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Rings Whose Invertible Elements Are Weakly Nil-Clean

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Abstract. We study those rings in which all invertible elements are weakly nil-clean, calling them *UWNC rings*. This somewhat extends results due to Karimi-Mansoub et al. in Contemp. Math. (2018), where rings in which all invertible elements are nil-clean were considered abbreviating them as *UNC rings*. Specifically, our main achievements are that the triangular matrix ring $\mathrm{T}_n(R)$ over a ring R is UWNC precisely when R is UNC. Besides, the notions UWNC and UNC do coincide when $2 \in J(R)$. We also describe UWNC 2-primal rings R by proving that R is a ring with $J(R) = \mathrm{Nil}(R)$ such that $U(R) = \pm 1 + \mathrm{Nil}(R)$. In particular, the polynomial ring $R[x]$ over some arbitrary variable x is UWNC exactly when R is UWNC. Likewise, we furthermore apply the obtained results to group rings showing that if G is a locally finite p-group and R is a UWNC ring such that the prime p is a nilpotent in R, then RG is too a UWNC ring.

Some other relevant assertions are proved in the present direction as well.

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UU-Ring.

1. Introduction and Major Concepts

Everywhere in the current paper, let R be an associative but *not* necessarily a commutative ring having an identity element, usually stated as 1. Standardly, for such a ring R, the letters $U(R)$, $\mathrm{Nil(R)}$ and $\mathrm{Id}(R)$ are designed for the set of invertible elements (also termed as the unit group of R), the set of nilpotent elements and the set of idempotent elements in R, respectively. Likewise, $J(R)$ denotes the Jacobson radical of R, and $\mathrm{Z}(R)$ denotes the center of R. The ring of $n \times n$ matrices over R and the ring of $n \times n$ upper triangular matrices over R are denoted by $\mathrm{M}_n(R)$ and $\mathrm{T}_n(R)$, respectively. Traditionally, a ring is said to be *abelian* if each of its idempotents is central, that is, $\mathrm{Id}(R) \subseteq \mathrm{Z}(R)$.

For all other unexplained explicitly notions and notations, we refer to the classical source [15] or to the cited in the bibliography research sources. However, for completeness of the exposition and for the reader's convenience, we recall the following basic notions.

Definition 1.1 ([11]). *Let R be a ring. An element $r \in R$ is said to be nil-clean if there is an idempotent $e \in R$ and a nilpotent $b \in R$ such that $r = e + b$. Such an element r is further called strongly nil-clean if the existing idempotent and nilpotent can be chosen such that $be = eb$. A ring is called nil-clean (respectively, strongly nil-clean) if each of its elements is nil-clean (respectively, strongly nil-clean).*

Definition 1.2 ([3],[6]). *A ring R is said to be weakly nil-clean provided that, for any $a \in R$, there exists an idempotent $e \in R$ such that $a - e$ or $a + e$ is nilpotent. A ring R is said to be strongly weakly nil-clean provided that, for any $a \in R$, a or $-a$ is strongly nil-clean.*

Definition 1.3 ([4],[9]). *A ring is called UU if all of its units are unipotent,that is, $U(R) \subseteq 1 + \mathrm{Nil}(R)$ (and so $1 + \mathrm{Nil}(R) = U(R)$).*

Definition 1.4 ([9]). *A ring R is called weakly UU, and abbreviated as WUU for short, if $U(R) = \mathrm{Nil}(R) \pm 1$. This is equivalent to the condition that every unit can be presented as either $n + 1$ or $n - 1$, where $n \in \mathrm{Nil}(R)$.*

Definition 1.5 ([12]). *A ring R is called UNC if every of its units is nil-clean.*

Our key working instrument is the following one.

Definition 1.6. *A ring R is called UWNC if each unit of R is weakly nil-clean.*

A brief retrospection of some more substantial achievements in the subject are as follows: in [12], the authors investigated UNC rings, i.e., those rings whose units are nil-clean. They were motivated by the results proven in [9] and succeeded to establish that in some special cases these rings are completely classified in terms of their fractions.

Our plan here is to expand this investigation substantially by developing a useful for this purpose. In fact, we use some non-standard techniques from ring and matrix theories as well as we also compare and distinguish the established results with these from [9].

On the other hand, in [18], [19] and [20] the author studied a few similar classes of rings which are completely relevant to the present topic. It is established there that in some specific cases these rings are totally characterized in terms of their divisions.

It is also worth noticing that some closely related background material can also be found in the source [16] and the given references therewith.

The next constructions that show some proper ring classes inclusions are worthwhile.

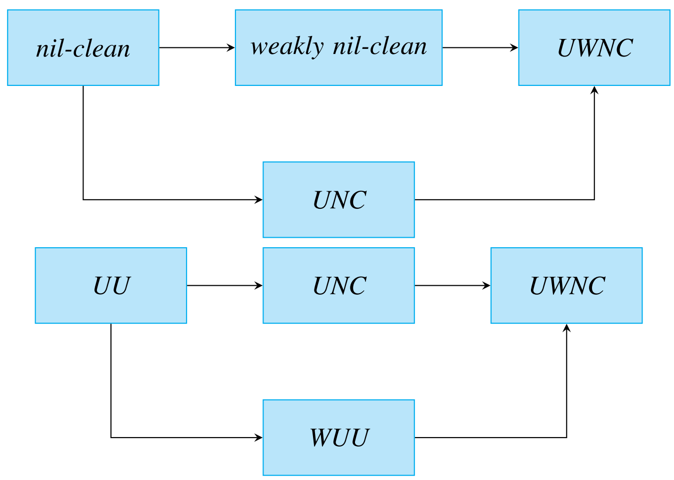
Example 1.7. (1) *Any nil-clean ring is weakly nil-clean, but the converse is* not *true in general. For instance,* $\mathbb{Z}_3$ *is weakly nil-clean but is not nil-clean.*

(2) *Any UU ring is WUU, but the converse is not true in general. For instance,* $\mathbb{Z}$ *is a WUU ring but is not UU.*

(3) *Any UU ring and nil-clean ring are UNC, but the converse is not true. For instance, the direct sum* $\mathbb{Z}_2[t] \oplus \mathrm{M}_2(\mathbb{Z}_2)$ *is a UNC ring which is neither UU nor nil-clean.*

(4) *Any WUU ring and weakly nil-clean ring are UWNC, but the converse is not true in general. For instance, the direct sum* $\mathbb{Z} \oplus M_2(\mathbb{Z}_2)$ *is a UWNC ring which is neither WUU nor weakly nil-clean.*

(5) *Any UNC ring is UWNC, but the converse is not true in general. For instance, all of the rings* $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_3 \oplus \mathrm{M}_2(\mathbb{Z}_2)$ *are UWNC but are not UNC.*



Our further programm is the following: in the next second section, we obtain some main properties of the newly defined class of UWNC rings – the main results here are Proposition 2.21 as well as Theorems 2.28, 2.30 and 2.38, respectively. In the subsequent third section, we explore UWNC group rings – the main results here are Propositions 3.3 and 3.5, respectively.

2. Basic Properties of UWNC Rings

We start here with the following obvious technicality whose proof is added only for the sake of completeness.

Proposition 2.1. *A unit* u *of a ring* R *is strongly weakly nil-clean if, and only if,* $u \in \pm 1 + \mathrm{Nil}(R)$. *In particular,* R *is a WUU ring if, and only if, every unit of* R *is strongly weakly nil-clean.*

Proof. (⇒). It is obvious, so we omit the details.
(⇐). Choosing $u \in U(R)$, by hypothesis, we have $u = q \pm e$, where q is a nilpotent element in R and e is an idempotent element in R such that $qe = eq$. Hence, it must be that $uq = qu$. Now, if $u = q + e$, then $e = u - q = u(1 - u^{-1}q) \in U(R)(1 + \mathrm{Nil}(R)) \subseteq U(R)$ and hence $e = 1$. If now $u = q - e$, then $e = q - u = -u(1 - u^{-1}q) \in U(R)(1 + \mathrm{Nil}(R)) \subseteq U(R)$ and hence $e = 1$, as required. □

We continue our work with the next two technical claims as follows.

Proposition 2.2. *Let R be a UWNC ring and S a UNC ring. Then, $R \times S$ is a UWNC ring.*

Proof. Choose $(u, v) \in U(R \times S) = U(R) \times U(S)$. Thus, there exists an idempotent $e \in R$ and a nilpotent $n \in R$ such that $u = e + n$ or $u = -e + n$. We now differ two cases:

Case I: Write $u = e + n$. Then, we have an idempotent $f \in S$ and a nilpotent $n' \in S$ such that $v = f + n'$. Thus, $(u, v) = (e, f) + (n, n')$, where $(e, f) \in \mathrm{Id}(R \times S)$ and $(n, n') \in \mathrm{Nil}(R \times S)$.

Case II: Write $u = -e + n$. Then, we have an idempotent $f \in S$ and a nilpotent $n' \in S$ such that $-v = f + n'$. Thus, $(u, v) = -(e, f) + (n, -n')$, where $(e, f) \in \mathrm{Id}(R \times S)$ and $(n, -n') \in \mathrm{Nil}(R \times S)$.

Therefore, (u, v) is either the sum or difference of a nilpotent and an idempotent in $R \times S$, whence we get the desired result. □

Proposition 2.3. *Let $\{R_i\}$ be a family of rings. Then, the direct product $R = \prod R_i$ of rings R_i is UWNC if, and only if, each R_i is UWNC and at most one of them is not UNC.*

Proof. (⇒). Obviously, each R_i is UWNC. Suppose now R_{i_1} and R_{i_2} $(i_1 \neq i_2)$ are *not* UNC. Then, there exist some $u_{i_j} \in U(R_{i_j}) \quad (j = 1, 2)$ such that $u_{i_1} \in U(R_{i_1})$ and $-u_{i_2} \in U(R_{i_2})$ are both *not* nil-clean. Choosing $u = (u_i)$, where $u_i = 0$ whenever $i \neq i_j \quad (j = 1, 2)$, we infer that u and $-u$ are it not the sum of an idempotent and a nilpotent, as required to get a contradiction. Consequently, each R_i is a UWNC ring as at most one of them is *not* UNC.
(⇐). Assume that R_{i_0} is a UWNC ring and all the others R_i are UNC. So, a simple check shows that $\prod_{i \neq i_0} R_i$ is UNC. According to Proposition 2.2, we conclude that R is a UWNC ring. □

As shown in the preceding Proposition 2.3, the property of being UWNC is *not* closed under taking (internal, external) direct sums. However, we will be more concrete as the next construction manifestly illustrates.

Example 2.4. *The ring $\mathbb{Z}_3$ is a UWNC ring, but the direct product $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not UWNC.*

Three further helpful affirmations are the following.

Corollary 2.5. *Let $L = \prod_{i \in I} R_i$ be the direct product of rings $R_i \cong R$ and $|I| \geq 2$. Then, L is a UWNC ring if, and only if, L is a UNC ring if, and only if, R is a UNC ring.*

Corollary 2.6. *For any $n \geq 2$, the ring R^n is UWNC if, and only if, R^n is UNC if, and only if, R is UNC.*

Proposition 2.7. *Let R be a UWNC ring. If T is a factor-ring of R such that all units of T lift to units of R, then T is a UWNC ring.*

Proof. Suppose that $f : R \to T$ is a surjective ring homomorphism. Let $v \in U(T)$. Then, there exists $u \in U(R)$ such that $v = f(u)$ and $u = \pm e + n$, where $e \in \mathrm{Id}(R)$ and $n \in \mathrm{Nil}(R)$. Therefore, we have $v = \pm f(e) + f(n)$, where $f(e) \in \mathrm{Id}(T)$ and $f(n) \in \mathrm{Nil}(T)$, as needed. □

We now offer the validity of the following statement.

Theorem 2.8. *Let R be a ring and I a nil-ideal of R.*

(1) *R is a UWNC ring if, and only if, $J(R)$ is nil and $\frac{R}{J(R)}$ is a UWNC ring.*

(2) *R is a UWNC ring if, and only if, $\frac{R}{I}$ is a UWNC ring.*

Proof. Set $\bar{R} := \frac{R}{J(R)}$.

(1) Let R be a UWNC ring and suppose $x \in J(R)$ and $x \notin \mathrm{Nil}(R)$. Since $1 + x \in U(R)$, it must be that $1 + x = -e + n$, where $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Id}(R)$, because if $1 + x = e + n$, we will have $x \in \mathrm{Nil}(R)$ that is a contradiction.

So, $2 + x \in \mathrm{Nil}(R)$. Similarly, since $1 + x^2 \in U(R)$, we deduce that $2 + x^2 \in \mathrm{Nil}(R)$. Hence,

$$(2 + x^2) - (2 + x) = x^2 - x = -x(1 - x) \in \mathrm{Nil}(R).$$

But $1 - x \in U(R)$ whence $x \in \mathrm{Nil}(R)$, a contradiction. Thus, $J(R)$ is nil.

Now, letting $\bar{u} \in U\left(\bar{R}\right)$, we obtain $u \in U(R)$, because units lift module $J(R)$. Therefore, as in the previous Proposition 2.7, one writes that $u = \pm e + n$, where $e \in \mathrm{Id}(R)$ and $n \in \mathrm{Nil}(R)$. So, we have $\bar{u} = \pm\bar{e} + \bar{n}$, where $\bar{e} \in \mathrm{Id}(\bar{R})$ and $\bar{n} \in \mathrm{Nil}(\bar{R})$. Thus, $\frac{R}{J(R)}$ is a UWNC ring, as promised.

Conversely, let $u \in U(R)$. Then, $\bar{u} \in U(\bar{R})$ and write $\bar{u} = \pm\bar{e} + \bar{n}$, where $\bar{e} \in \mathrm{Id}(\bar{R})$ and $\bar{n} \in \mathrm{Nil}(\bar{R})$. As $J(R)$ is nil, idempotents of $\frac{R}{J(R)}$ can be lifted to idempotents of R. So, we can assume that $e^2 = e \in R$. Moreover, one inspects that $n \in R$ is nilpotent. Thus, for some $j \in J(R)$,

$$u = \pm e + n + j = \pm e + (n + j)$$

is weakly nil-clean, because $n + j \in \mathrm{Nil}(R)$, as expected.

(2) The proof is similar to (i), so we omit the details.

□

Let $\mathrm{Nil}_*(R)$ denote the prime radical of a ring R, i.e., the intersection of all prime ideals of R. We know that $\mathrm{Nil}_*(R)$ is a nil-ideal of R.

We can now extract the next two consequences.

Corollary 2.9. *Let R be a ring. Then, the following two items are equivalent:*

(1) *R is UWNC.*

(2) $\dfrac{R}{\mathrm{Nil}_*(R)}$ *is UWNC.*

Corollary 2.10. *Let I be an ideal of a ring R. Then, the following are equivalent:*

(1) *R/I is UWNC.*
(2) *R/I^n is UWNC for all $n \in \mathbb{N}$.*
(3) *R/I^n is UWNC for some $n \in \mathbb{N}$.*

Proof. (i) $\Rightarrow$ (ii). For any $n \in \mathbb{N}$, the isomorphism $\dfrac{R/I^n}{I/I^n} \cong R/I$ holds. Since I/I^n is a nil-ideal of R/I^n and R/I is UWNC, Theorem 2.8 suggests us that R/I^n is a UWNC ring.
(ii) $\Rightarrow$ (iii). This is trivial.
(iii) $\Rightarrow$ (i). For any ideal I of R, we have as above that $\dfrac{R/I^n}{I/I^n} \cong R/I$. Since I/I^n is a nil-ideal of R/I^n and R/I^n is UWNC, Theorem 2.8 enables us that R/I is a UWNC ring. □

Given a ring R and a bi-module ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operation are used.

As two immediate consequences, we yield:

Corollary 2.11. *Let R be a ring and M a bi-module over R. Then, the following hold:*

(1) *The trivial extension $\mathrm{T}(R, M)$ is a UWNC ring if, and only if, R is a UWNC ring.*
(2) *For $n \geq 2$, the quotient-ring $\dfrac{R[x]}{\langle x^n \rangle}$ is a UWNC ring if, and only if, R is a UWNC ring.*
(3) *For $n \geq 2$, the quotient-ring $\dfrac{R[[x]]}{\langle x^n \rangle}$ is a UWNC ring if, and only if, R is a UWNC ring.*

Proof. (1) Set $A = \mathrm{T}(R, M)$ and consider $I := \mathrm{T}(0, M)$. It is not too hard to verify that I is a nil-ideal of A such that $\dfrac{A}{I} \cong R$. So, the result follows directly from Theorem 2.8.

(2) Put $A = \dfrac{R[x]}{\langle x^n \rangle}$. Considering $I := \dfrac{\langle x \rangle}{\langle x^n \rangle}$, we obtain that I is a nil-ideal of A such that $\dfrac{A}{I} \cong R$. So, the result follows automatically Theorem 2.8.

(3) Knowing that the isomorphism $\dfrac{R[x]}{\langle x^n \rangle} \cong \dfrac{R[[x]]}{\langle x^n \rangle}$ is true, point (iii) follows automatically from (ii).

□

Corollary 2.12. *Let R be a ring and M a bi-module over R. Then, the following statements are equivalent:*

(1) *R is a UWNC ring.*
(2) $\mathrm{T}(R, M)$ *is a UWNC ring.*
(3) $\mathrm{T}(R, R)$ *is a UWNC ring.*
(4) $\dfrac{R[x]}{\langle x^2 \rangle}$ *is a UWNC ring.*

Consider R to be a ring and M to be a bi-module over R. Let

$$\mathrm{DT}(R, M) := \{(a, m, b, n) | a, b \in R, m, n \in M\}$$

with addition defined componentwise and multiplication defined by

$$(a_1, m_1, b_1, n_1)(a_2, m_2, b_2, n_2) = (a_1a_2, a_1m_2+m_1a_2, a_1b_2+b_1a_2, a_1n_2+m_1b_2+b_1m_2+n_1a_2).$$

Then, $\mathrm{DT}(R, M)$ is a ring which is isomorphic to $\mathrm{T}(\mathrm{T}(R, M), \mathrm{T}(R, M))$. Also, we have

$$\mathrm{DT}(R, M) = \left\{ \begin{pmatrix} a & m & b & n \\ 0 & a & 0 & b \\ 0 & 0 & a & m \\ 0 & 0 & 0 & a \end{pmatrix} | a, b \in R, m, n \in M \right\}.$$

We now consider the following isomorphism of rings $\dfrac{R[x, y]}{\langle x^2, y^2 \rangle} \to \mathrm{DT}(R, R)$ defined by

$$a + bx + cy + dxy \mapsto \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}.$$

We, thereby, detect the following.

Corollary 2.13. *Let R be a ring and M a bi-module over R. Then the following statements are equivalent:*

(1) *R is a UWNC ring.*
(2) $\mathrm{DT}(R, M)$ *is a UWNC ring.*
(3) $\mathrm{DT}(R, R)$ *is a UWNC ring.*
(4) $\dfrac{R[x, y]}{\langle x^2, y^2 \rangle}$ *is a UWNC ring.*

Let α be an endomorphism of R and n a positive integer. It was defined by Nasr-Isfahani in [17] the *skew triangular matrix ring* like this:

$$\mathrm{T}_n(R, \alpha) = \left\{ \left. \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \right| a_i \in R \right\}$$

with addition point-wise and multiplication given by:

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ 0 & b_0 & b_1 & \cdots & b_{n-2} \\ 0 & 0 & b_0 & \cdots & b_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & b_0 \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ 0 & c_0 & c_1 & \cdots & c_{n-2} \\ 0 & 0 & c_0 & \cdots & c_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & c_0 \end{pmatrix},$$

where

$$c_i = a_0\alpha^0(b_i) + a_1\alpha^1(b_{i-1}) + \cdots + a_i\alpha^i(b_0),\ 1 \le i \le n-1.$$

We denote the elements of $\mathrm{T}_n(R,\alpha)$ by $(a_0, a_1, \ldots, a_{n-1})$. If α is the identity endomorphism, then $\mathrm{T}_n(R,\alpha)$ is a subring of upper triangular matrix ring $\mathrm{T}_n(R)$.

All of the mentioned above guarantee the truthfulness of the following statement.

Corollary 2.14. *Let R be a ring. Then, the following are equivalent:*

(1) *R is a UWNC ring.*
(2) *$\mathrm{T}_n(R,\alpha)$ is a UWNC ring.*

Proof. Choose

$$I := \left\{ \begin{pmatrix} 0 & a_{12} & \ldots & a_{1n} \\ 0 & 0 & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \middle| \, a_{ij} \in R \quad (i \le j) \right\}.$$

Then, one easily verifies that $I^n = 0$ and $\dfrac{\mathrm{T}_n(R,\alpha)}{I} \cong R$. Consequently, Theorem 2.8 applies to get the wanted result. □

Let α be an endomorphism of R. We denote by $R[x,\alpha]$ the *skew polynomial ring* whose elements are the polynomials over R, the addition is defined as usual, and the multiplication is defined by the equality $xr = \alpha(r)x$ for any $r \in R$. Thus, there is a curious ring isomorphism

$$\varphi : \frac{R[x,\alpha]}{\langle x^n \rangle} \to \mathrm{T}_n(R,\alpha),$$

given by

$$\varphi(a_0 + a_1x + \ldots + a_{n-1}x^{n-1} + \langle x^n \rangle) = (a_0, a_1, \ldots, a_{n-1})$$

with $a_i \in R$, $0 \le i \le n-1$. So, one finds that $\mathrm{T}_n(R,\alpha) \cong \dfrac{R[x,\alpha]}{\langle x^n \rangle}$, where $\langle x^n \rangle$ is the ideal generated by x^n.

We, thus, discover the following two claims.

Corollary 2.15. *Let R be a ring with an endomorphism α such that $\alpha(1) = 1$. Then, the following are equivalent:*

(1) *R is a UWNC ring.*
(2) *$\dfrac{R[x,\alpha]}{\langle x^n \rangle}$ is a UWNC ring.*
(3) *$\dfrac{R[[x,\alpha]]}{\langle x^n \rangle}$ is a UWNC ring.*

Corollary 2.16. *Let R be a ring, and let*

$$\mathrm{S}_n(R) = \{(a_{ij}) \in \mathrm{T}_n(R) \mid a_{11} = a_{22} = \cdots = a_{nn}\}.$$

Then, the following two points are equivalent:

(1) *R is a UWNC ring.*
(2) *$\mathrm{S}_n(R)$ is a UWNC ring.*

Proof. We set $I := \{(a_{ij}) \in S_n(R) : a_{11} = 0\}$, so evidently I is a nil-ideal of $\mathrm{S}_n(R)$ and $\mathrm{S}_n(R)/I \cong R$. Thus, Theorem 2.8 is applicable to get the wanted result. □

Wang in [23] introduced the matrix ring $\mathrm{S}_{n,m}(R)$. Supposing R is a ring, then the matrix ring $\mathrm{S}_{n,m}(R)$ can be represented as

$$\left\{ \begin{pmatrix} a & b_1 & \cdots & b_{n-1} & c_{1n} & \cdots & c_{1n+m-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a & b_1 & c_{n-1,n} & \cdots & c_{n-1,n+m-1} \\ 0 & \cdots & 0 & a & d_1 & \cdots & d_{m-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & a & d_1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & a \end{pmatrix} \in \mathrm{T}_{n+m-1}(R) : a, b_i, d_j, c_{i,j} \in R \right\}.$$

Also, let $\mathrm{T}_{n,m}(R)$ be

$$\left\{ \left(\begin{array}{ccccc|ccccc} a & b_1 & b_2 & \cdots & b_{n-1} & & & & & \\ 0 & a & b_1 & \cdots & b_{n-2} & & & & & \\ 0 & 0 & a & \cdots & b_{n-3} & & & \mathbf{0} & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & & & & & \\ 0 & 0 & 0 & \cdots & a & & & & & \\ \hline & & & & & a & c_1 & c_2 & \cdots & c_{m-1} \\ & & & & & 0 & a & c_1 & \cdots & c_{m-2} \\ & & \mathbf{0} & & & 0 & 0 & a & \cdots & c_{m-3} \\ & & & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & & 0 & 0 & 0 & \cdots & a \end{array}\right) \in \mathrm{T}_{n+m}(R) : a, b_i, c_j \in R \right\},$$

and

$$\mathrm{U}_n(R) = \left\{ \begin{pmatrix} a & b_1 & b_2 & b_3 & b_4 & \cdots & b_{n-1} \\ 0 & a & c_1 & c_2 & c_3 & \cdots & c_{n-2} \\ 0 & 0 & a & b_1 & b_2 & \cdots & b_{n-3} \\ 0 & 0 & 0 & a & c_1 & \cdots & c_{n-4} \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a \end{pmatrix} \in \mathrm{T}_n(R) : a, b_i, c_j \in R \right\}.$$

So, we obtain:

Corollary 2.17. *Let R be a ring. Then, the following statements are equivalent:*

(1) *R is a UWNC ring.*
(2) *$\mathrm{S}_{n,m}(R)$ is a UWNC ring.*
(3) *$\mathrm{T}_{n,m}(R)$ is a UWNC ring.*
(4) *$\mathrm{U}_n(R)$ is a UWNC ring.*

Letting R be an arbitrary ring. Danchev et al. defined in [10] the aforementioned rings as follows:

$$A_{n,m}(R) = R[x, y | x^n = xy = y^m = 0],$$
$$B_{n,m}(R) = R\langle x, y | x^n = xy = y^m = 0\rangle,$$
$$C_n(R) = R\langle x, y | x^2 = \underbrace{xyxyx...}_{n-1 \text{ words}} = y^2 = 0\rangle.$$

We also recollect the following.

Lemma 2.18. [10, Lemma 5.1]. *Let R be a ring and $m, n \in \mathbb{N}$. Then, the next three isomorphisms of rings are fulfilled:*

(1) $\mathrm{A}_{n,m}(R) \cong \mathrm{T}_{n,m}(R)$.
(2) $\mathrm{B}_{n,m}(R) \cong \mathrm{S}_{n,m}(R)$.
(3) $\mathrm{C}_n(R) \cong \mathrm{U}_n(R)$.

So, we receive two more relevant assertions as follows:

Corollary 2.19. *Let R be a ring. Then, the following four statements are equivalent:*

(1) *R is a UWNC ring.*
(2) *$\mathrm{B}_{n,m}(R)$ is a UWNC ring.*
(3) *$\mathrm{A}_{n,m}(R)$ is a UWNC ring.*
(4) *$\mathrm{C}_n(R)$ is a UWNC ring.*

Example 2.20. *Let R be a ring. Then, the following three items hold:*

(1) $R\left[x, y \mid x^2 = xy = y^2 = 0\right] \cong \left\{ \begin{pmatrix} a_1 & a_2 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_1 & a_3 \\ 0 & 0 & 0 & a_1 \end{pmatrix} : a_i \in R \right\}$.

(2) $R\left\langle x, y \mid x^2 = xy = y^2 = 0\right\rangle \cong \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & a_4 \\ 0 & 0 & a_1 \end{pmatrix} : a_i \in R \right\}$.

(3) $R\left\langle x,y \mid x^2 = xyx = y^2 = 0\right\rangle \cong \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 & a_5 & a_6 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_1 \end{pmatrix} : a_i \in R \right\} \cong \mathrm{T}(\mathrm{T}(R,R), \mathrm{M}_2(R)).$

The next assertion is pivotal.

Proposition 2.21. *Let R be a ring. Then, the following are equivalent:*

(1) *R is a UNC ring.*
(2) *$\mathrm{T}_n(R)$ is a UNC ring for all $n \in \mathbb{N}$.*
(3) *$\mathrm{T}_n(R)$ is a UNC ring for some $n \in \mathbb{N}$.*
(4) *$\mathrm{T}_n(R)$ is a UWNC ring for some $n \geq 2$.*

Proof. (i) $\Rightarrow$ (ii). This follows employing [12, Corollary 2.6].
(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). These two implications are trivial, so we remove the details.
(iv) $\Rightarrow$ (i). We manage to give an evidence of two different methods for verification like these:

Method 1: Let $u \in U(R)$ and choose

$$A = \begin{pmatrix} u & & & * \\ & -u_1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \in U(\mathrm{T}_n(R)).$$

By hypothesis, we can find an idempotent $\begin{pmatrix} e_1 & & & * \\ & e_2 & & \\ & & \ddots & \\ 0 & & & e_n \end{pmatrix}$ and a nilpotent $\begin{pmatrix} w_1 & & & * \\ & w_2 & & \\ & & \ddots & \\ 0 & & & w_n \end{pmatrix}$ such that

$$A = \pm \begin{pmatrix} e_1 & & & * \\ & e_2 & & \\ & & \ddots & \\ 0 & & & e_n \end{pmatrix} + \begin{pmatrix} w_1 & & & * \\ & w_2 & & \\ & & \ddots & \\ 0 & & & w_n \end{pmatrix}.$$

It now follows that $u = e_1 + w_1$ or $u = e_2 - w_2$. Clearly, e_1, e_2 are idempotents and w_1, w_2 are nilpotents in R, thus proving point (i).

Method 2: Setting $I := \{(a_{ij}) \in T_n(R) | a_{ii} = 0\}$, we obtain that it is a nil-ideal in $\mathrm{T}_n(R)$ with $\dfrac{\mathrm{T}_n(R)}{I} \cong R^n$. Therefore, Theorem 2.8 and Corollary 2.6 are applicable to get the pursued result. $\square$

We know that the direct sum $\mathbb{Z}_2[x] \oplus \mathrm{M}_2(\mathbb{Z}_2)$ is a UWNC ring that is neither WUU nor weakly nil-clean. In this vein, the following example concretely demonstrates an indecomposable UWNC ring that is neither WUU nor weakly nil-clean.

Example 2.22. *Let $R = \mathrm{M}_n(\mathbb{Z}_2)$ with $n \geq 2$, $S = \mathbb{Z}_2[x]$ and $M = S^n$. Then, the formal triangular matrix ring $T := \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is an indecomposable UNC ring invoking* [12, Example 2.7]. *So, T is an indecomposable UWNC ring. But since R is not a WUU ring and S is not a weakly nil-clean ring, one plainly follows that T is neither a WUU ring nor a weakly nil-clean ring, as claimed.*

It was proved in [9, Proposition 2.25] that any unital subring of a WUU ring is again a WUU ring. But, curiously, a subring of a UWNC ring may *not* be a UWNC ring as the next example unambiguously shows.

Example 2.23. *Let $T = \mathrm{M}_2(\mathbb{Z}_2)$ and $u = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in T$. Then, one sees that $u^3 = 1$. Now, let R be the unital subring of T generated by u. Therefore, one calculates that*

$$R = \{a1 + bu + cu^2 | a, b, c \in \mathbb{Z}\} = \{0, 1, u, u^2, 1 + u, 1 + u^2, 1 + u + u^2, u + u^2\}.$$

But, we have that

$$1 + u = u^2, 1 + u^2 = u, 1 + u + u^2 = 0 \text{ and } u + u^2 = 1,$$

so we deduce that $R = \{0, 1, u, u^2\}$. It is now easy to see that $\mathrm{Nil}(R) = \{0\}$, which means R is reduced. As $u^2 \neq u$, the element u is manifestly not weakly nil-clean utilizing [3, Theorem 20]*, and so T is a UWNC ring, as asserted.*

We continue our examination with the following series of technicalities.

Proposition 2.24. *The following two statements are valid:*

(1) *If R is a weakly nil-clean ring, then $\mathrm{Z}(R)$ is strongly weakly nil-clean.*
(2) *If R is a UWNC ring, then $\mathrm{Z}(R)$ is a WUU ring.*

Proof. (1) Let $a \in \mathrm{Z}(R)$. Then, $a \in R$ is weakly nil-clean and central, so a is strongly weakly nil-clean in R. Thus, $a \pm a^2 \in \mathrm{Nil}(R)$ by [5, Theorem 2.1]. But, $a \pm a^2 \in \mathrm{Z}(R)$, so that

$$a \pm a^2 \in \mathrm{Nil}(R) \cap \mathrm{Z}(R) \subseteq \mathrm{Nil}(\mathrm{Z}(R)).$$

Hence, $\mathrm{Z}(R)$ is strongly weakly nil-clean, as required.

(2) The proof is analogous to (i).

□

Proposition 2.25. *For any ring R, the power series ring $R[[x]]$ is not UWNC.*

Proof. Note the principal fact that the Jacobson radical of $R[[x]]$ is *not* nil (see, e.g., [15]). Thus, in view of Theorem 2.8, $R[[x]]$ is really *not* a UWNC ring, as expected. □

Lemma 2.26. *Let R be a ring. Then, the following two points are equivalent:*

(1) *R is a UNC ring.*
(2) *R is a UWNC ring and $2 \in J(R)$.*

Proof. (i) $\Longrightarrow$ (ii). Evidently, R is a UWNC ring. Also, we have $2 \in J(R)$ in virtue of [12, Lemma 2.4].
(ii) $\Rightarrow$ (i). Notice that $\dfrac{R}{J(R)}$ is of characteristic 2, because $2 \in J(R)$, and so $a = -a$ for every $a \in \dfrac{R}{J(R)}$. That is why, $\dfrac{R}{J(R)}$ is a UNC ring, and thus we can apply [12, Theorem 2.5] since $J(R)$ is nil in view of Theorem 2.8. □

Recall that an element r in a ring R is said to be an *unipotent* if $r - 1$ is a nilpotent. The following technical claim is elementary, but rather applicable in the sequel.

Lemma 2.27. *Let R be a ring and let $r \in R$ be the sum of an idempotent and a nilpotent. If $r^2 = 1$, then r is unipotent.*

Proof. Write $r = e + n$ with $e \in \mathrm{Id}(R)$ and $n \in \mathrm{Nil}(R)$. Set $f := 1 - e$ and $x := n(n+1) \in \mathrm{Nil}(R)$. Taking into account the equality $fn = f(r - e) = fr$, we compute that

$$fx = fn(n+1) = fr(r - e + 1) = fr(r + f) = fr^2 + frf = f + frf,$$

and, similarly, that $xf = f + frf$. Hence, $fx = xf$, so that x is a nilpotent which commutes with f, e, n and r, respectively. Accordingly,

$$f = fr^2 = fr \cdot r = fnr = fx(1+n)^{-1}r = f(1+n)^{-1}r \cdot x$$

is a nilpotent and hence $f = 0$, as desired. □

Our next main result, which sounds quite surprising, is the following one.

Theorem 2.28. *Let R be a ring and $2 \in U(R)$. Then, the following two items are equivalent:*

(1) *R is a UWNC ring.*
(2) *R is a WUU ring.*

Proof. (ii) $\Rightarrow$ (i). This is pretty obvious, so we leave the argumentation.
(i) $\Rightarrow$ (ii). First, we show that R is an abelian ring. To this goal, let $e^2 = e \in R$, and let $a = 1 - 2e$. Then, it is obviously true that $a^2 = 1$. Since R is UWNC, either a or $-a$ is nil-clean. By virtue of Lemma 2.27, one has that $a \in 1 + \mathrm{Nil}(R)$ or $a \in -1 + \mathrm{Nil}(R)$. If $a \in 1 + \mathrm{Nil}(R)$, then $2e \in \mathrm{Nil}(R)$, and so $e \in \mathrm{Nil}(R)$. This implies that $e = 0$. If, however, $a \in -1 + \mathrm{Nil}(R)$, then $2(1 - e) \in \mathrm{Nil}(R)$, whence $1 - e \in \mathrm{Nil}(R)$. This forces that $e = 1$. Therefore, R has only trivial idempotents. Thus, R is abelian, as asserted.

Now, let $u \in U(R)$, so $u = \pm e + n$, where $e \in \mathrm{Id}(R)$ and $n \in \mathrm{Nil}(R)$. If $u = e + n$, so $e = u - n \in U(R)$, then $e = 1$. If, however, $u = -e + n$, so $e = -u + n \in U(R)$ and then $e = 1$. Therefore, $u \in \pm 1 + \mathrm{Nil}(R)$. Finally, one concludes that R is WUU, as formulated. □

As an immediate consequence, we derive:

Corollary 2.29. *Let R be an abelian ring. Then, the following are equivalent:*

(1) *R is a UWNC ring.*
(2) *R is a WUU ring.*

Appealing to [9], a commutative ring $R[x]$ is a WUU ring if, and only if, so is R. In what follows, we present a generalization of this result.

Standardly, the prime radical $\mathrm{N}(R)$ of a ring R is defined to be the intersection of the prime ideals of R. It is know that $\mathrm{N}(R) = \mathrm{Nil}_*(R)$, the lower nil-radical of R. A ring R is called a 2-primal ring if $\mathrm{N}(R)$ coincides with $\mathrm{Nil}(R)$. For an endomorphism α of a ring R, R is said to be *α-compatible* if, for any $a, b \in R$, $ab = 0 \Longleftrightarrow a\alpha(b) = 0$, and in this case α is clearly injective.

We now arrive at our third chief result.

Theorem 2.30. *Let R be a 2-primal ring and α an endomorphism of R such that R is α-compatible. The following issues are equivalent:*

(1) *$R[x, \alpha]$ is a UWNC ring.*
(2) *$R[x, \alpha]$ is a WUU ring.*
(3) *R is a WUU ring.*
(4) *R is a UWNC ring.*
(5) *$J(R) = \mathrm{Nil}(R)$ and $U(R) = 1 \pm J(R)$.*

Proof. (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (iv). Straightforward.
(i) $\Rightarrow$ (iv). As $\dfrac{R[x, \alpha]}{\langle x \rangle} \cong R$ and all units of $\dfrac{R[x, \alpha]}{\langle x \rangle}$ are lifted to units of $R[x, \alpha]$, the implication easily holds.
(ii) $\Longrightarrow$ (iii). We argue as in the proof of (i) $\Longrightarrow$ (iv).
(iv) $\Rightarrow$ (v). As R is 2-primal, we have $\mathrm{Nil}(R) \subseteq J(R)$, so that $J(R) = \mathrm{Nil}(R)$ bearing in mind Theorem 2.8. Let $u \in U(R)$, so by hypothesis we have $u = \pm e + n$, where $e \in \mathrm{Id}(R)$ and $n \in \mathrm{Nil}(R) = J(R)$. If $u = e + n$, so $e = u - n \in U(R)$, and thus $e = 1$. If, however, $u = -e + n$, so $e = n - u \in U(R)$, and thus $e = 1$. Therefore, we receive $u \in \pm 1 + J(R)$, and hence $U(R) = \pm 1 + J(R)$, as required.
(v) $\Rightarrow$ (ii). As R is a 2-primal ring, with the aid of (v) we have $J(R) = \mathrm{Nil}(R) = \mathrm{Nil}_*(R)$. Thus, the quotient-ring $\dfrac{R}{J(R)}$ is a reduced ring. Moreover, it is easy to see that $\alpha(\mathrm{Nil}(R)) \subseteq \mathrm{Nil}(R)$, so $\alpha(J(R)) \subseteq J(R)$ and $\bar{\alpha} : \dfrac{R}{J(R)} \to \dfrac{R}{J(R)}$, defined by $\bar{\alpha}(\bar{a}) = \overline{\alpha(a)}$, is an endomorphism of $\dfrac{R}{J(R)}$.

We next show that $\dfrac{R}{J(R)}$ is $\bar{\alpha}$-compatible. That is, we must show that, for any $a + J(R), b + J(R) \in \dfrac{R}{J(R)}$, the equivalence

$$(a + J(R))(b + J(R)) = J(R) \Leftrightarrow (a + J(R))\bar{\alpha}(b + J(R)) = J(R)$$

holds. Equivalently, we have to show that, for any $a, b \in R$, the equivalence $ab \in \mathrm{Nil}(R) \Leftrightarrow a\alpha(b) \in \mathrm{Nil}(R)$ is true. But this equivalence has been established in the proof of Claims 1 and 2 in [1, Theorem 3.6]. As $\dfrac{R}{J(R)}$ is a reduced factor-ring and also is

$\bar{\alpha}$-compatible, with [7, Corollary 2.12] at hand we have

$$U\left(\frac{R}{J(R)}[x,\bar{\alpha}]\right) = U\left(\frac{R}{J(R)}\right),$$

which is equal to $\{\pm\bar{1}\}$ by assumption. So,

$$\frac{R}{J(R)}[x,\bar{\alpha}] \cong \frac{R[x,\alpha]}{J(R)[x,\alpha]}$$

is a WUU ring. Also, [7, Lemma 2.2] tells us that

$$\mathrm{Nil}_*(R[x,\alpha]) = \mathrm{Nil}_*(R)[x,\alpha].$$

Therefore,

$$J(R)[x,\alpha] = \mathrm{Nil}_*(R[x,\alpha]),$$

which is manifestly nil. Hence, [9, Proposition 2.5 and 2.6] ensures that $R[x,\alpha]$ is a WUU ring, as asked for. □

As a direct consequence, we deduce:

Corollary 2.31. *Let R be a 2-primal ring. Then, the following are equivalent:*

(1) *R is a UWNC ring.*
(2) *$R[x]$ is a UWNC ring.*
(3) *$J(R) = \mathrm{Nil}(R)$ and $U(R) = \pm 1 + J(R)$.*

The following criterion is worthy of documentation.

Proposition 2.32. *Suppose R is a commutative ring. Then, $R[x]$ is a UWNC ring if, and only if, R is a UWNC ring.*

Let A, B be two rings and M, N be (A,B)-bi-module and (B,A)-bi-module, respectively. Also, we consider the bilinear maps $\phi : M \otimes_B N \to A$ and $\psi : N \otimes_A M \to B$ that apply to the following properties.

$$Id_M \otimes_B \psi = \phi \otimes_A Id_M, Id_N \otimes_A \phi = \psi \otimes_B Id_N.$$

For $m \in M$ and $n \in N$, define $mn := \phi(m \otimes n)$ and $nm := \psi(n \otimes m)$. Now the 4-tuple $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ becomes an associative ring with obvious matrix operations that is called a *Morita context ring*. Denote two-side ideals $Im\phi$ and $Im\psi$ to MN and NM, respectively, that are called the *trace ideals* of the Morita context (compare with [2] as well).

We now have at our disposal all the ingredients necessary to establish the following statement.

Proposition 2.33. *Let $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context ring such that MN and NM are nilpotent ideals of A and B, respectively. If R is a UWNC ring, then A and B are UWNC rings. The converse holds provided one of the A or B is UNC and the other is UWNC.*

Proof. Apparently, since $MN \subseteq J(A)$ and $NM \subseteq J(B)$, by using [21, Lemma 3.1(1)], we have $J(R) = \begin{pmatrix} J(A) & M \\ N & J(B) \end{pmatrix}$ and $R/J(R) \cong A/J(A) \times B/J(B)$. Since R is a UWNC ring, then the factor $\dfrac{R}{J(R)}$ is a UWNC ring and $J(R)$ is nil consulting with Theorem 2.8, so it follows that both $\dfrac{A}{J(A)}$ and $\dfrac{B}{J(B)}$ are UWNC. As $J(R)$ is nil, $J(A)$ and $J(B)$ are nil too. Thus, A and B are UWNC as well.

Oppositely, assuming that A is a UNC ring and B is a UWNC ring, we conclude that $\dfrac{R}{J(R)}$ is a UWNC ring by a combination of [12, Theorem 2.5], Theorem 2.8 and Proposition 2.2. It then suffices to show that $J(R)$ is nil. To this target, suppose $r = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in J(R)$. Then, $a \in J(A)$, $b \in J(B)$. In virtue of Theorem 2.8, both ideals $J(A)$ and $J(B)$ are nil. Thus, we can find $n \in \mathbb{N}$ such that $a^n = 0$ in A and $b^n = 0$ in B. So,

$$\begin{pmatrix} a & m \\ n & b \end{pmatrix}^{n+1} \subseteq \begin{pmatrix} MN & M \\ N & NM \end{pmatrix}.$$

Clearly,

$$\begin{pmatrix} MN & M \\ N & NM \end{pmatrix}^2 = \begin{pmatrix} MN & (MN)M \\ (NM)N & NM \end{pmatrix}.$$

Moreover, for any $j \in \mathbb{N}$, one easily checks that

$$\begin{pmatrix} MN & M \\ N & NM \end{pmatrix}^{2j} = \begin{pmatrix} MN & (MN)M \\ (NM)N & NM \end{pmatrix}^j = \begin{pmatrix} (MN)^j & (MN)^j M \\ (NM)^j N & (NM)^j \end{pmatrix}.$$

By hypothesis, we may assume that $(MN)^p = 0$ in A and $(NM)^p = 0$ in B. Therefore,

$$\begin{pmatrix} MN & M \\ N & NM \end{pmatrix}^{2p} = 0.$$

Consequently, $\begin{pmatrix} a & m \\ n & b \end{pmatrix}^{2p(n+1)} = 0$, and so $J(R)$ is indeed nil, as desired. □

Now, let R, S be two rings, and let M be an (R, S)-bi-module such that the operation $(rm)s = r(ms)$ is valid for all $r \in R$, $m \in M$ and $s \in S$. Given such a bi-module M, we can set

$$\mathrm{T}(R, S, M) = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : r \in R, m \in M, s \in S \right\},$$

where it forms a ring with the usual matrix operations. Thus, the so-stated formal matrix $\mathrm{T}(R, S, M)$ is called a *formal triangular matrix ring*. In Proposition 2.33, if we set $N = \{0\}$, then we will obtain the following statement.

Corollary 2.34. *Let R, S be rings and let M be an (R, S)-bi-module. If $\mathrm{T}(R, S, M)$ is a UWNC ring, then both R,S are UWNC rings. The converse holds provided one of the rings R or S is UNC and the other is UWNC.*

We now state the following: given a ring R and a central elements s of R, the 4-tuple $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$ becomes a ring with addition component-wise and with multiplication defined by

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & x_2 \\ y_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + sx_1y_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_2 & sy_1x_2 + b_1b_2 \end{pmatrix}.$$

This ring is denoted by $\mathrm{K}_s(R)$. A Morita context $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ with $A = B = M = N = R$ is called a *generalized matrix ring* over R. It was observed by Krylov in [14] that a ring S is a generalized matrix ring over R if, and only if, $S = \mathrm{K}_s(R)$ for some $s \in \mathrm{Z}(R)$. Here $MN = NM = sR$, so $MN \subseteq J(A) \iff s \in J(R)$, $NM \subseteq J(B) \iff s \in J(R)$, and MN, NM are nilpotent $\iff s$ is a nilpotent.

As an automatic consequence, we derive:

Corollary 2.35. *Let R be a ring and $s \in \mathrm{Z}(R) \cap \mathrm{Nil}(R)$. If $\mathrm{K}_s(R)$ is a UWNC ring, then R is a UWNC ring. The converse holds, provided R is a UNC ring.*

Following Tang and Zhou (cf. [22]), for $n \geq 2$ and for $s \in \mathrm{Z}(R)$, the $n \times n$ formal matrix ring over R defined by s, and denoted by $\mathrm{M}_n(R; s)$, is the set of all $n \times n$ matrices over R with usual addition of matrices and with multiplication defined below:
For (a_{ij}) and (b_{ij}) in $\mathrm{M}_n(R; s)$,

$$(a_{ij})(b_{ij}) = (c_{ij}), \quad \text{where } (c_{ij}) = \sum s^{\delta_{ikj}} a_{ik} b_{kj}.$$

Here, $\delta_{ijk} = 1 + \delta_{ik} - \delta_{ij} - \delta_{jk}$, where δ_{jk}, δ_{ij}, δ_{ik} are the Kroncker delta symbols.

We, therefore, are ready to proceed by proving the following.

Corollary 2.36. *Let R be a ring and $s \in \mathrm{Z}(R) \cap \mathrm{Nil}(R)$. If $\mathrm{M}_n(R; s)$ is a UWNC ring, then R is a UWNC ring. The converse holds, provided R is a UNC ring.*

Proof. If $n = 1$, then $\mathrm{M}_n(R; s) = R$. So, in this case, there is nothing to prove. Let $n = 2$. By the definition of $\mathrm{M}_n(R; s)$, we have $\mathrm{M}_2(R; s) \cong \mathrm{K}_{s^2}(R)$. Apparently, $s^2 \in \mathrm{Nil}(R) \cap \mathrm{Z}(R)$, so the claim holds for $n = 2$ with the help of Corollary 2.35.

To proceed by induction, assume now that $n > 2$ and that the claim holds for $\mathrm{M}_{n-1}(R; s)$. Set $A := \mathrm{M}_{n-1}(R; s)$. Then, $\mathrm{M}_n(R; s) = \begin{pmatrix} A & M \\ N & R \end{pmatrix}$ is a Morita context, where

$$M = \begin{pmatrix} M_{1n} \\ \vdots \\ M_{n-1,n} \end{pmatrix} \quad \text{and} \quad N = (M_{n1} \dots M_{n,n-1})$$

with $M_{in} = M_{ni} = R$ for all $i = 1, \dots, n-1$, and

$$\psi : N \otimes M \to N, \quad n \otimes m \mapsto snm$$
$$\phi : M \otimes N \to M, \quad m \otimes n \mapsto smn.$$

Besides, for $x = \begin{pmatrix} x_{1n} \\ \vdots \\ x_{n-1,n} \end{pmatrix} \in M$ and $y = (y_{n1} \dots y_{n,n-1}) \in N$, we write

$$xy = \begin{pmatrix} s^2 x_{1n} y_{n1} & s x_{1n} y_{n2} & \dots & s x_{1n} y_{n,n-1} \\ s x_{2n} y_{n1} & s^2 x_{2n} y_{n2} & \dots & s x_{2n} y_{n,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ s x_{n-1,n} y_{n1} & s x_{n-1,n} y_{n2} & \dots & s^2 x_{n-1,n} y_{n,n-1} \end{pmatrix} \in sA$$

and

$$yx = s^2 y_{n1} x_{1n} + s^2 y_{n2} x_{2n} + \dots + s^2 y_{n,n-1} x_{n-1,n} \in s^2 R.$$

Since s is nilpotent, we see that MN and NM are nilpotent too. Thus, we obtain that

$$\frac{\mathrm{M}_n(R; s)}{J(\mathrm{M}_n(R; s))} \cong \frac{A}{J(A)} \times \frac{R}{J(R)}.$$

Finally, the induction hypothesis and Proposition 2.33 yield the claim after all. ☐

A Morita context $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is called *trivial*, if the context products are trivial, i.e., $MN = 0$ and $NM = 0$. We now have

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong \mathrm{T}(A \times B, M \oplus N),$$

where $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is a trivial Morita context consulting with [13].

What we can now offer is the following.

Corollary 2.37. *If the trivial Morita context* $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ *is a UWNC ring, then A, B are UWNC rings. The converse holds if one of the rings A or B is UWNC and the other is UNC.*

Proof. It is apparent to see that the isomorphisms

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong \mathrm{T}(A \times B, M \oplus N) \cong \begin{pmatrix} A \times B & M \oplus N \\ 0 & A \times B \end{pmatrix}$$

are fulfilled. Then, the rest of the proof follows combining Corollary 2.11 and Proposition 2.3. ☐

We now intend to prove the following.

Theorem 2.38. *Let R be a local ring. Then, the following are equivalent:*

(1) *R is a UWNC ring.*
(2) *R is a weakly nil-clean ring.*

Proof. (i) $\Rightarrow$ (ii). As R is local, one finds that $\mathrm{Id}(R) = \{0, 1\}$, and so R is abelian. Therefore, R is WUU owing to Corollary 2.29. Also, R is clean, because every local ring is clean. Thus, R is a clean WUU and we apply [9, Corollary 2.15] to find that R is strongly weakly nil-clean and so it is weakly nil-clean.
(ii) $\Rightarrow$ (i). It is clear. ☐

3. UWNC Group Rings

We begin here with the following simple but useful technicality.

Lemma 3.1. *Let R and S be rings and $i : R \to S$, $\varepsilon : S \to R$ be ring homomorphisms such that $\varepsilon i = id_R$.*

(1) $\varepsilon(\mathrm{Nil}(S)) = \mathrm{Nil}(R)$, $\varepsilon(U(S)) = U(R)$ *and* $\varepsilon(\mathrm{Id}(S)) = \mathrm{Id}(R)$.
(2) *If S is a UWNC ring, then R is a UWNC ring.*
(3) *If R is a UWNC ring and* $\ker\varepsilon \subseteq \mathrm{Nil}(S)$*, then S is a UWNC ring.*

Proof. (1) Clearly, the inclusions $\varepsilon(\mathrm{Nil}(S)) \subseteq \mathrm{Nil}(R)$, $\varepsilon(U(S)) \subseteq U(R)$ and $\varepsilon(\mathrm{Id}(S)) \subseteq \mathrm{Id}(R)$ are valid. On the other hand, we also have that

$$\mathrm{Nil}(R) = \varepsilon i(\mathrm{Nil}(R)) \subseteq \varepsilon(\mathrm{Nil}(S)), U(R) = \varepsilon i(U(R)) \subseteq \varepsilon(U(S))$$

and

$$\mathrm{Id}(R) = \varepsilon i(\mathrm{Id}(R)) \subseteq \varepsilon(\mathrm{Id}(S)).$$

(2) Two methods occur for proving this point like these:

Method 1: Let S be a UWNC ring. Choose $u \in U(R) = \varepsilon(U(S))$, so $u = \varepsilon(v)$, where $v \in U(S)$. Thus, we can write $v = \pm e + q$, where $e \in \mathrm{Id}(S)$, $q \in \mathrm{Nil}(S)$. Therefore,

$$u = \varepsilon(v) = \varepsilon(\pm e + q) = \pm\varepsilon(e) + \varepsilon(q),$$

where $\varepsilon(e) \in \mathrm{Id}(R)$ and $\varepsilon(q) \in \mathrm{Nil}(R)$ exploiting (i).

Method 2: Let S be a UWNC ring. Hence, $U(S) = \pm\mathrm{Id}(S) + \mathrm{Nil}(S)$, whence by (i) one has that

$$U(R) = \varepsilon(U(S)) = \varepsilon(\pm\mathrm{Id}(S) + \mathrm{Nil}(S)) = \pm\varepsilon(\mathrm{Id}(S)) + \varepsilon(\mathrm{Nil}(S)) = \pm\mathrm{Id}(R) + \mathrm{Nil}(R),$$

as required.

(3) If R is a UWNC ring, point (i) enables us that

$$U(S) = \varepsilon^{-1}(U(R)) = \varepsilon^{-1}(\pm\mathrm{Id}(R) + \mathrm{Nil}(R)) = \pm\mathrm{Id}(S) + \mathrm{Nil}(S) + \ker\varepsilon = \pm\mathrm{Id}(S) + \mathrm{Nil}(S),$$

as required.

□

Remark 3.2. *It is a routine technical exercise to see that Lemma 3.1 (iii) and (iii) is true also for WUU rings.*

Suppose now that G is an arbitrary group and R is an arbitrary ring. As usual, RG stands for the group ring of G over R. The homomorphism $\varepsilon : RG \to R$, defined by $\varepsilon(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g$, is called the *augmentation map* of RG and its kernel, denoted by $\Delta(G)$, is called the *augmentation ideal* of RG.

We are now prepared to establish the following two assertions.

Proposition 3.3. *Let R be a ring and G a group. If RG is a UWNC ring, then R is a UWNC ring. The converse holds if $\Delta(G) \subseteq \mathrm{Nil}(RG)$.*

Proof. Let us consider the inclusion $i : R \to RG$, given by $i(r) = \sum_{g \in G} a_g g$, where $a_{1G} = r$ and $a_g = 0$ provided $g \neq 1_G$. It is easy to check that the map i is a ring monomorphism and thus R can also be viewed as a subring of RG. Furthermore, it is only enough to apply Lemma 3.1 (ii) to get the claim. □

Incidentally, we are able to prove the following statement.

Corollary 3.4. *Let R be a ring and G a group. If RG is a WUU ring, then R is a WUU ring. The converse holds if $\Delta(G) \subseteq \mathrm{Nil}(RG)$.*

Proof. It follows at once by Proposition 3.3 and Lemma 3.1. □

A group G is called *locally finite* if every finitely generated subgroup of G is finite. Let p be a prime number. A group G is called a *p-group* if the order of each element of G is a power of p.

We finish off our results with the following statement.

Proposition 3.5. *Let R be a UWNC ring with $p \in \mathrm{Nil}(R)$ and let G be a locally finite p-group, where p is a prime. Then, RG is a UWNC ring.*

Proof. Referring to [8, Proposition 16], one verifies that $\Delta(G)$ is nil. Now, the assertion follows from the obvious isomorphism $\dfrac{RG}{\Delta(G)} \cong R$ and Theorem 2.8. □

Remark 3.6. *It is easily seen that Proposition 3.5 holds also for WUU rings.*

4. Open Questions

We close the work with the following challenging conjectures and problems.

A ring R is called *uniquely weakly nil-clean*, provided that R is a weakly nil-clean ring in which every nil-clean element is uniquely nil-clean (see [6] for more account).

Conjecture 4.1. *A ring R is a WUU ring if, and only if, every unit of R is uniquely weakly nil-clean.*

Conjecture 4.2. *A ring R is a strongly weakly nil-clean if, and only if, it is a semi-potent WUU ring.*

Problem 4.3. *Is a clean, UWNC ring a weakly nil-clean ring?*

Problem 4.4. *Characterize semi-perfect UWNC rings. Are they weakly nil-clean?*

Problem 4.5. *Suppose that R is a ring and $n \in \mathbb{N}$. Find a criterion when the full $n \times n$ matrix ring $\mathrm{M}_n(R)$ is UWNC.*

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