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On Pillai's Problem With Balancing Numbers and Powers of 2

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Abstract. An open conjecture of Pillai asserts that the equation $x^m - y^n = k$ has a finite number of solutions if m, n, x, y, k are integers with $m \geq 3, n, x, y \geq 2$ and $k \neq 0$ is fixed. Baker's theory of linear forms is employed to solve a variant of the Pillai problem for Balancing numbers and powers of 2. More precisely, all the integer numbers c which can be expressed in the form $B_r - 2^s$ for non-negative integers r and s in at least two ways are determined. The strategy of solution depends mainly on Matveev's fundamental inequality and on a reduction theorem of A. Dujella and A. Pethö.

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1. Introduction

Catalan, in 1844, addressed the following problem: Does the equation $x^m - y^n = 1$, for integers $x, y, m, n \geq 1$, have only the solution $(x, y, m, n) = (3, 2, 2, 3)$?, see [7]. Mihăilescu, in 2002, solved completely the problem, see [25]. Linear forms in logarithms were not used in Mihăilescu solution. In 1976, Tijdeman had used linear forms in logarithms to prove that there are only finite number of solutions of the equation $x^m - y^n = 1$, [29].

In 1936, Pillai generalized Catalan's problem by formulating, for $(a = 1, b = 1)$, the following conjecture, see [26].

Conjecture 1.1. *(Pillai's conjecture) The equation*

$$ax^m - by^n = k, \tag{1. 1}$$

where a, b, and k are non-zero integers, has only a finite number of integer solutions (x, y, m, n) with $m \geq 3$ and $n, x, y \geq 2$.

Eq. (1. 1) is known as the Pillai equation. It is not known that Eq. (1. 1) has a finite number of solutions if a, b, k are relatively prime and $|abk| \geq 2$.
One the most important settings of the Pillai equation is the equation $U_n - V_m = k$, where $(U_n)_n$ and $(V_m)_m$ are fixed linear recurrent sequences and k is a fixed integer. In [9], Chim et al. proved that this equation, under some conditions, has at most two distinct solutions (n, m). A special case of this equation is the equation $U_n - b^m = k$ with b a fixed integer. In [20], Heintze et al. showed that there exist effectively computable constants B and N_0 such that for any integers b, c with $b > B$ the equation $U_n - b^m = k$ has at most two distinct solutions $(n, m) \in \mathbb{N}^2$ with $n \geq N_0$ and $m \geq 1$. Now, we mention some recent works concerning these cases of the Pillai equation. In [3], Bravo et al. found all integers numbers that can be written as a difference of a Tribonacci number and a power of 2 in more than one way. Their work was a continuation of [11], in which they solved the same problem for Fibonacci numbers and a powers of 2. In [8], Chim et al. investigated differences between a Fibonacci number and a Tribonacci number that produce an integer in at least two ways. In [9], they generalized their results in [8]. In [23], Lomeli et al. solved $P_n - F_m = P_{n_1} - F_{m_1}$, where P_n and F_n are the nth Padovan and Fibonacci numbers, respectively. In [14], Erazo et al. showed that, for integers c, n, m, the equation $c = X_n - 2^m$, where X_n is the X-coordinate of the solution to the Pell equation, has at most 3 solutions with $n \geq 1$ and $m \geq 0$. In [15], they continued the study. In [10], Ddamulira and Luca solved the previous problem for k-generalized Fibonacci numbers instead of Fibonacci numbers. In [18], García and Gómez specified the integers that expressible as a difference of a perfect power and a k–Pell number it at least two ways. In [16], Faye and Edjeou solved the problem for Pell numbers, Pell–Lucas numbers and a powers of 3. In [21], Hernández et al. studied, for non-negative integer pairs $(n, m) \neq (n_1, m_1)$, the equation $F_n - P_m = F_{n_1} - P_{m_1}$ where P_n is the nth Pell number. In [1], Batte et al. proved that there exist at most 4 distinct representations to write an integer in the form $F_k - p^l$ with p prime, $k \geq 2$ and $l \geq 0$. For more applications of Diophantine equations we refer to [22] and [28]. This paper is a contribution to the previous works in the literature.

We aim to completely determine the numbers a which can be expressed in more than one way as a difference of a balancing number and a power of 2. In other words, we study the equation:

$$B_r - 2^s = a \tag{1. 2}$$

for fixed integer a and non negative integers r and s. The following result gathers the solutions. The only integers that can be represented in the form $B_r - 2^s$,with $r, s \geq 0$, in at least two ways are -1 and -2. In addition, the representations are

$$-1 = B_0 - 2^0 = B_1 - 2^1, -2 = B_0 - 2^1 = B_2 - 2^3$$

The approach is as follows: we upper bound all the implied variables by a single variable. Then Matveev's theorem is employed to obtain an upper bound for this variable. The obtained bound is too large and so it is difficult to probe all the possible cases. As a result, we try to obtain certain linear forms to utilize the result of Dujella and Pethö to reduce that bound. Lastly, Sage computations are used to determine the solutions.

2. NUMBER THEORETIC BACKGROUNDS

2.1. **Balancing numbers.** Balancing numbers are defined recursively by $B_0 = 0, B_1 = 1$ and $B_{n+1} = 6B_n - B_{n-1}$ for $n \geq 1$. So, the initial terms are

$$0, 1, 6, 35, 204, 1189, 6930, ...$$

Balancing numbers are characterized by the equation

$$\Theta(\nu) := \nu^2 - 6\nu + 1 = 0.$$

Assuming $\sigma = 3 + \sqrt{8}$ and $\varrho = 3 - \sqrt{8}$ are the solutions, then the Binet formula is

$$B_n = \frac{\sigma^n - \varrho^n}{2\sqrt{8}} \quad \text{for all} \quad n \geq 0. \tag{2. 3}$$

One can show that

$$\sigma^{n-1} \leq B_n < \sigma^n \quad \text{for all} \quad n \geq 1. \tag{2. 4}$$

For details on balancing numbers and related concepts, see [2], [17], [27] and the references mentioned therein.

2.2. **Linear forms in logarithms.** Assume that the minimal polynomial (over $\mathbb{Z}$) of an algebraic number κ is

$$c_0 x^m + c_1 x^{m-1} + ... + c_m = c_0 \prod_{i=1}^{m} (x - \kappa^{(i)}),$$

where $c_0 > 0$ is integer and $\kappa^{(i)}$'s are the conjugates of κ. Over an algebraic real field, the logarithmic Weil height of κ is defined by

$$\mathfrak{h}(\alpha) := \frac{1}{m}\left(\log c_0 + \sum_{i=1}^{m} \log\left(\max\left\{\left|\kappa^{(i)}\right|, 1\right\}\right)\right).$$

The following properties are satisfied by $\mathfrak{h}$ (see [5] for proofs):

$$\begin{aligned} \mathfrak{h}(\kappa_1 \pm \kappa_2) &\leq \mathfrak{h}(\kappa_1) + \mathfrak{h}(\kappa_2) + \log 2; \\ \mathfrak{h}\left(\kappa_1 \kappa_2^{\pm 1}\right) &\leq \mathfrak{h}(\kappa_1) + \mathfrak{h}(\kappa_1); \\ \mathfrak{h}(\kappa^s) &= |s|\, \mathfrak{h}(\kappa) \quad (s \in \mathbb{Z}). \end{aligned}$$

Using Matveev's theorem, see [24], Bugeaud, Mignotte and Siksek deduced the following result, see [4], .

Let $d_{\mathbb{A}}$ be the degree of a real algebraic number field $\mathbb{A}$, $\kappa_1, ..., \kappa_m \in \mathbb{A}$ with $\kappa_i > 0$ for $i = 1, ..., m$ and let $t_1, ..., t_m$ be non zero integers such that

$$\Omega := \kappa_1^{t_1} \kappa_2^{t_2} ... \kappa_m^{t_m} - 1 \neq 0.$$

For $i = 1, ..., m$, let

$$H_i \geq \max\{d_{\mathbb{A}} \mathfrak{h}(\kappa_i), |\log \kappa_i|, 0.16\}, \quad .$$
$$and\ \beta \geq \max\{|t_1|, ..., |t_m|\}.$$

Then

$$\log |\Omega| > -1.4 \cdot 30^{m+3} \cdot m^{4.5} \cdot d_{\mathbb{A}}^2 \cdot (1 + \log d_{\mathbb{A}}) \cdot (1 + \log \beta) H_1 ... H_m. \tag{2. 5}$$

The following result is a part of Lemma (2.2) in de Wager[12].

Lemma 2.3. *Let $a, x \in \mathbb{R}$ and $|x| < a < 1$. Then $|x| < \frac{a}{1-e^{-a}}|e^x - 1|$.*

Immediately, if $a = \frac{1}{2}$ then $|x| < 2|e^x - 1|$. This is the case that will be used throughout the paper.

3. Bounding the variables

Assume that there exist r, s, r_1, s_1 with $(r, s) \neq (r_1, s_1)$ and $B_r - 2^s = B_{r_1} - 2^{s_1}$. The equation is symmetric. So we can assume that $s \geq s_1$. But $s = s_1$ leads to $r = r_1$, contradicting the assumption. Therefore, $s > s_1$. This implies that $r > r_1$. Thus, $r \geq 2$ and $r_1 \geq 0$. Bounding the variable r in the equation

$$B_r - B_{r_1} = 2^s - 2^{s_1} \tag{3. 6}$$

is our goal in this section. We note that $B_r - B_{r-2} = 6B_{r-1} - 2B_{r-2} > 4B_{r-1} > B_{r_1}$. Combining this with (2. 4) we see that, for $r \geq 3$,

$$\sigma^{r-3} \leq B_{r-2} \leq B_r - B_{r_1} = 2^s - 2^{s_1} < 2^s \tag{3. 7}$$

and

$$\sigma^r \geq B_r \geq B_r - B_{r_1} = 2^s - 2^{s_1} \geq 2^{s-1}. \tag{3. 8}$$

These imply that

$$\frac{\log 2}{\log \sigma}(s-1) \leq r \leq \frac{\log 2}{\log \sigma}s + 3. \tag{3. 9}$$

If $r < 150$, then $s \leq 600$. Solving Eq.(3. 6) for $0 \leq r_1 < r \leq 150$ and $0 \leq s_1 < s \leq 600$ we obtain the solutions in Theorem 1. From now on, we assume that $r \geq 150$. From inequality (3. 9) we have $s < 3r$.

Inserting Binet formulas into Eq.(1. 2), we get

$$\frac{\sigma^r - \varrho^r}{4\sqrt{2}} - \frac{\sigma^{r_1} - \varrho^{r_1}}{4\sqrt{2}} = 2^s - 2^{s_1}. \tag{3. 10}$$

Then

$$\left|\frac{\sigma^r}{4\sqrt{2}} - 2^s\right| = \left|\frac{\sigma^{r_1} - \varrho^{r_1}}{4\sqrt{2}} + \frac{\varrho^r}{4\sqrt{2}} - 2^{s_1}\right|.$$

Since $|\varrho| < 1$, it follows that

$$\left|\frac{\sigma^r}{4\sqrt{2}} - 2^s\right| < \frac{\sigma^{r_1} + 2}{4\sqrt{2}} + 2^{s_1} < \frac{\sigma^{r_1}}{\sqrt{2}} + 2^{s_1} < 2\max\{\sigma^{r_1}, 2^{s_1}\}. \tag{3. 11}$$

Then

$$\left|(\sqrt{2})^{-1}\sigma^r 2^{-s-2} - 1\right| < 2\max\{\sigma^{r_1}2^{-s}, 2^{s_1-s}\} < \max\{\sigma^{r_1-r+4}, 2^{s_1-s+1}\}, \tag{3. 12}$$

where in the last inequality we used $\sigma > 2$. Let $\Omega_1 = (\sqrt{2})^{-1}\sigma^r 2^{-s-2} - 1$. If $\Omega_1 = 0$, then $\sigma^{2r} = 2^{2s+5}$. Conjugation in $\mathbb{Q}(\sqrt{2})$ yields $\varrho^{2r} = 2^{2s+5} \in \mathbb{Q}$ which is impossible since $\varrho < 1$. Therefore, $\Omega_1 \neq 0$. Taking $\mathbb{A} = \mathbb{Q}(\sigma) = \mathbb{Q}(\sqrt{2})$, we find that $d_{\mathbb{A}} = 2$. Let

$$m = 3, \quad \kappa_1 = \sqrt{2}, \quad \kappa_2 = \sigma, \quad \kappa_3 = 2, \quad t_1 = -1, \quad t_2 = r, \quad t_3 = -(s+2).$$

Then

$$\mathfrak{h}(\kappa_1) = \frac{1}{2}\log 2;$$

$$\mathfrak{h}(\kappa_2) = \frac{1}{2}\log\sigma;$$
$$\mathfrak{h}(\kappa_3) = \log 2.$$

Let

$$H_1 = \log 2, \quad H_2 = \log\sigma, \quad \text{and} \quad H_3 = 2\log 2.$$

As $4r \geq \max\{1, r, s+2\}$, we take $\beta = 4r$. Then Theorem (2.2) shows that

$$\log|\Omega_1| > -c_1(1+\log 4r),$$

where $c_1 = 1.4\cdot 30^6\cdot 3^{4.5}\cdot 4\cdot(1+\log 2)(\log 2)(2\log 2\log\sigma)$. Therefore

$$\log|\Omega_1| > -1.7\cdot 10^{12}(1+\log 4r). \tag{3. 13}$$

A combination with (3. 12) yields

$$\min\{(r-r_1-4)\log\sigma, (s-s_1-1)\log 2\} < 1.7\cdot 10^{12}(1+\log 4r).$$

Thus

$$\min\{(r-r_1)\log\sigma, (s-s_1)\log 2\} < 1.8\cdot 10^{12}(1+\log 4r). \tag{3. 14}$$

Case I: $\min\{(r-r_1)\log\sigma, (s-s_1)\log 2\} = (r-r_1)\log\sigma$.
From Eq.(3. 6) we deduce that

$$\left|\frac{\sigma^{r-r_1}-1}{4\sqrt{2}}\sigma^{r_1} - 2^s\right| = \left|\frac{\varrho^r - \varrho^{r_1}}{4\sqrt{2}} - 2^{s_1}\right| < 2^{s_1}+1 < 2^{s_1+1}.$$

Then

$$\left|\left(\frac{\sigma^{r-r_1}-1}{4\sqrt{2}}\right)\sigma^{r_1}2^{-s} - 1\right| < 2^{s_1-s+1}. \tag{3. 15}$$

Let $\Omega_2 = \left(\frac{\sigma^{r-r_1}-1}{4\sqrt{2}}\right)\sigma^{r_1}2^{-s} - 1$. If $\Omega_2 = 0$, then

$$(\sigma^{r-r_1}-1)\sigma^{r_1} = 4\sqrt{2}\cdot 2^s$$

Conjugating in $\mathbb{Q}(\sqrt{2})$ yields

$$(\sigma^{r-r_1}-1)\sigma^{r_1} = (1+\varrho^{r-r_1})\varrho^{r_1}. \tag{3. 16}$$

We have $(\sigma^{r-r_1}-1)\sigma^{r_1} = \sigma^r - \sigma^{r_1} \geq \sigma^r - \sigma^{r-1} \geq \sigma^{r-1} \geq \sigma^{149}$. In addition, $|(1+\varrho^{r-r_1})\varrho^{r_1}| \leq (1+|\varrho|^{r-r_1})\,|\varrho^{r_1}| < 2$. This is a contradiction. Thus $\Omega_2 \neq 0$. Set

$$\kappa_1 = \frac{\sigma^{r-r_1}-1}{4\sqrt{2}}, \quad \kappa_2 = \sigma, \quad \kappa_3 = 2, \quad m = 3, \quad t_1 = 1, \quad t_2 = r_1, \quad t_3 = -s.$$

Note that the minimal polynomial of κ_1 divides $32x^2 - 32B_{r-r_1}x + 2(C_{r-r_1}-1)$, where $C_r = \frac{\sigma^r+\varrho^r}{2}$ is the rth Lucas-balancing numbers. Then,

$$\begin{aligned}\mathfrak{h}(\kappa_1) &\leq \frac{1}{2}\left(\log 32 + \log\left(\frac{\sigma^{r-r_1}+1}{4\sqrt{2}}\right)\right) < \frac{1}{2}\log(8\sqrt{2}\sigma^{r-r_1})\\ &< \frac{1}{2}\log(\sigma^{r-r_1+2}) < \frac{1}{2}(r-r_1+2)\log\sigma\\ &< 0.9\cdot 10^{12}(1+\log 4r)\end{aligned}$$

Moreover, $\mathfrak{h}(\kappa_2) = \frac{1}{2}\log\sigma,$ and $\mathfrak{h}(\kappa_3) = \log 2$. So, we take

$$H_1 = 1.8\cdot 10^{12}(1+\log 4r), \quad H_2 = \log\sigma, \quad \text{and} \quad H_3 = 2\log 2.$$

Also, $3r \geq \max\{1, r_1, s\}$. We take $\beta = 3r$. Thanks to Theorem(2.2), we have

$$\log|\Omega_2| > -c_2(1+\log 4r)(1+\log 3r),$$

where $c_2 = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 4 \cdot 1.8 \cdot 10^{12} \cdot (1+\log 2)(2\log 2 \log\sigma)$. Then

$$\log|\Omega_2| > -4.3 \cdot 10^{24}(1+\log 4r)^2. \tag{3. 17}$$

This and (3. 15) give

$$(s-s_1)\log 2 < 4.4 \cdot 10^{24}(1+\log 4r)^2. \tag{3. 18}$$

Case II: $\min\{(r-r_1)\log\sigma, (s-s_1)\log 2\} = (s-s_1)\log 2$.
From Eq.(3. 6) we deduce that

$$\left|\frac{\sigma^r}{4\sqrt{2}} - 2^{s_1}(2^{s-s_1}-1)\right| = \left|\frac{\sigma^{r_1}+\varrho^r-\varrho^{r_1}}{4\sqrt{2}}\right| < \frac{\sigma^{r_1}+2}{4\sqrt{2}} < \sigma^{r_1}.$$

Then

$$\left|\sqrt{2}\left(2^{s-s_1}-1\right)^{-1}\sigma^r 2^{-s_1-2} - 1\right| < \frac{\sigma^{r_1}}{s-s_1} \leq \frac{2\sigma^{r_1}}{2^s}.$$

Since $2^s > \sigma^{r-3}$ and $\sigma > 2$, then

$$\left|\left(\sqrt{2}\left(2^{s-s_1}-1\right)\right)^{-1}\sigma^r 2^{-s_1-2} - 1\right| < \sigma^{r_1-r+4}. \tag{3. 19}$$

Let $\Omega_3 = \left(\sqrt{2}\,(2^{s-s_1}-1)\right)^{-1}\sigma^r 2^{-s_1-2} - 1$. Assume that $\Omega_3 = 0$. Then we have $\sigma^r = \sqrt{2}\cdot 2^{s_1+2}(2^{s-s_1}-1)$. Conjugation in $\mathbb{Q}(\sqrt{2})$ yields $\varrho^r = \sqrt{2}\cdot 2^{s_1+2}(2^{s-s_1}-1)$. Therefore, $|\varrho^r| > 1$, a contradiction. So, $\Omega_3 \neq 0$. Let

$$\kappa_1 = \sqrt{2}(2^{s-s_1}-1),\quad \kappa_2 = \sigma,\quad \kappa_3 = 2,\quad m = 3,\quad t_1 = -1,\quad t_2 = r,\quad t_3 = -s_1-2.$$

The minimal polynomial of κ_1 is $x^2 - 2(2^{s-s_1}-1)^2$. Consequently,

$$\mathfrak{h}(\kappa_1) = \log\left(\sqrt{2}\left(2^{s-s_1}-1\right)\right) < (s-s_1+1)\log 2 < 1.9 \cdot 10^{12}\left(1+\log 4r\right).$$

Let

$$H_1 = 3.8 \cdot 10^{13}(1+\log 4r),\quad H_2 = \log\sigma,\quad \text{and}\quad H_3 = 2\log 2.$$

Also, $3r \geq \max\{1, s_1+2, r\}$. So, taking $\beta = 3r$ and employing Theorem(2.2) again, we get

$$\log|\Omega_3| > -c_3(1+\log 4r)(1+\log 3r), \tag{3. 20}$$

where $c_2 = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 4 \cdot 3.8 \cdot 10^{13} \cdot (1+\log 2)(2\log 2\log\sigma)$. Then

$$\log|\Omega_3| > -9.1 \cdot 10^{25}(1+\log 4r)^2. \tag{3. 21}$$

This and (3. 19) give

$$(r-r_1)\log\sigma < 9.2 \cdot 10^{25}(1+\log 4r)^2. \tag{3. 22}$$

Thus, by (3. 18) and (3. 22), we arrive at

$$\min\{(r-r_1)\log\sigma, (s-s_1)\log 2\} < 4.4 \cdot 10^{24}(1+\log 4r)^2 \tag{3. 23}$$

and

$$\max\{(r-r_1)\log\sigma, (s-s_1)\log 2\} < 9.2 \cdot 10^{25}(1+\log 4r)^2. \tag{3. 24}$$

From Eq.(3. 6) we see that

$$\left|\left(\frac{\sigma^{r-r_1}-1}{4\sqrt{2}}\right)\sigma^{r_1}-2^{s_1}\left(2^{s-s_1}-1\right)\right|=\left|\frac{\varrho^{r}-\varrho^{r_1}}{4\sqrt{2}}\right|<|\varrho|^{r_1}=\frac{1}{\sigma^{r_1}}.$$

Then

$$\left|\left(\frac{\sigma^{r-r_1}-1}{\sqrt{2}\left(2^{s-s_1}-1\right)}\right)\sigma^{r_1}2^{-s_1-2}-1\right|<\frac{1}{\sigma^{r_1}\left(2^{s}-2^{s_1}\right)}\leq\frac{2}{\sigma^{r_1}2^{s}}<\sigma^{4-r}. \qquad (3.\ 25)$$

Let $\Omega_4=\left(\frac{\sigma^{r-r_1}-1}{\sqrt{2}(2^{s-s_1}-1)}\right)\sigma^{r_1}2^{-s_1-2}-1$. If $\Omega_4=0$, then conjugation produces a contradiction. Therefore, $\Omega_4\neq 0$. We take

$$\kappa_1=\frac{\sigma^{r-r_1}-1}{\sqrt{2}(2^{s-s_1}-1)},\ \ \kappa_2=\sigma,\ \ \kappa_3=2,\ \ m=3,\ \ t_1=-1,\ \ t_2=r_1,\ \ t_3=-s_1-2.$$

We have

$$\begin{aligned}\mathfrak{h}(\kappa_1) &\leq \mathfrak{h}\left(\frac{\sigma^{r-r_1}-1}{\sqrt{2}}\right)+\mathfrak{h}\left(2^{s-s_1}-1\right)\\ &< \frac{1}{2}\left(r-r_1+2\right)\log\sigma+\left(s-s_1\right)\log 2\\ &< 1.4\cdot 10^{26}\left(1+\log 4r\right)^2.\end{aligned}$$

Take

$$H_1=2.8\cdot 10^{26}(1+\log 4r)^2,\ \ H_2=\log\sigma,\ \ \text{and}\ \ H_3=2\log 2.$$

Also, $3r\geq \max\{1,s_1+2,r_1\}$. We take $\beta=3r$. Then

$$\log|\Omega_4|>-c_4(1+\log 4r)^2(1+\log 3r),$$

where $c_4=1.4\cdot 30^6\cdot 3^{4.5}\cdot 4\cdot 5\cdot 10^{24}\cdot(1+\log 2)(2\log 2\log\sigma)$. Then

$$\log|\Omega_3|>-1.2\cdot 10^{37}(1+\log 4r)^3. \qquad (3.\ 26)$$

A combination of (3. 25) and (3. 26) yields

$$r-4<0.7\cdot 10^{37}(1+\log 4r)^3.$$

Hence

$$r<7.3\cdot 10^{42}.$$

4. Reducing the upper bound

Set $||\lambda||:=min\{|\lambda-n|:n\in\mathbb{Z}\}$ for any real number λ . Dujella and Pethö proved the next result, see [13].

Lemma 4.1. *Let $K>0$ be an integer, λ be an irrational number, θ,A,B be given real numbers with $A>0,B>1$. Let $\frac{p}{q}$ be a convergent of λ and $q>6K$. If $s,t,\omega>0$ are positive integers that satisfy*

$$0<|s\lambda-t+\theta|<\frac{A}{B^{\omega}}$$

with $s\leq K$, and $\epsilon:=||\theta q||-K||\lambda q||>0$, then

$$\omega<\frac{\log\left(\frac{Aq}{\epsilon}\right)}{\log B}.$$

Let

$$\Delta_1 = r\log\sigma - (s+2)\log 2 - \log(4\sqrt{2}).$$

By Eq.(3. 12), we have

$$|\Omega_1| = \left|e^{\Delta_1} - 1\right| < \max\{\sigma^{r_1-r+4}, 2^{s_1-s+1}\}. \tag{4. 27}$$

So, by lemma (2.3), we have

$$|\Delta_1| < 2\max\{\sigma^{r_1-r+4}, 2^{s_1-s+1}\} < \max\{\sigma^{r_1-r+5}, 2^{s_1-s+2}\}. \tag{4. 28}$$

If $\Delta_1 > 0$, then

$$0 < r\left(\frac{\log\sigma}{\log 2}\right) - s - \frac{9}{2} < \max\{\frac{\sigma^5}{\log 2}\sigma^{r_1-r}, \frac{4}{\log 2}2^{s_1-s}\} < \max\{9704\sigma^{r_1-r}, 6\cdot 2^{s_1-s}\}. \tag{4. 29}$$

Let $K = 7.3\cdot 10^{42}$ $(K > r > 3s)$, $\lambda = \frac{\log\sigma}{\log 2}$, $\theta = \frac{-9}{2}$, $A = 9704$, $B = \sigma$ or $A = 6$, $B = 2$. The continued fraction of λ entails that $q_{94} > 6K$. Then

$$\epsilon = ||\theta q_{94}|| - K\,||\lambda q_{94}|| > 0.49.$$

Lemma (4.1) entails that $r - r_1 < 64$, or $s - s_1 < 151$.

If $\Delta_1 < 0$, then

$$0 < (s+2)\left(\frac{\log 2}{\log\sigma}\right) - s + \frac{\log(4\sqrt{2})}{\log\sigma} = s\left(\frac{\log 2}{\log\sigma}\right) - s + \frac{\log(16\sqrt{2})}{\log\sigma}$$
$$< \max\{\frac{\sigma^5}{\log\sigma}\sigma^{r_1-r}, \frac{4}{\log\sigma}2^{s_1-s}\} < \max\{3816\sigma^{r_1-r}, 3\cdot 2^{s_1-s}\}.$$

Let $K = 4.32\cdot 10^{43}$, $\lambda = \frac{\log 2}{\log\sigma}$, $\theta = \frac{\log(16\sqrt{2})}{\log\sigma}$, $A = 3816$, $B = \sigma$ or $A = 3$, $B = 2$. For λ, we find that $q_{95} > 6K(q_{95} = 315888950006391451014443556349119477725131236)$. We compute

$$\epsilon = ||\theta q_{95}|| - K\,||\lambda q_{95}|| > 0.49.$$

Lemma (4.1) entails that $r - r_1 < 64$, or $s - s_1 < 151$. Hence, $\Delta_1 \neq 0$ implies that $r - r_1 < 64$, or $s - s_1 < 151$.

We investigate each case separately. Let $r - r_1 < 64$. Assume that $s - s_1 \geq 20$. Set

$$\Delta_2 = r_1\log\sigma - s\log 2 + \log\left(\frac{\sigma^{r-r_1} - 1}{4\sqrt{2}}\right).$$

By Eq.(3. 15) and lemma (2.3), we obtain

$$|\Delta_2| < \frac{4}{2^{s-s_1}}. \tag{4. 30}$$

Since $\Omega_2 \neq 0$, we deduce that $\Delta_2 \neq 0$. If $\Delta_2 > 0$, then

$$0 < r_1\left(\frac{\log\sigma}{\log 2}\right) - s + \frac{\log\left(\frac{\sigma^{r-r_1}-1}{4\sqrt{2}}\right)}{\log 2} < \frac{4}{(\log 2)2^{s-s_1}} < \frac{6}{2^{s-s_1}}. \tag{4. 31}$$

Let $K = 7.2\cdot 10^{42}$ $(K > r > 3s)$, $\lambda = \frac{\log\sigma}{\log 2}$, $\theta_t = \frac{\log\left(\frac{\sigma^t-1}{4\sqrt{2}}\right)}{\log 2}$, where $t = r - r_1$, $A = 6$, $B = 2$. As before, $q_{94} > 6K$. For $1 \leq t \leq 63$ and $t \neq 2$, we compute

$$\epsilon = ||\theta_t q_{94}|| - K\,||\lambda q_{94}|| > 0.007.$$

Lemma (4.1) entails that $s - s_1 \leq 157$.

If $\Delta_2 < 0$, we obtain the same result.

For investigating the case $t = 2$, we use a theorem of Legendre, see Theorem 184 in [19] for details. Let $z = [a_0, a_1, ...]$ be a real number and $p, q \in \mathbb{Z}$. If

$$\left| \frac{p}{q} - z \right| < \frac{1}{2q^2},$$

then $\frac{p}{q}$ is a convergent of the continued fraction of z. Let t be a non negative integer such that $1 \leq q \leq q_t$, then

$$\frac{1}{(b+2)\, q^2} < \left| \frac{p}{q} - z \right|,$$

where $b = \max_{0 \leq i \leq t}\{a_i\}$

Now we finish our investigation of this part by considering the case $t = 2$. The reason behind excluding $t = 2$ is that $\frac{\sigma^2 - 1}{4\sqrt{2}} = \sigma$. If $t = 2$, then

$$\Delta_2 = (r_1 + 1)\log\sigma - s\log 2.$$

Consequently, by Eq.(3. 15),

$$\left| \frac{\log\sigma}{\log 2} - \frac{s}{r_1 + 1} \right| < \frac{3}{2^{s-s_1}(r_1 + 1)}.$$

Assume that $s - s_1 > 150$. Then $2^{s-s_1} > 6 \cdot 8 \cdot 10^{42} > 6 \cdot (r_1 + 1)$. Thus

$$\frac{3}{2^{s-s_1}(r_1 + 1)} < \frac{1}{2(r_1 + 1)^2}.$$

Therefore

$$\left| \frac{\log\sigma}{\log 2} - \frac{s}{r_1 + 1} \right| < \frac{1}{2(r_1 + 1)^2}.$$

Then, by Legendre's theorem, we deduce that $\frac{s}{r_1+1}$ is a convergent of $\frac{\log\sigma}{\log 2} = \lambda$. Actually, $q_{90} = 8876525780054568437394852482864205881249240 > 8 \cdot 10^{42} > r_1 + 1$. Then $b = 200$. Hence

$$\frac{1}{202(r_1 + 1)^2} < \frac{3}{2^{s-s_1}(r_1 + 1)}.$$

This implies that

$$2^{s-s_1} < 3 \cdot 202(r_1 + 1) < 3 \cdot 202 \cdot 8 \cdot 10^{42}.$$

Then

$$s - s_1 \leq 151.$$

Now, we turn our attention to the case $s - s_1 \leq 151$. Let

$$\Delta_3 = r\log\sigma - s_1\log 2 + \log\left(\frac{1}{4\sqrt{2}(2^{s-s_1} - 1)}\right).$$

Assume that $r - r_1 \geq 20$. Then, by lemma (2.3), we have

$$|\Delta_3| < \frac{2\sigma^4}{\sigma^{r-r_1}}.$$

If $\Delta_3 > 0$, then

$$0 < r\left(\frac{\log\sigma}{\log 2}\right) - s_1 + \frac{\log\left(\frac{1}{4\sqrt{2}(2^{s-s_1}-1)}\right)}{\log 2} < \frac{2\sigma^4}{(\log 2)\sigma^{r-r_1}} < \frac{3330}{\sigma^{r-r_1}}.$$

Let $K = 7.3\cdot 10^{42}$, $\lambda = \frac{\log\sigma}{\log 2}$, $\theta_u = \frac{\log\left(\frac{1}{4\sqrt{2}(2^u-1)}\right)}{\log 2}$, where $u = s - s_1$, $A = 3330$, $B = \sigma$. As before, $q_{94} > 6K$. For $1 \le u \le 150$, we find that

$$\epsilon = ||\theta_u q_{94}|| - K\,||\lambda q_{94}|| > 0.005.$$

Hence, by Lemma (4.1), $r - r_1 \le 65$. The same result is obtained if $\Delta_3 < 0$.
In summary, all the previous cases always give the bounds $r - r_1 \le 65$ and $s - s_1 \le 157$.

Now, let

$$\Delta_4 = r_1\log\sigma - s_1\log 2 + \log\left(\frac{\sigma^{r-r_1}-1}{4\sqrt{2}\,(2^{s-s_1}-1)}\right).$$

By (3. 25), we deduce that

$$|\Delta_4| < \frac{2\sigma^4}{\sigma^r}.$$

If $\Delta_4 > 0$, then

$$0 < r_1\left(\frac{\log\sigma}{\log 2}\right) - s_1 + \frac{\log\left(\frac{\sigma^t-1}{4\sqrt{2}(2^u-1)}\right)}{\log 2} < \frac{2\sigma^4}{(\log 2)\,\sigma^r} < \frac{3330}{\sigma^r},$$

where $t = r - r_1$ and $u = s - s_1$.
Let $K = 7.2\cdot 10^{42}$, $\lambda = \frac{\log\sigma}{\log 2}$, $\theta_{t,u} = \frac{\log\left(\frac{\sigma^t-1}{4\sqrt{2}(2^u-1)}\right)}{\log 2}$, $A = 3330$, $B = \sigma$. As before, $q_{94} > 6K$. We find that

$$\epsilon = ||\theta_{t,u} q_{94}|| - K\,||\lambda q_{94}|| > 0.0001,$$

for all $1 \le t \le 65$, $1 \le u \le 157$, except the pairs $(t,u) = (2,1), (4,2), (10,83), (28,99)$. Excluding these pairs, as they give negative values of ϵ, we get $r < 68$. This contradicts the main assumption that $r \ge 150$.
The case $t = 2, u = 1$**:** can be investigated as we did in treating Δ_2.
The case $t = 10, u = 83$**:** Using q_{95}, we find that

$$\epsilon = ||\theta_{10,83} q_{95}|| - K\,||\lambda q_{95}|| > 0.09,$$

Hence, by Lemma (4.1), $r \le 66$.
The case $t = 28, u = 99$**:** Using q_{95} again, we find that

$$\epsilon = ||\theta_{28,99} q_{95}|| - K\,||\lambda q_{95}|| > 0.03,$$

Hence, by Lemma (4.1), $r \le 67$.

The case $t = 4, u = 2$**:** We have
Since $\frac{\sigma^4-1}{4\sqrt{2}(2^2-1)} = 2\sigma^2$. Therefore,

$$(r_1+2)\frac{\log\sigma}{\log 2} - (s_1 - 1) = r_1\left(\frac{\log\sigma}{\log 2}\right) - s_1 + \frac{\log 2\sigma^2}{\log 2}.$$

Then

$$\left|(r_1+2)\frac{\log\sigma}{\log 2}-(s_1-1)\right|<\frac{3330}{\sigma^r}.$$

Therefore

$$\left|\frac{\log\sigma}{\log 2}-\frac{s_1-1}{r_1+2}\right|<\frac{3330}{\sigma^r(r_1+2)}.$$

As before $\sigma^r > 2\cdot 3330\cdot 8\cdot 10^{42} > 2\cdot 3330\cdot(r_1+2)$. Thus $\frac{s_1-1}{r_1+2}$ is a convergent of $\frac{\log\sigma}{\log 2}$. Similar calculations as before gives $r < 64$.

As a result, we see that the all the solutions lie inside the ranges $0 \le r_1 < r \le 150$ and $0 \le s_1 < s \le 600$. This completes the proof.

5. Conclusion

We completely determined the solutions of the Pillai's problem with Balancing numbers and powers of 2. We used a result of Bugeaud, Mignotte and Siksek that deduced from Matveev's theorem. A reduction lemma is then applied to decrease the too large upper bound to an investigatable level. We found that there are two numbers (-1 and -2) which have at least two representations of the form $B_r - 2^s$. This can be continued to investigate the problem of Pillai same problem with other integer sequences.

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