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Addition Theorems for the Product of Two Generalized Hermite Polynomials and Their Consequences

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Abstract. This paper aims to acquire two theorems (believed to be new and not recorded in the literature) for the product of two generalized Hermite polynomials, using suitable generating relations, decomposition of infinite series, Cauchy double-series identity, and series rearrangement technique. Further, we discuss some special cases related to generalized and classical Hermite polynomials.

AMS (MOS) Subject Classification Codes: 33C20; 33C45; 42C05
Key Words: Classical Hermite polynomials; Generalized Hermite polynomials; Generating relations.

1. Introduction and Preliminaries

Indeed, the Hermite polynomials were proposed more than a century ago. These polynomials were defined by Pierre-Simon Laplace in 1810, though in scarcely recognizable form, and studied in detail by Pafnuty Chebyshev in 1859. Since then, they have been recognized as a unique tool in pure and applied mathematics. Chebyshev's work was overlooked and they were named later after Charles Hermite [8], who wrote on these polynomials in 1864, describing them as new. He suggested a class of generalized polynomials that are less known than they should be and than the ordinary case. The Hermite polynomials belong to the system of classical orthogonal polynomials. In 1990, P. R. Subramanian [20] studied a class of Hermite polynomials $H_n(x)$ in the sense that one of the above-mentioned four properties implies the other three. The Hermite polynomials are orthogonal in the interval $(-\infty, \infty)$ or the entire x-axis with respect to the normal distribution $w(x) = e^{-x^2}$. The classical real Hermite polynomials are extensively studied in the mathematics literature. They have found interesting applications in various branches of mathematics, theoretical physics, technology, chemistry, approximation theory and several other mathematical branches. Hermite functions have been an important tool in the development of elementary quantum mechanics as solutions of the quantum non-relativistic harmonic oscillator.

• In the year 1993, Subuhi Khan [10, p.84, Eq.(4.2.8), Eq.(4.2.9)] see also[14, p.60, Eq.(2.9)] defined a generating relation associated with generalized Hermite polynomials in the following form:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left\{ H_{n,\alpha,\beta}(x) \right\} t^n = \exp\left(\frac{xt}{\alpha}\right) \exp\left(-\beta t^2\right); \ \forall \text{ finite values of } \alpha,\ \beta,\ x \text{ and } t, \tag{1. 1}$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n)!}H_{2n,\alpha,\beta}(x)t^{n}= \ e^{-\beta t}\cosh\left(\frac{x\sqrt{t}}{\alpha}\right), \tag{1. 2}$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}H_{2n+1,\alpha,\beta}(x)t^{n}=(t^{\frac{-1}{2}})e^{-\beta t}\sinh\left(\frac{x\sqrt{t}}{\alpha}\right). \tag{1. 3}$$

By using the rather obvious decomposition identity (1. 13) in left hand side of equation(1. 1) and setting $t=iT$ together with Euler's formula, we thus find that

$$\sum_{n=0}^{\infty}\frac{1}{(2n)!}H_{2n,\alpha,\beta}(x)(-1)^{n}\left(T\right)^{2n}+(iT)\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}H_{2n+1,\alpha,\beta}(x)(-1)^{n}\left(T\right)^{2n}=$$

$$=\left[\cos\left(\frac{xT}{\alpha}\right)+i\sin\left(\frac{xT}{\alpha}\right)\right]e^{\beta T^{2}}. \tag{1. 4}$$

If α, β and x are purely real numbers, then equating the real and imaginary parts of equation (1. 4) and setting $T=i\sqrt{t}$ which, by virtue of the various properties of hyperbolic functions, yields the generating relations (1. 2),(1. 3) for even and odd degree polynomials of Subuhi.

• In the year 2019, Shabana Khan [9, p.80, Eq.(8.2.1)] obtained the hypergeometric form of generalized Hermite polynomials earlier defined by Subuhi khan

$$H_{n,\alpha,\beta}(x)=\left(\frac{x}{\alpha}\right)^{n}{}_{2}F_{0}\left[\begin{array}{ll}\frac{-n}{2},\ \frac{-n+1}{2}\ ; & \\ & \frac{-4\alpha^{2}\beta}{x^{2}}\\ -\ ; & \end{array}\right], \tag{1. 5}$$

$$H_{n,\alpha,\beta}(-x)=(-1)^{n}H_{n,\alpha,\beta}(x), \tag{1. 6}$$

$$H_{2n,\alpha,\beta}(-x)=H_{2n,\alpha,\beta}(x), \tag{1. 7}$$

$$H_{2n+1,\alpha,\beta}(-x)=-H_{2n+1,\alpha,\beta}(x). \tag{1. 8}$$

If we set $\alpha=\frac{1}{2}$ and $\beta=1$ in equation (1. 5), we find the classical Hermite polynomials as follows:

$$H_{n,\frac{1}{2},1}(x)=H_{n}(x)=(2x)^{n}{}_{2}F_{0}\left[\begin{array}{ll}\frac{-n}{2},\ \frac{-n+1}{2}\ ; & \\ & \frac{-1}{x^{2}}\\ -\ ; & \end{array}\right]. \tag{1. 9}$$

• Similarly, if we set $\alpha=\frac{1}{2}$ and $\beta=1$ in equation (1. 1) which, leads us readily to a generating relation [16, p.187, Eq.(1)] associated with classical Hermite polynomials

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left\{H_{n}(x)\right\}t^{n}=\exp\left(2xt-t^{2}\right)\ ;\ \forall \text{ finite values of } x \text{ and } t. \tag{1. 10}$$

•Auxiliary result

Consider the following generating relation :

$$\sum_{N=0}^{\infty}\frac{1}{N!}\left\{H_{N,A,B}(z)\right\}T^{N}=\exp\left(\frac{zT}{A}\right)\exp\left(-BT^{2}\right). \tag{1. 11}$$

When we put $B=\omega^2, T=t, A=\frac{\alpha}{\gamma}, z=\frac{y}{\zeta}$ in generating relation(1. 11), we find that

$$\exp\left(\frac{yt\gamma}{\alpha\zeta}\right)\exp(-\omega^2t^2)=\sum_{N=0}^{\infty}\frac{1}{N!}\left\{H_{N,\frac{\alpha}{\gamma},\omega^2}\left(\frac{y}{\zeta}\right)\right\}t^N. \tag{1. 12}$$

- **Decomposition of Unilateral infinite series**[18, p.200, Eq.(1)]

$$\sum_{r=0}^{\infty}\Phi(r)=\sum_{r=0}^{\infty}\Phi(2r)+\sum_{r=0}^{\infty}\Phi(2r+1), \tag{1. 13}$$

provided that each infinite series of both sides is absolutely convergent.

- **Decomposition of finite series identities**

$$\sum_{m=0}^{2n}\Xi(m)=\sum_{m=0}^{n}\Xi(2m)+\sum_{m=0}^{n-1}\Xi(2m+1), \tag{1. 14}$$

$$\sum_{m=0}^{2n+1}\Xi(m)=\sum_{m=0}^{n}\Xi(2m)+\sum_{m=0}^{n}\Xi(2m+1). \tag{1. 15}$$

- **Cauchy's double series identity** [18, p.100, Eq.(1)]**:**

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Psi(n,m)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\Psi(n-m,m), \tag{1. 16}$$

provided that the associated double series are absolutely convergent.

- **Multiple series identities** [18, p.102, Lemma(4), Eq.(16), Eq.(17)]**:**

$$\sum_{n=0}^{\infty}\sum_{m_1,\ldots,m_r=0}^{\infty}\Theta(m_1,\ldots,m_r;n)=\sum_{n=0}^{\infty}\sum_{m_1,\ldots,m_r=0}^{k_1m_1+\cdots+k_rm_r\leq n}\Theta(m_1,\ldots,m_r;n-k_1m_1-\cdots-k_rm_r), \tag{1. 17}$$

where $k_1,\ldots,k_r$ are positive integers and $r\geq 1$.

$$\sum_{n=0}^{\infty}\sum_{m_1,\ldots,m_r=0}^{k_1m_1+\cdots+k_rm_r\leq n}\Psi(m_1,\ldots,m_r;n)=\sum_{n=0}^{\infty}\sum_{m_1,\ldots,m_r=0}^{\infty}\Psi(m_1,\ldots,m_r;n+k_1m_1+\cdots+k_rm_r), \tag{1. 18}$$

where $k_1,\ldots,k_r$ are positive integers and $r\geq 1$.

TABLE 1. $H_n(\pm\frac{1}{2})$

$H_n\left(\frac{1}{2}\right)$	Values	$H_n\left(\frac{-1}{2}\right)$	Values
$H_0\left(\frac{1}{2}\right)$	1	$H_0\left(\frac{-1}{2}\right)$	1
$H_1\left(\frac{1}{2}\right)$	1	$H_1\left(\frac{-1}{2}\right)$	-1
$H_2\left(\frac{1}{2}\right)$	1	$H_2\left(\frac{-1}{2}\right)$	-1
$H_3\left(\frac{1}{2}\right)$	-5	$H_3\left(\frac{-1}{2}\right)$	5
$H_4\left(\frac{1}{2}\right)$	1	$H_4\left(\frac{-1}{2}\right)$	1
$H_5\left(\frac{1}{2}\right)$	41	$H_5\left(\frac{-1}{2}\right)$	-41
$H_6\left(\frac{1}{2}\right)$	31	$H_6\left(\frac{-1}{2}\right)$	31
$H_7\left(\frac{1}{2}\right)$	-461	$H_7\left(\frac{-1}{2}\right)$	461
$H_8\left(\frac{1}{2}\right)$	-895	$H_8\left(\frac{-1}{2}\right)$	-895
$H_9\left(\frac{1}{2}\right)$	6481	$H_9\left(\frac{-1}{2}\right)$	-6481
$H_{10}\left(\frac{1}{2}\right)$	22591	$H_{10}\left(\frac{-1}{2}\right)$	22591

TABLE 2. $H_{n,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)$,

$H_{n,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)$	Values
$H_{0,4,\frac{7}{16}}\left(\frac{2.5}{2}\right)$	1
$H_{1,4,\frac{7}{16}}\left(\frac{2.5}{2}\right)$	0.3125
$H_{2,4,\frac{7}{16}}\left(\frac{2.5}{2}\right)$	-0.777344
$H_{3,4,\frac{7}{16}}\left(\frac{2.5}{2}\right)$	-0.789795
$H_{4,4,\frac{7}{16}}\left(\frac{2.5}{2}\right)$	1.79372
$H_{5,4,\frac{7}{16}}\left(\frac{2.5}{2}\right)$	3.32482
$H_{6,4,\frac{7}{16}}\left(\frac{2.5}{2}\right)$	-6.8085
$H_{7,4,\frac{7}{16}}\left(\frac{2.5}{2}\right)$	-19.583
$H_{8,4,\frac{7}{16}}\left(\frac{2.5}{2}\right)$	35.5824
$H_{9,4,\frac{7}{16}}\left(\frac{2.5}{2}\right)$	148.2
$H_{10,4,\frac{7}{16}}\left(\frac{2.5}{2}\right)$	-233.899

$x = 2.5, \alpha = 4$ and $\beta = \frac{7}{2}$

TABLE 3. $H_{n,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)$

$H_{n,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)$	Values
$H_{0,4,\frac{7}{16}}\left(\frac{3.7}{2}\right)$	1
$H_{1,4,\frac{7}{16}}\left(\frac{3.7}{2}\right)$	0.4625
$H_{2,4,\frac{7}{16}}\left(\frac{3.7}{2}\right)$	-0.661094
$H_{3,4,\frac{7}{16}}\left(\frac{3.7}{2}\right)$	-1.11513
$H_{4,4,\frac{7}{16}}\left(\frac{3.7}{2}\right)$	1.21962
$H_{5,4,\frac{7}{16}}\left(\frac{3.7}{2}\right)$	4.46703
$H_{6,4,\frac{7}{16}}\left(\frac{3.7}{2}\right)$	-3.26985
$H_{7,4,\frac{7}{16}}\left(\frac{3.7}{2}\right)$	-24.9642
$H_{8,4,\frac{7}{16}}\left(\frac{3.7}{2}\right)$	8.48186
$H_{9,4,\frac{7}{16}}\left(\frac{3.7}{2}\right)$	178.672
$H_{10,4,\frac{7}{16}}\left(\frac{3.7}{2}\right)$	15.8414

$y = 3.7, \alpha = 4$ and $\beta = \frac{7}{2}$

TABLE 4. $H_{n,\alpha,\beta}(x+y)$

$H_{n,\alpha,\beta}(x+y)$	Values
$H_{0,4,\frac{7}{2}}(6.2)$	1
$H_{1,4,\frac{7}{2}}(6.2)$	1.55
$H_{2,4,\frac{7}{2}}(6.2)$	-4.5975
$H_{3,4,\frac{7}{2}}(6.2)$	-28.8261
$H_{4,4,\frac{7}{2}}(6.2)$	51.867
$H_{5,4,\frac{7}{2}}(6.2)$	887.525
$H_{6,4,\frac{7}{2}}(6.2)$	-439.681
$H_{7,4,\frac{7}{2}}(6.2)$	-37957.6
$H_{8,4,\frac{7}{2}}(6.2)$	-37289.9
$H_{9,4,\frac{7}{2}}(6.2)$	2.06782×10^6
$H_{10,4,\frac{7}{2}}(6.2)$	5.55439×10^6

$x =$ 2.5, $y = 3.7$, $\alpha = 4$ and $\beta = \frac{7}{2}$

Our present investigation is motivated essentially by several interesting and widespread developments on various families of Orthogonal polynomials scattered in the literature, by (for example) Beals and Wong [1, Ch.(4)(5)], Belafhal[2], Bell[3, Ch.(3)(5)(6)(7)(8)], Brychkov[4], Cesarano, C.[5, 6], Gradshteyn and Ryzhik[7, Ch.(8.9)], Khan and Ali[11], Latif et al. (see [12] and [13]), Qureshi et al. [15], Rainville [16, Ch.(9)(10)], Srivastava et al. (see [17] and [19]), Szegö [21, Ch.(2)(3)], Temme[22, Ch.(6)], Vasileva and Maria [23] .

The article is organized as follows: In **section 2**, we derive an Addition theorem (2. 19) related to generalized

Hermite polynomials with the aid of above mentioned suitable generating relations, decomposition of infinite series, Cauchy double-series identities and series rearrangement technique. In **section 3**, we establish some applications which are directly produced from the Addition theorem (2. 19). In **section 4**, we provide some applications for the product of Classical Hermite Polynomials. Finally, in **section 5**, we procure one more Addition theorem (5. 45).

Remark 1. Any values of numerator and denominator parameters in **sections 2-5** leading to the results which donot make sense are tacitly excluded. It is very interesting to mention here that we have verified the Addition theorems using *MATHEMATICA* software, a general system of doing mathematics by computer.

2. Addition Theorem of the product of two Generalized hermite Polynomials

In this section, we establish a result for the finite sum of the product of two generalized Hermite polynomials as in the following theorem:

Theorem 2. 19 $\forall$ finite values of x, α, β, the following Addition theorem holds true:

$$H_{n,\alpha,\beta}(x+y) = \left(\sqrt{\alpha}\right)^n \sum_{m=0}^{n} \frac{1}{m!}(-1)^m(-n)_m \left\{H_{n-m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\right\}\left\{H_{m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\}. \quad (2.\ 19)$$

Proof of assertion(2. 19)

We are familiar with the following generating relation [10, 14]:

$$\exp\left(\frac{xt}{\alpha} - \beta t^2\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left\{H_{n,\alpha,\beta}(x)\right\} t^n, \quad (2.\ 20)$$

which, upon replacing x by $x+y$ in equation (2. 20), yields

$$\exp\left[\frac{(x+y)t}{\alpha} - \beta t^2\right] = \exp\left[\left(\frac{xt}{\alpha} - \frac{\beta t^2}{2}\right) + \left(\frac{yt}{\alpha} - \frac{\beta t^2}{2}\right)\right] = \sum_{n=0}^{\infty} \frac{1}{n!}\left\{H_{n,\alpha,\beta}(x+y)\right\} t^n, \quad (2.\ 21)$$

$$\text{or } \exp\left[\left(\frac{xt}{\alpha} - \frac{\beta t^2}{2}\right)\exp\left(\frac{yt}{\alpha} - \frac{\beta t^2}{2}\right)\right] = \sum_{n=0}^{\infty} \frac{1}{n!}\left\{H_{n,\alpha,\beta}(x+y)\right\} t^n, \quad (2.\ 22)$$

if x is replaced by $\frac{x}{\sqrt{\alpha}}$, t by $t\sqrt{\alpha}$ and β by $\frac{\beta}{2\alpha}$ in equation(2. 20), we deduce that

$$\exp\left(\frac{xt}{\alpha} - \frac{\beta t^2}{2}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left\{H_{n,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\right\}\left(\sqrt{\alpha}\right)^n t^n. \quad (2.\ 23)$$

If we now replace x by y in the last equation (2. 23), we find that

$$\exp\left(\frac{yt}{\alpha} - \frac{\beta t^2}{2}\right) = \sum_{m=0}^{\infty} \frac{1}{m!}\left\{H_{m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\}\left(\sqrt{\alpha}\right)^m t^m. \quad (2.\ 24)$$

Now using equations (2. 23) and (2. 24), in equation(2. 22), we get

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left\{\, H_{n,\alpha,\beta}(x+y)\right\} t^n = \left[\sum_{n=0}^{\infty} \frac{1}{n!}\left\{\, H_{n,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\right\}\left(\sqrt{\alpha}\right)^n\right] \times$$

$$\times\left[\sum_{m=0}^{\infty} \frac{1}{m!}\left\{\, H_{m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\}\left(\sqrt{\alpha}\right)^m\right] t^{n+m}. \quad (2.\ 25)$$

Replacing n by $(n-m)$ and using Cauchy's double series identity (1. 16) in right hand side of equation (2. 25), we obtain

$$\sum_{n=0}^{\infty}\{\, H_{n,\alpha,\beta}(x+y)\}\, t^n =$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\frac{(\sqrt{\alpha})^n}{m!}\left\{(-1)^m(-n)_m\, H_{n-m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\, H_{m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\}\right\} t^n. \tag{2. 26}$$

Finally, if we equate the coefficients of t^n in equation (2. 26), which leads us readily to the Addition Theorem (2. 19).

Note:- We have verified the addition theorem (2. 19) numerically by using Mathematica program and Tables 2, 3 and 4.

3. Some Consequences of Addition Theorem (2. 19)

Case (i) In the Addition Theorem (2. 19), if we replace n by $2n$, interchange x and y and apply the rather obvious decomposition identity(1. 14) together with equations(1. 7)(1. 8), it takes the following form:

$$H_{2n,\alpha,\beta}(x+y) = \alpha^n\left[\sum_{m=0}^{n}\frac{(-2n)_{2m}}{(2m)!}\left\{H_{2m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\right\}\left\{H_{2n-2m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\} - \right.$$

$$\left. - \sum_{m=0}^{n-1}\frac{(-2n)_{2m+1}}{(2m+1)!}\left\{H_{2m+1,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\right\}\left\{H_{2n-2m-1,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\}\right]. \tag{3. 27}$$

Again if we replace y by $-y$ in last equation (3. 27) and use the hypergeometric forms (1. 7) (1. 8) and after simplification, we find that

$$H_{2n,\alpha,\beta}(x-y) = \alpha^n\left[\sum_{m=0}^{n}\frac{(-2n)_{2m}}{(2m)!}\left\{H_{2m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\right\}\left\{H_{2n-2m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\} + \right.$$

$$\left. + \sum_{m=0}^{n-1}\frac{(-2n)_{2m+1}}{(2m+1)!}\left\{H_{2m+1,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\right\}\left\{H_{2n-2m-1,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\}\right]. \tag{3. 28}$$

Now adding the equations(3. 27) and (3. 28), we find a new result as follows:

$$H_{2n,\alpha,\beta}(x+y) + H_{2n,\alpha,\beta}(x-y) = 2\alpha^n\sum_{m=0}^{n}\frac{(-2n)_{2m}}{(2m)!}\left\{H_{2m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\right\}\left\{H_{2n-2m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\}. \tag{3. 29}$$

Case (ii) Subtracting equation(3. 27) from equation(3. 28), we find one more new result as follows:

$$H_{2n,\alpha,\beta}(x-y) - H_{2n,\alpha,\beta}(x+y) =$$

$$= 2\alpha^n\sum_{m=0}^{n-1}\frac{(-2n)_{2m+1}}{(2m+1)!}\left\{H_{2m+1,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\right\}\left\{H_{2n-2m-1,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\}. \tag{3. 30}$$

Case (iii) Replacing n by $2n+1$ in equation(2. 19) and using decomposition identity(1. 15) together with equations(1. 7)(1. 8), we get

$$H_{2n+1,\alpha,\beta}(x+y) = \alpha^{n+\frac{1}{2}}\left[\sum_{m=0}^{n}\frac{(-2n-1)_{2m}}{(2m)!}\left\{H_{2n-2m+1,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\right\}\left\{H_{2m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\} - \right.$$
$$\left. - \sum_{m=0}^{n}\frac{(-2n-1)_{2m+1}}{(2m+1)!}\left\{H_{2n-2m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\right\}\left\{H_{2m+1,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\}\right]. \quad (3.\ 31)$$

Replacing y by $-y$ in equation(3. 31) and using equations(1. 7)(1. 8) after simplification, we get

$$H_{2n+1,\alpha,\beta}(x-y) = \alpha^{n+\frac{1}{2}}\left[\sum_{m=0}^{n}\frac{(-2n-1)_{2m}}{(2m)!}\left\{H_{2n-2m+1,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\right\}\left\{H_{2m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\} + \right.$$
$$\left. + \sum_{m=0}^{n}\frac{(-2n-1)_{2m+1}}{(2m+1)!}\left\{H_{2n-2m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\right\}\left\{H_{2m+1,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\}\right]. \quad (3.\ 32)$$

After adding equations (3. 31) and (3. 32), we get the new result as follows:

$$H_{2n+1,\alpha,\beta}(x+y) + H_{2n+1,\alpha,\beta}(x-y) =$$
$$= 2\alpha^{n+\frac{1}{2}}\sum_{m=0}^{n}\frac{(-2n-1)_{2m}}{(2m)!}\left\{H_{2n-2m+1,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\right\}\left\{H_{2m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\}. \quad (3.\ 33)$$

Case (iv) Subtracting equation(3. 31) from equation(3. 32), we get the new result

$$H_{2n+1,\alpha,\beta}(x-y) - H_{2n+1,\alpha,\beta}(x+y) =$$
$$= 2\alpha^{n+\frac{1}{2}}\sum_{m=0}^{n}\frac{(-2n-1)_{2m+1}}{(2m+1)!}\left\{H_{2n-2m,\alpha,\frac{\beta}{2\alpha}}\left(\frac{x}{\sqrt{\alpha}}\right)\right\}\left\{H_{2m+1,\alpha,\frac{\beta}{2\alpha}}\left(\frac{y}{\sqrt{\alpha}}\right)\right\}. \quad (3.\ 34)$$

4. Some Consequences Related with Classical Hermite Polynomials

Case (v) Put $\beta = 1$ and $\alpha = \frac{1}{2}$ in equation (2. 19), we get the following result:

$$\left(\sqrt{2}\right)^{n} H_n(x+y) = \sum_{m=0}^{n}\frac{1}{m!}(-1)^m(-n)_m\ \left\{H_{n-m}\left(x\sqrt{2}\right)\right\}\ \left\{H_m\left(y\sqrt{2}\right)\right\}. \quad (4.\ 35)$$

• Now replacing x by $\frac{x}{\sqrt{2}}$ and y by $\frac{y}{\sqrt{2}}$ in equation (5. 52), we get

$$\left(\sqrt{2}\right)^{n} H_n\left(\frac{x+y}{\sqrt{2}}\right) = \sum_{m=0}^{n}\frac{1}{m!}(-1)^m(-n)_m\ \{H_{n-m}(x)\}\ \{H_m(y)\}. \quad (4.\ 36)$$

• In equations(3. 27)-(3. 34) put $\alpha = \frac{1}{2}$, $\beta = 1$ and with the aid of the property (1. 9), we get the following new results:

$$2^n H_{2n}(x+y) = \sum_{m=0}^{n}\frac{(-2n)_{2m}}{(2m)!}\left\{H_{2m}\left(x\sqrt{2}\right)\right\}\left\{H_{2n-2m-1}\left(y\sqrt{2}\right)\right\} -$$

$$-\sum_{m=0}^{n-1}\frac{(-2n)_{2m+1}}{(2m+1)!}\left\{H_{2m+1}\left(x\sqrt{2}\right)\right\}\left\{H_{2n-2m-1}\left(y\sqrt{2}\right)\right\}, \tag{4. 37}$$

$$2^nH_{2n}(x-y)=\sum_{m=0}^{n}\frac{(-2n)_{2m}}{(2m)!}\left\{H_{2m}\left(x\sqrt{2}\right)\right\}\left\{H_{2n-2m}\left(y\sqrt{2}\right)\right\}+$$

$$+\sum_{m=0}^{n-1}\frac{(-2n)_{2m+1}}{(2m+1)!}\left\{H_{2m+1}\left(x\sqrt{2}\right)\right\}\left\{H_{2n-2m-1}\left(y\sqrt{2}\right)\right\}, \tag{4. 38}$$

$$2^{(n-1)}\left\{H_{2n}(x+y)+H_{2n}(x-y)\right\}=\sum_{m=0}^{n}\left(\begin{array}{c}2n\\2m\end{array}\right)\left\{H_{2m}(x\sqrt{2})\right\}\left\{H_{2n-2m}(y\sqrt{2})\right\}, \tag{4. 39}$$

$$2^{(n-1)}[H_{2n}(x-y)-H_{2n}(x+y)]=\sum_{m=0}^{n-1}\frac{(-2n)_{2m+1}}{(2m+1)!}\left\{H_{2m+1}\left(x\sqrt{2}\right)\right\}\left\{H_{2n-2m-1}\left(y\sqrt{2}\right)\right\}, \tag{4. 40}$$

$$(\sqrt{2})^{2n+1}H_{2n+1}(x+y)=\sum_{m=0}^{n}\frac{(-2n-1)_{2m}}{(2m)!}\left\{H_{2n-2m+1}\left(x\sqrt{2}\right)\right\}\left\{H_{2m}\left(y\sqrt{2}\right)\right\}-$$

$$-\sum_{m=0}^{n}\frac{(-2n-1)_{2m+1}}{(2m+1)!}\left\{H_{2n-2m}\left(x\sqrt{2}\right)\right\}\left\{H_{2m+1}\left(y\sqrt{2}\right)\right\}, \tag{4. 41}$$

$$(\sqrt{2})^{2n+1}H_{2n+1}(x-y)=\sum_{m=0}^{n}\frac{(-2n-1)_{2m}}{(2m)!}\left\{H_{2n-2m+1}\left(x\sqrt{2}\right)\right\}\left\{H_{2m}\left(y\sqrt{2}\right)\right\}+$$

$$+\sum_{m=0}^{n}\frac{(-2n-1)_{2m+1}}{(2m+1)!}\left\{H_{2n-2m}\left(x\sqrt{2}\right)\right\}\left\{H_{2m+1}\left(y\sqrt{2}\right)\right\}, \tag{4. 42}$$

$$2^{\left(n-\frac{1}{2}\right)}[H_{2n+1}(x+y)+H_{2n+1}(x-y)]=\sum_{m=0}^{n}\frac{(-2n-1)_{2m}}{(2m)!}\left\{H_{2n-2m+1}\left(x\sqrt{2}\right)\right\}\left\{H_{2m}\left(y\sqrt{2}\right)\right\}, \tag{4. 43}$$

$$2^{\left(n-\frac{1}{2}\right)}[H_{2n+1}(x-y)-H_{2n+1}(x+y)]=\sum_{m=0}^{n}\frac{(-2n-1)_{2m+1}}{(2m+1)!}\left\{H_{2n-2m}\left(x\sqrt{2}\right)\right\}\left\{H_{2m+1}\left(y\sqrt{2}\right)\right\}. \tag{4. 44}$$

5. Finite Sum of the Product of Generalized Hermite Polynomials

In this section, we establish a result for the finite sum of the product of generalized Hermite polynomials, as in the following theorem:

Theorem 5. 45 $\forall\,\mu,\sigma\in\mathbb{R}^+$, the following Addition theorem holds true:

$$\frac{1}{N!}\left\{H_{\left(N,\alpha,\omega^2+\sum\limits_{i=1}^{r}(\mu_i^2)\right)}\left(\frac{\gamma y+\sum\limits_{i=1}^{r}\lambda_i y_i}{\sqrt[m]{\sigma_1^{p_1}+\sigma_2^{p_2}+\cdots+\sigma_q^{p_q}}}\right)\right\}=\sum_{m_1,m_2,\ldots,m_r=0}^{m_1+m_2+\cdots+m_r\leq N}\frac{1}{\left(N-\sum\limits_{i=1}^{r}m_i\right)!\prod\limits_{i=1}^{r}(m_i)!}\times$$

$$\times\left\{H_{\left(N-\sum\limits_{i=1}^{r}m_i,\frac{\alpha}{\gamma},\omega^2\right)}\left(\frac{y}{\sqrt[m]{\sigma_1^{p_1}+\sigma_2^{p_2}+\cdots+\sigma_q^{p_q}}}\right)\right\}\left\{\prod_{j=1}^{r}H_{\left(m_j,\frac{\alpha}{\lambda_j},\mu_j^2\right)}\left(\frac{y_j}{\sqrt[m]{\sigma_1^{p_1}+\sigma_2^{p_2}+\cdots+\sigma_q^{p_q}}}\right)\right\} \tag{5. 45}$$

Proof of result (5. 45)

Consider the following generating relation:

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left\{H_{n,\alpha,\beta}(x)\right\}t^n=\exp\left(\frac{xt}{\alpha}\right)\exp\left(-\beta t^2\right);\ \forall \text{ finite values of } \alpha,\ \beta,\ x \text{ and } t, \tag{5. 46}$$

put $x=\dfrac{\gamma y+\sum\limits_{i=1}^{r}\lambda_i y_i}{\zeta}$ and $\beta=\omega^2+\sum\limits_{i=1}^{r}\mu_i^2,$ in both sides of the Subuhi generating function (5. 46),

where $\zeta=\sqrt[m]{\sigma_1^{p_1}+\sigma_2^{p_2}+\cdots+\sigma_q^{p_q}}$; $m,p_1,p_2,\ldots,p_q\in\mathbb{R}^+$ and γ, λ_i, ω, μ_i,σ_i are constants

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left\{H_{n,\alpha,\omega^2+\sum\limits_{i=1}^{r}\mu_i^2}\left(\frac{\gamma y+\sum\limits_{i=1}^{r}\lambda_i y_i}{\zeta}\right)\right\}t^n=\exp\left(\frac{\gamma yt+\sum\limits_{i=1}^{r}\lambda_i y_i t}{\alpha\zeta}\right)\exp\left(-\omega^2t^2-\sum_{i=1}^{r}\mu_i^2t^2\right) \tag{5. 47}$$

$$=\exp\left(\frac{\gamma yt}{\alpha\zeta}\right)\exp\left(-\omega^2t^2\right)\exp\left(\frac{\lambda_1y_1t}{\alpha\zeta}\right)\exp\left(-\mu_1^2t^2\right)\ldots\exp\left(\frac{\lambda_ry_rt}{\alpha\zeta}\right)\exp\left(-\mu_r^2t^2\right). \tag{5. 48}$$

Using the result (1. 12) in right hand side of equation (5. 48), we get

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left\{H_{n,\alpha,\omega^2+\sum\limits_{i=1}^{r}\mu_i^2}\left(\frac{\gamma y+\sum\limits_{i=1}^{r}\lambda_i y_i}{\zeta}\right)\right\}t^n=$$

$$=\sum_{N=0}^{\infty}\frac{1}{N!}\left\{H_{N,\frac{\alpha}{r},\omega^2}\left(\frac{y}{\zeta}\right)\right\}t^N\sum_{m_1=0}^{\infty}\frac{1}{(m_1)!}\left\{H_{m_1,\frac{\alpha}{\lambda_1},\mu_1^2}\left(\frac{y_1}{\zeta}\right)\right\}t^{m_1}\times$$

$$\times\sum_{m_2=0}^{\infty}\frac{1}{(m_2)!}\left\{H_{m_2,\frac{\alpha}{\lambda_2},\mu_2^2}\left(\frac{y_2}{\zeta}\right)\right\}t^{m_2}\cdots\sum_{m_r=0}^{\infty}\frac{1}{(m_r)!}\left\{H_{m_r,\frac{\alpha}{\lambda_r},\mu_r^2}\left(\frac{y_r}{\zeta}\right)\right\}t^{m_r}, \tag{5. 49}$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left\{H_{n,\alpha,\omega^2+\sum\limits_{i=1}^{r}\mu_i^2}\left(\frac{\gamma y+\sum\limits_{i=1}^{r}\lambda_i y_i}{\zeta}\right)\right\}t^n=$$

$$=\sum_{N=0}^{\infty}\sum_{m_1,m_2,\ldots,m_r=0}^{\infty}\left\{\frac{1}{N!}H_{N,\frac{\alpha}{\gamma},\omega^2}\left(\frac{y}{\zeta}\right)\right\}\left\{\frac{1}{(m_1)!}H_{m_1,\frac{\alpha}{\lambda_1},\mu_1^2}\left(\frac{y_1}{\zeta}\right)\right\}\cdots$$

$$\cdots\left\{\frac{1}{(m_r)!}H_{m_r,\frac{\alpha}{\lambda_r},\mu_r^2}\left(\frac{y_r}{\zeta}\right)\right\}t^{N+m_1+m_2+\cdots+m_r}. \tag{5. 50}$$

Now replacing N by $N-\sum\limits_{i=1}^{r} m_i$ and using the multiple series identity (1. 17) in right hand side of equation (5. 50), we get

$$\sum_{N=0}^{\infty}\frac{1}{N!}\left\{H_{\left(N,\alpha,\omega^2+\sum\limits_{i=1}^{r}\mu_i^2\right)}\left(\frac{\gamma y+\sum\limits_{i=1}^{r}\lambda_i y_i}{\zeta}\right)\right\}t^N=$$

$$=\sum_{N=0}^{\infty}\left[\sum_{m_1,m_2,\dots,m_r=0}^{m_1+m_2+\cdots+m_r\leq N}\frac{1}{\left(N-\sum\limits_{i=1}^{r}m_i\right)!\prod_{i=1}^{r}(m_i)!}\left\{H_{\left(N-\sum\limits_{i=1}^{r}m_i,\frac{\alpha}{r},\omega^2\right)}\left(\frac{y}{\zeta}\right)\right\}\left\{\prod_{j=1}^{r}H_{\left(m_j,\frac{\alpha}{\lambda_j},\mu_j^2\right)}\left(\frac{y_j}{\zeta}\right)\right\}t^N\right. . \tag{5. 51}$$

Finally, equating the coefficients of t^N in equation (5. 51), we arrive at the result (5. 45).

Remark 2. With a view to encouraging and motivating further researches emerging from the present investigation, we have chosen to draw the attention of the interested readers toward some related development to establish a generalized result for the finite sum of the product of several classical Hermite polynomials as in the following form :

$$\frac{\left(\alpha_1^2+\alpha_2^2+\cdots+\alpha_r^2\right)^{\frac{n}{2}}}{n!}H_n\left(\frac{\sum\limits_{i=1}^{r}(\alpha_i x_i)}{\sqrt{\sum\limits_{i=1}^{r}(\alpha_i^2)}}\right)$$

$$=\sum_{m_1,m_2,\dots,m_{r-1}=0}^{m_1+m_2+\cdots+m_{r-1}\leq n}\frac{(\alpha_r)^{\left(n-\sum\limits_{i=1}^{r-1}m_i\right)}}{\left(n-\sum\limits_{i=1}^{r-1}m_i\right)!}H_{\left(n-\sum\limits_{i=1}^{r-1}m_i\right)}(x_r)\prod_{j=1}^{r-1}\left\{\frac{(\alpha_j)^{m_j}}{(m_j)!}H_{m_j}(x_j)\right\}. \tag{5. 52}$$

6. CONCLUSION

To conclude our present investigation in above article, we recognize that finite sums of the product of two generalized Hermite polynomials, and classical Hermite polynomials are obtained by using suitable generating relations, decomposition of infinite series, Cauchy double-series identity, and series rearrangement technique. Moreover, various new particular cases are derived from the Addition theorem (2. 19). Further, on using new generating functions for other types of hypergeometric polynomials, like Laguerre polynomials for $r\in\mathbb{N}$, the following Addition theorem holds true:

$$L_n^{(\gamma+r+\beta_1+\beta_2+\cdots+\beta_r)}\left(y+\sum_{i=1}^{r}x_i\right)=\sum_{m_1,m_2,\dots,m_r=0}^{(m_1+m_2+\cdots+m_r)\leq n}\left\{L^{(\gamma)}_{n-\sum\limits_{i=1}^{r}m_i}(y)\right\}\prod_{j=1}^{r}\{L_{m_j}^{(\beta_j)}(x_j)\}, \tag{6. 53}$$

where $\min\{\Re(\gamma),\Re(\beta_1),\Re(\beta_2),\dots,\Re(\beta_r)\}>-1$.

Similarly, on using other polynomials like Legendre, Chebyshev, Sonine polynomials, similar types of Addition theorems can be derived in an analogous manner. Besides the derived results are quite significant, we believe that these would be able to find wide range of applications and advantages in obtaining the exact addition theorems in place of already provided mathematical expressions scattered in the fields of Applied

Mathematics and Engineering sciences to improve the accuracy.
Conflicts of Interests: The authors declare that they have no conflicts of interest.

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