)}{2} - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(t)dt \right| \leq \frac{\zeta_2 - \zeta_1}{4} \max\{h, 1-h\} (\Gamma - \gamma). \quad (3.\ 24)$$

Remark 3.10. *If we put* $h = 0$ *in* (3.22)*,* (3.23) *and* (3.24)*, then we get respectively:*

$$\left| f\left(\frac{\zeta_1+\zeta_2}{2}\right) - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(t)dt \right| \leq \frac{\zeta_2 - \zeta_1}{2} (S - \gamma),$$

$$\left| f\left(\frac{\zeta_1+\zeta_2}{2}\right) - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(t)dt \right| \leq \frac{\zeta_2 - \zeta_1}{2} (\Gamma - S),$$

and

$$\left| f\left(\frac{\zeta_1+\zeta_2}{2}\right) - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(t)dt \right| \leq \frac{\zeta_2 - \zeta_1}{4} (\Gamma - \gamma),$$

which are bounds of Hermite−Hadamard left inequality (or mid-point inequality).

Remark 3.11. *If we put* $h = 1/2$ *in* (3.22), (3.23) *and* (3.24) *we get respectively:*

$$\left|\frac{1}{2}\left(f\left(\frac{\zeta_1+\zeta_2}{2}\right)+\frac{f(\zeta_1)+f(\zeta_2)}{2}\right)-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right| \leq \frac{\zeta_2-\zeta_1}{4}(S-\gamma),$$

$$\left|\frac{1}{2}\left(f\left(\frac{\zeta_1+\zeta_2}{2}\right)+\frac{f(\zeta_1)+f(\zeta_2)}{2}\right)-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right| \leq \frac{\zeta_2-\zeta_1}{4}(\Gamma-S),$$

and

$$\left|\frac{1}{2}\left(f\left(\frac{\zeta_1+\zeta_2}{2}\right)+\frac{f(\zeta_1)+f(\zeta_2)}{2}\right)-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right| \leq \frac{\zeta_2-\zeta_1}{8}(\Gamma-\gamma),$$

which are average of mid-point and trapezoidal inequality.

Remark 3.12. *If we put* $h = 1/3$*, in* (3.22), (3.23) *and* (3.24) *we get:*

$$\left|\frac{1}{6}\left[f(\zeta_1)+4f\left(\frac{\zeta_1+\zeta_2}{2}\right)+f(\zeta_2)\right]-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right| \leq \frac{\zeta_2-\zeta_1}{3}(S-\gamma),$$

$$\left|\frac{1}{6}\left[f(\zeta_1)+4f\left(\frac{\zeta_1+\zeta_2}{2}\right)+f(\zeta_2)\right]-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right| \leq \frac{\zeta_2-\zeta_1}{3}(\Gamma-S),$$

and

$$\left|\frac{1}{6}\left[f(\zeta_1)+4f\left(\frac{\zeta_1+\zeta_2}{2}\right)+f(\zeta_2)\right]-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right| \leq \frac{\zeta_2-\zeta_1}{6}(\Gamma-\gamma).$$

These inequalities are called Simpson's inequality.

Corollary 3.13. *Under the assumptions of Theorem* 3.1*, we have the inequalities:*

$$\begin{aligned}&\left|(1-h)\frac{f\left(a+h\frac{\zeta_2-\zeta_1}{2}\right)+f\left(b-h\frac{\zeta_2-\zeta_1}{2}\right)}{2}+h\frac{f(\zeta_1)+f(\zeta_2)}{2}\right.\\&\qquad\left.-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right|\\&\leq \frac{\zeta_2-\zeta_1}{4}\left[(h-1)^2+h+\max\{1-h^2,|(h-1)^2-h|\}\right](S-\gamma), \quad (3.\ 25)\end{aligned}$$

and

$$\begin{aligned}&\left|(1-h)\frac{f\left(a+h\frac{\zeta_2-\zeta_1}{2}\right)+f\left(b-h\frac{\zeta_2-\zeta_1}{2}\right)}{2}+h\frac{f(\zeta_1)+f(\zeta_2)}{2}\right.\\&\qquad\left.-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right|\\&\leq \frac{\zeta_2-\zeta_1}{4}\left[(h-1)^2+h+\max\{1-h^2,|(h-1)^2-h|\}\right](\Gamma-S), \quad (3.\ 26)\end{aligned}$$

where $h \in [0,1]$.

Proof. Substituting $\chi = \zeta_1 + h\frac{\zeta_2-\zeta_1}{2}$ and then $\chi = \zeta_2 - h\frac{\zeta_2-\zeta_1}{2}$ in inequality (3.13), we get respectively:

$$\Bigg|(1-h)\left(f\left(\zeta_1 + h\frac{\zeta_2-\zeta_1}{2}\right) + (1-h)\frac{f(\zeta_2)-f(\zeta_1)}{2}\right) + h\frac{f(\zeta_1)+f(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\Bigg| \leq \frac{\zeta_2-\zeta_1}{2}\left((h-1)^2+h\right)(S-\gamma), \quad (3.\ 27)$$

and

$$\Bigg|(1-h)\left(f\left(\zeta_2 - h\frac{\zeta_2-\zeta_1}{2}\right) - (1-h)\frac{f(\zeta_2)-f(\zeta_1)}{2}\right) + h\frac{f(\zeta_1)+f(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\Bigg| \leq \frac{\zeta_2-\zeta_1}{2}\max\left\{1-h^2, \left|(h-1)^2-h\right|\right\}(S-\gamma). \quad (3.\ 28)$$

By adding the inequalities (3.27) and (3.28) and then using the triangular property of absolute value and dividing by two we get our required result.

Since proof of the second inequality is similar so we omit the details. ☐

Remark 3.14. *Corollary* 3.13 *is far more accurate than Corollary* 2.3 *of* [14]. *As when we put* $h = 1$ *in Corollary* 2.3 *of* [14] *we get*

$$\left|\frac{f(\zeta_1)+f(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\right| \leq 0,$$

which is impossible.

Remark 3.15. *If we add both the inequalities* (3.25) *and* (3.26)*, then we have:*

$$\Bigg|(1-h)\frac{f\left(a+h\frac{\zeta_2-\zeta_1}{2}\right)+f\left(b-h\frac{\zeta_2-\zeta_1}{2}\right)}{2} + h\frac{f(\zeta_1)+f(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} f(t)dt\Bigg| \leq \frac{\zeta_2-\zeta_1}{8}\left[(h-1)^2+h+\max\{1-h^2, |(h-1)^2-h|\}\right](\Gamma-\gamma), \quad (3.\ 29)$$

where $h \in [0,1]$.

Remark 3.16. *For Corollary* 3.13 *and Remark* 3.15*, we can state plenty of different results and special cases by choosing different values of* $h \in [0,1]$.

Remark 3.17. *Under the assumptions of Theorem* 3.1, *choosing* $C = \frac{\Gamma+\gamma}{2}$, *in* (3.18)*, then we obtain:*

$$\left|(1-h)\left(f(\chi)-\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right)S\right)+h\frac{f(\zeta_1)+f(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right|$$
$$\leq \frac{\zeta_2-\zeta_1}{2}\max\left\{\left|(\chi-\zeta_2)h+\frac{\zeta_2-\zeta_1}{2}\right|,\left|(\chi-\zeta_1)h+\left(\frac{\zeta_1+\zeta_2}{2}-\chi\right)\right|\right\}(\Gamma-\gamma).$$

Remark 3.18. *If in Remark* 3.17 *we let* $|f'| \leq M$*, in that case for* $\Gamma = M$ *and* $\gamma = -M$*, we have:*

$$\left|(1-h)\left(f(\chi)-\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right)S\right)+h\frac{f(\zeta_1)+f(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right|$$
$$\leq (\zeta_2-\zeta_1)M\max\left\{\left|(\chi-\zeta_2)h+\frac{\zeta_2-\zeta_1}{2}\right|,\left|(\chi-\zeta_1)h+\left(\frac{\zeta_1+\zeta_2}{2}-\chi\right)\right|\right\}.$$

Remark 3.19. *For Remark* 3.17 *we can state numerous distinct results and special cases by choosing different values of* h *and* χ.

Theorem 3.20. *Let* $f : J \to \mathbb{R}$ *be a function such that its first derivative is absolutely continuous and* $f'' \in L_2(\zeta_1, \zeta_2)$ *where* $\zeta_1, \zeta_2 \in J^0$, $\zeta_1 < \zeta_2$*. Then we have:*

$$\left|(1-h)\left(f(\chi)-\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right)S\right)+h\frac{f(\zeta_1)+f(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right|$$
$$\leq \frac{(\zeta_2-\zeta_1)^{1/2}}{\pi}\left[h(1-h)\left(\chi-\frac{\zeta_1+\zeta_2}{2}\right)^2+\frac{1}{12}(\zeta_2-\zeta_1)^2(h^3+(1-h)^3)\right]^{1/2}\times$$
$$\times\|f''\|_2, \quad (3.\ 30)$$

where $\chi \in \left[\zeta_1 + h\frac{\zeta_2-\zeta_1}{2}, \zeta_2 - h\frac{\zeta_2-\zeta_1}{2}\right]$ *for* $h \in [0,1]$.

Proof. For proof kindly see [11]. □

Remark 3.21. *There are two main differences between Theorem* 3.20 *and Theorem* 1.10*, namely in both theorems* χ *belongs to different intervals and here in assumptions absolute continuity of first derivative is used instead of second derivative.*

Remark 3.22. *By putting* $h = 0$ *in* (3.30) *we capture Theorem* 1.6.

Remark 3.23. *By putting* $h = 1$ *in* (3.30) *(in this case* $\chi = \frac{\zeta_1+\zeta_2}{2}$*), we get:*

$$\left|\frac{f(\zeta_1)+f(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right| \leq \frac{(\zeta_2-\zeta_1)^{3/2}}{2\sqrt{3}\pi}\|f''\|_2,$$

which is bound of Hermite−Hadamard right inequality (or trapezoidal inequality).

Remark 3.24. *Under the assumptions of Theorem* 3.20*, if we put* $\chi = \frac{\zeta_1+\zeta_2}{2}$ *in* (3.30)*, then we get:*

$$\left|(1-h)f\left(\frac{\zeta_1+\zeta_2}{2}\right)+h\frac{f(\zeta_1)+f(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right| \leq \frac{(\zeta_2-\zeta_1)^{3/2}}{2\sqrt{3}\pi}(h^3+(1-h)^3)^{1/2}\|f''\|_2. \quad (3.\ 31)$$

Remark 3.25. *If we put* $h = 0$ *in* (3.31)*, then we get:*

$$\left|f\left(\frac{\zeta_1+\zeta_2}{2}\right)-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right| \leq \frac{(\zeta_2-\zeta_1)^{3/2}}{2\sqrt{3}\pi}\|f''\|_2,$$

which is bound of Hermite−Hadamard left inequality (or mid-point inequality).

Remark 3.26. *Further, if we put* $h = 1/2$ *in* (3.31) *we get:*

$$\left|\frac{1}{2}\left(f\left(\frac{\zeta_1+\zeta_2}{2}\right)+\frac{f(\zeta_1)+f(\zeta_2)}{2}\right)-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right| \leq \frac{(\zeta_2-\zeta_1)^{3/2}}{4\sqrt{3}\pi}\|f''\|_2,$$

which is average of mid-point and trapezoidal inequality.

Remark 3.27. *If we put* $h = 1/3$*, in* (3.31)*, then we get following Simpson's inequality:*

$$\left|\frac{1}{6}\left[f(\zeta_1)+4f\left(\frac{\zeta_1+\zeta_2}{2}\right)+f(\zeta_2)\right]-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right| \leq \frac{(\zeta_2-\zeta_1)^{3/2}}{6\pi}\|f''\|_2.$$

Corollary 3.28. *Under the assumptions of Theorem* 3.20*, we have the following inequality:*

$$\left|(1-h)\frac{f\left(a+h\frac{\zeta_2-\zeta_1}{2}\right)+f\left(b-h\frac{\zeta_2-\zeta_1}{2}\right)}{2}+h\frac{f(\zeta_1)+f(\zeta_2)}{2} -\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right| \leq \frac{(\zeta_2-\zeta_1)^{3/2}}{2\sqrt{2}\pi}\left(\frac{1}{3}-h^2(1-h)(2-h)\right)^{1/2}\|f''\|_2, \quad (3.\ 32)$$

where $h \in [0, 1]$.

Remark 3.29. *If we choose* $h = 0$ *or* $h = 1$ *in* (3.32)*, then we get following trapezoid inequality which is better than result given in Remark* 3.23*:*

$$\left|\frac{f(\zeta_1)+f(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}f(t)dt\right| \leq \frac{(\zeta_2-\zeta_1)^{3/2}}{2\sqrt{6}\pi}\|f''\|_2.$$

Remark 3.30. *Numerous results can be derived from* (3.32) *by varying* h *in* $[0, 1]$.

4. Conclusion

In this article, we critically examined prior results related to the Ostrowski−Grüss inequality, identifying numerous inaccuracies and inconsistencies, particularly in [14] and [11]. Through detailed analysis and commentary, we corrected these errors, broadened the assumptions to include absolute, and presented version of key inequalities. These corrected results not only enhance the theoretical framework but also ensure greater accuracy and broader applicability. To verify the validity of our proposed result, we also provided an example along with its graphical illustrations. Special cases such as the Hermite−Hadamard inequalities, Simpson-type inequalities, and trapezoidal inequalities have been systematically derived. We have also captured many results − and, in fact, more accurate versions compared to those − stated in [5], [16], [14], and [11] as special cases of our main results. This work thus contributes significant refinements to the study of Ostrowski−Grüss type inequalities and establishes a stronger foundation for further research and applications in numerical integration and related areas.

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6. Conflict of Interest

The author declares that there is no conflict of interest, as this is a sole-authored article.

7. Author's Contribution

As the sole author of this article, I confirm that the entire work, including conceptualization, methodology, analysis, and writing, is my own contribution.

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