Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 48(2)(2016) pp. 1-10

ON Fuzzy Infra-Semiopen Sets

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Received: 08 February, 2016 / Accepted: 15 April, 2016 / Published online: 19 April, 2016

Abstract. The notions of $closure*$ and $interior*$ operators are extended to a fuzzy topology. We start with new concepts called fuzzy infra-semiopen (infra-semiclosed) sets. In applying these concepts, new sorts of fuzzy continuous functions, namely, fuzzy infra-semicontinuous, infra-semi-irresolute and fuzzy infra-semiopen (infra-semiclosed) functions are introduced. A detail study is carried out on properties of these new concepts. The relationship between these concepts and others are discussed and the inverse of these relations are illustrated with examples. Finally, the concept of fuzzy infra-semiconnected are presented.

AMS (MOS) Subject Classification Codes: 54A40, 54C08, 54C10.
Key Words: fuzzy $closure*$; fuzzy $interior*$; fuzzy infra-semiopen set; fuzzy infra-semiclosed set; fuzzy infra-semicontinuous function, infra-semi-irresolute function; fuzzy infra-semiopen function; fuzzy infra-semiclosed function; fuzzy infra-semiconnected.

1. Introduction

In fuzzy topology and fuzzy analysis, fuzzy open set and fuzzy continuity and different types of sets and functions which are weaker or stronger than fuzzy open sets and fuzzy continuity are found to be extremely useful and constitute an important investigating area. Since then, various types of open sets and continuous functions in fuzzy topology have been introduced and extensively investigated such as to their characterization, mutual interrelations and applications to the study of different fuzzy topological concepts, by a large number of mathematicians e.g. see [1], [2], [8] and [4].

In 2012, Rebert and Missier [9] have defined the notion of semi*open (infra-semiopen) sets. In this research, we generalize the classes of infra-semiopen sets and closure* and interior* operators to fuzzy topology space and the relations among fuzzy infra-semiopen sets and other kinds of fuzzy open sets are studied. Further, the notions of fuzzy infra-semicontinuous, fuzzy infra-semiopen and fuzzy infra-semiclosed function are introduced and discussed. We investigate the relations of these fuzzy function with other weak types of fuzzy continuous functions. Finally, the concept of fuzzy infra-semiconnected are introduced and some of interesting results about these new concepts are investigated.

2. Preliminaries

Everywhere in this paper, by X we mean a fuzzy topological space (fts, shortly) and a fuzzy function f from a fuzzy topological space X to a fuzzy topological space Y denoted by $f : X \to Y$.

We review some of results and basic definitions that will be used in this paper.

Definition 2.1. *A fuzzy set $B \in X$ is called:*

- *Fuzzy pre-open* [8] *if $B \leq Int(ClB)$.*
- *Fuzzy semiopen* [1] *if $B \leq Cl(IntB)$.*
- *Fuzzy α-open* [2] *if $B \leq Int(Cl(IntB))$.*

The family of all fuzzy α-open (resp. fuzzy preopen, fuzzy semi open) sets of X is denoted by $F\alpha O(X)$ (resp. $FPO(X)$, $FSO(X)$).

Definition 2.2. *A fuzzy set $\omega \in X$ is called:*

- *Fuzzy supra-pre-open* [5] *if $\omega \leq Int^*(Cl(\omega))$.*
- *Fuzzy infra-α-open* [6] *if $\omega \leq Int(Cl^*(Int(\omega)))$.*

The family of all fuzzy supra-pre-open and fuzzy infra-α-open sets in X will be denoted by $SFPO(X)$ and $IF\alpha O(X)$, respectively.

Definition 2.3. *A function $f : X \to Y$ is said to be:*

- *Fuzzy α-continuous* [2] *if for every fuzzy closed (open) set $u \in Y$, then $f^{-1}(u) \in F\alpha C(X)$ ($F\alpha O(X)$).*
- *Fuzzy semicontinuous* [1] *if for every fuzzy closed (open) set $u \in Y$, then $f^{-1}(u) \in FSC(X)$ ($FSO(X)$).*
- *Fuzzy pre-continuous* [2] *if for every fuzzy closed (open) set $u \in Y$, then $f^{-1}(u) \in FPC(X)$ ($FPO(X)$).*

Definition 2.4. *A fuzzy function $f : X \to Y$ called:*

- *Fuzzy semiopen* [1] *if $f(\eta) \in FSO(Y)$, $\forall\, \eta \in FO(X)$.*
- *Fuzzy semiclosed* [1] *if $f(\eta) \in FSC(Y)$, $\forall\, \eta \in FC(X)$.*

Definition 2.5. *A function $f : X \to Y$ is said to be:*

- *Fuzzy infra-α-continuous* [6] *if for every fuzzy closed (open) set $u \in Y$, then $f^{-1}(u) \in IF\alpha C(X)(IF\alpha O(X))$.*
- *Fuzzy supra-pre-continuous* [5] *if for every fuzzy closed (open) set $u \in Y$, then $f^{-1}(u) \in SFPC(X)(SFPO(X))$.*

3. Fuzzy infra-semiopen set

Definition 3.1. *Let η any fuzzy set. Then,*

- *Fuzzy closure**
  $(Cl^*\eta) = \wedge\{\omega : \omega \geq\ \eta,\ \omega\, is\ a\ fuzzy\ generalized - closed\ set\ of\ X\}$.
- *Fuzzy Interior**
  $(Int^*\eta) = \vee\{\omega : \omega \leq\ \eta,\ \omega\, is\ a\ fuzzy\ generalized - open\ set\ of\ X\}$.

Lemma 3.2. *Let η any fuzzy set. Then,*

- $\eta \leq Cl^*\eta \leq Cl\, \eta$.

- $Int\ \eta \leq Int^{*}\eta \leq \eta$.

Example 3.3. *If $X = \{c, d\}$ and h_1, h_2, h_3 and h_4 be fuzzy sets of X:*

$$h_1(c) = 0.6 \quad h_1(d) = 0.4$$
$$h_2(c) = 0.3 \quad h_2(d) = 0.5$$
$$h_3(c) = 0.3 \quad h_3(d) = 0.4$$
$$h_4(c) = 0.6 \quad h_4(d) = 0.5$$

Let $\tau = \{0_x, h_1, h_2, h_1 \vee h_2, h_1 \wedge h_2, 1_x\}$. Then,
$Int^{*}(h_4^c) = \bigvee \begin{pmatrix} 0.3 \leq x < 4 \\ 0.5 \end{pmatrix}$ *but $Int(h_4^c) = h_2$.*
Then, $Int(h_4^c) \leq Int^{*}(h_4^c) \leq h_4^c$ and
$Cl^{*}(h_2) = \bigwedge \begin{pmatrix} 0.3 < x \leq 4 \\ 0.5 \end{pmatrix}$ *but $Cl(h_2) = h_4^c$.*
Then, $h_2 \leq Cl^{*}(h_2) \leq Cl\,(h_2)$.

Definition 3.4. *A fuzzy set $\omega \in X$ is called:*

- *Fuzzy infra-semiopen if $\delta \leq \omega \leq Cl^{*}(\delta)$, where δ is a fuzzy open set.*
- *Fuzzy infra-semiclosed if $Int^{*}(\delta) \leq \omega \leq \delta$, where δ is a fuzzy closed set.*

The family of all fuzzy infra-semiopen and fuzzy infra-semiclosed sets in X will be denoted by IFSO(X) and IFSC(X), respectively.

Theorem 3.5. *Let η any fuzzy subset of X, then the next properties are equipollent:*

(i): $\eta \in IFSO(X)$.
(ii): $\eta \leq Cl^{*}Int\ \eta$.
(iii): $Cl^{*}\eta = Cl^{*}(Int\ \eta)$.
(iv): η^c *is fuzzy infra-semiclosed set.*
(v): $Int^{*}Cl\ \eta^c \leq \eta^c$.
(vi): $Int^{*}(Cl\ \eta^c) = Int^{*}\eta^c$.

Proof.

$(i) \Rightarrow (ii)$.**:** Let $\eta \in IFSO(X)$. Then, there exists $\omega \in FO(X)$ such that $\omega \leq \eta \leq Cl^{*}\omega$. Hence, $\omega \leq Int\ \eta$, then $\eta \leq Cl^{*}(Int\ \eta)$.

$(ii) \Rightarrow (iii)$.**:** We have $\eta \leq Cl^{*}(Int\ \eta)$ and we know that $Cl^{*}(Int\ \eta) \leq Cl^{*}\eta$, then $Cl^{*}\eta = Cl^{*}(Int\ \eta)$.

$(iii) \Rightarrow (iv)$.**:** we have $\eta \leq Cl^{*}(Int\eta)$, then $Int\ \eta \leq \eta \leq Cl^{*}(Int\ \eta)$. Take $\omega = Int\ \eta$, then we have $\omega \leq \eta \leq Cl^{*}\omega$, where η is fuzzy infra-semiopen set. Therefore, $Int^{*}\omega^c \leq \eta^c \leq \omega^c$, where $\omega^c \in FC(X)$. Then $\eta^c \in IFSO(X)$.

$(iv) \Rightarrow (v)$.**:** we have $Int^{*}\omega^c \leq \eta^c \leq \omega^c$, where $\omega^c \in FC(X)$. Hence, $Cl\ \eta^c \leq \omega^c$ and $Int^{*}(Cl\ \eta^c) \leq Int^{*}\omega^c \leq \eta^c$. Then, $Int^{*}(Cl\ \eta^c) \leq \eta^c$

$(v) \Rightarrow (vi)$.**:** we have $Int^{*}(Cl\ \eta^c) \leq \eta^c$. This implies that, $Int^{*}(Cl\ \eta^c) \leq Int^{*}\eta^c$ and we know that $Int^{*}\eta^c \leq Int^{*}(Cl\eta^c)$, then $Int^{*}(Cl\ \eta) = Int^{*}\eta$.

$(vi) \Rightarrow (i)$.**:** we have $Int^{*}(Cl\ \eta^c) \leq \eta^c$, then $(Int^{*}(Cl\ \eta^c)^c) \leq (\eta^c)^c$. This implies that $\eta \leq Cl^{*}(Int\ \eta)$. Therefore, $Int\eta \leq \eta \leq Cl^{*}(Int\ \eta)$. Put $\omega = Int\ \eta$. Then, $\omega \leq \eta \leq Cl^{*}\omega$ and $\eta \in IFSO(X)$. □

Definition 3.6. *For η fuzzy set in fts X. Then,*

- *Fuzzy infra-semiclosure*
  $IsCl\ \eta = \wedge\{\omega : \omega \geq \eta,\ \omega \in IFSC(X)\}$.
- *Fuzzy infra-semiInterior*
  $IsInt\ \eta = \vee\{\omega : \omega \leq\ \eta, \omega \in IFSO(X)\}$.

Theorem 3.7. *Let η any fuzzy subset of X. Then the next properties are true:*

(a): $(IsInt\ \eta)^c = IsCl\ \eta$.
(b): $(IsCl\ \eta)^c = IsInt\ \eta$.
(c): $IsInt\ \eta = \eta \wedge Cl^*(Int\ \eta)$.
(d): $IsCl\ \eta = \eta \vee Int^*(Cl\ \eta)$.

Proof. We will prove only (b) and (c).

$$\begin{aligned} (\mathbf{b})\ \ (IsCl\ \eta)^c &= \big(\wedge\{\omega : \omega \geq\ \eta,\ \omega \in IFSC(X)\}\big)^c \\ &= \vee\{\omega^c : \omega^c \leq\ \eta,\ \omega^c \in IFSO(X)\} \\ &= IsInt\ \eta \end{aligned}$$

(c) We know that $IsInt$ is fuzzy infra-semiopen,
then $IsInt(\eta) \leq Cl^*(Int(IsInt(\eta))) \leq Cl^*(Int(\eta))$.
So, $IsInt(\eta) \leq \eta \wedge Cl^*(Int(\eta)) \rightarrow (1)$
We have $Int(\eta) \leq \eta \wedge Cl^*(Int(\eta)) \leq Cl^*(Int(\eta))$. By Definition 3.4
$(\eta \wedge Cl^*(Int(\eta))) \in IFSO(X)$ and $\eta \wedge Cl^*(Int(\eta)) \leq \eta$,
then $\eta \wedge Cl^*(Int(\eta)) \leq IsInt(\eta) \rightarrow (2)$
From (1) and (2), we get $IsInt(\eta) = \eta \wedge Cl^*(Int(\eta))$. □

Proposition 3.8. *If η and ω are any fuzzy subsets in X and $\eta \leq \omega$, then the next properties are true:*

(1) *$IsInt\ \eta$ is the largest fuzzy infra-semiopen set contained in η.*
(2) *$IsInt\ \eta \leq \eta$.*
(3) *$IsInt\ \eta \leq IsInt\ \omega$.*
(4) *$IsInt(IsInt\ \eta) = IsInt\ \eta$.*
(5) *$\eta \in IFS0(X)\ \Leftrightarrow IsInt\ \eta = \eta$.*
(6) *$IsInt\ \eta \vee IsInt\ \omega \leq IsInt(\eta \vee \omega)$.*
(7) *$IsInt\ \eta \wedge IsInt\ \omega \geq IsInt(\eta \wedge \omega)$.*

Proposition 3.9. *Let the fuzzy sets η and ω be in fts X and $\eta \leq \omega$. Then, the next properties hold:*

(1) *$IsCl\ \eta$ is the smallest fuzzy infra-semiclosed set containing η.*
(2) *$\eta \leq IsCl\ \eta$.*
(3) *$IsCl\ \eta \leq IsCl\ \omega$.*
(4) *$IsCl(IsCl\ \eta) = IsCl\ \eta$.*
(5) *$\eta \in IFSC(X)\ \Leftrightarrow IsCl\ \eta = \eta$.*
(6) *$IsCl\ \eta \vee IsCl\ \omega = IsCl(\eta \vee \omega)$.*
(7) *$IsCl\ \eta \wedge IsCl\ \omega \geq IsCl(\eta \wedge \omega)$.*

Theorem 3.10. *If $\eta \in IFSO(X)$, then $IsCl(\eta) \in IFSO(X)$.*

Proof. If $\eta \in IFSO(X)$, then $\eta \leq Cl^*(Int(\eta)) \leq Cl^*(Int(IsCl(\eta)))$. This implies that $IsCl(\eta) \leq IsCl(Cl^*(Int(IsCl(\eta)))) = Cl^*(Int(IsCl(\eta)))$.
This shows that $IsCl(\eta) \in IFSO(X)$ □

Theorem 3.11. *For a fts X, then*

(i): *If $\omega \in IFSO(X)$ and $\omega \leq \eta \leq Cl^*\omega$, then $\eta \in IFSO(X)$.*
(ii): *If $\omega \in IFSC(X)$ and $Int^*\omega \leq \eta \leq \omega$, then $\eta \in IFSC(X)$.*

Proof.

(i): If $\omega \in IFSO(X)$ such that $\delta \leq \omega \leq Cl^*(\delta)$. Then, $\delta \leq \eta$ and $\omega \leq Cl^*\delta$. This implies that $Cl^*\omega \leq Cl^*\delta$. Therefore, $\delta \leq \eta \leq Cl^*\delta$. Then, $\eta \in IFSO(X)$.

(ii): Obivous from (1). □

Theorem 3.12. *Let η any fuzzy subset of X, then* $Int^*\eta \leq IsInt\,\eta \leq \eta \leq IsCl\,\eta \leq Cl^*\eta$.

Proof. We know that, $Int^*\eta \leq \eta$. This implies that $IsInt(Int^*\eta) \leq IsInt\,\eta$. Then $Int^*\eta \leq IsInt\,\eta \leq \eta \rightarrow (1)$.
Also, we know that $\eta \leq IsCl\,\eta \leq Cl^*\eta \rightarrow (2)$.
From (1) and (2), we get $Int^*\eta \leq IsInt\,\eta \leq \eta \leq IsCl\,\eta \leq Cl^*\eta$. □

Theorem 3.13. *A fuzzy set $\eta \in IFSO(X)$ iff for every fuzzy singleton $p \leq \eta$, there exists a fuzzy set $\lambda \in IFSO(X)$ such that $p \leq \lambda \leq \eta$.*

Proof. It is obvious.. □

Theorem 3.14. *Let η any fuzzy subset of X, then the next properties are true:*

(a): *If η is a fuzzy infra-semiclosed (infra-semiopen) set, then η is a fuzzy semiclosed (semiopen) set.*

(b): *If η is a fuzzy infra-α-closed (infra-α-open) set, then η is a fuzzy infra-semiclosed (infra-semiopen) set.*

(c): *If η is a fuzzy closed (open) set, then η is a fuzzy infra-semiclosed (infra-semiopen) set.*

Proof. It is clear from Definition 3.4 Definition 2.2 Definition 2.1 and basic relations. The proof is completed. □

The relation among various sorts of fuzzy open sets are illustrated by the following "Implication Figure 1".

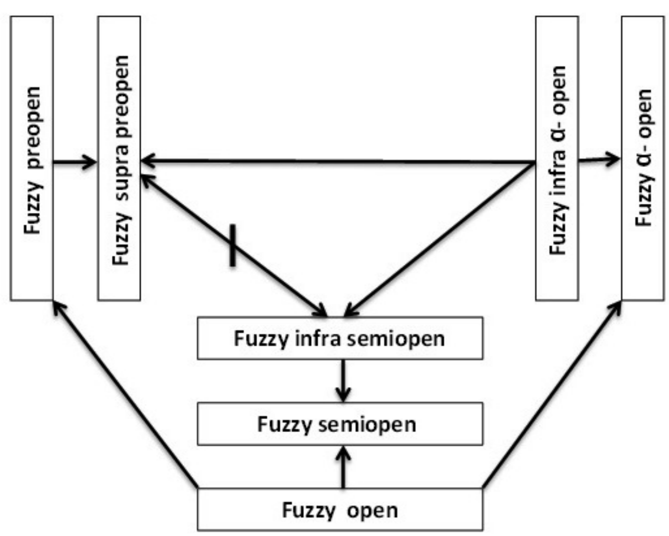

FIGURE 1

Remark 3.15. *In general, the reverse of the above implication (Figure 1) is not true as shown by the next examples:*

Example 3.16. *Suppose that $X = \{c, d\}$ and h_1, h_2 and h_3 be fuzzy sets of X:*

$$h_1(c) = 0.3 \quad h_1(d) = 0.5$$
$$h_2(c) = 0.6 \quad h_2(d) = 0.7$$
$$h_3(c) = 0.7 \quad h_3(d) = 0.5$$

If $\tau = \{0_x, h_1, 1_x\}$. Then, h_2 is not a fuzzy open (infra-semiopen) set but it is a fuzzy supra-pre-open set.

Example 3.17. *Suppose that $X = \{c, d\}$ and h_1, h_2 and h_3 be fuzzy sets of X:*

$$h_1(c) = 0.4 \quad h_1(d) = 0.7$$
$$h_2(c) = 0.4 \quad h_2(d) = 0.5$$
$$h_3(c) = 0.5 \quad h_3(d) = 0.5$$

If $\tau = \{0_x, h_1, h_2, 1_x\}$. Then, h_3 is not a fuzzy infra-semiopen set but it is a fuzzy semi open set and fuzzy supra-pre-open set.

Example 3.18. *Suppose that $X = \{c, d\}$ and h_1, h_2 and h_3 be fuzzy sets of X:*

$$h_1(c) = 0.4 \quad h_1(d) = 0.5$$
$$h_2(c) = 0.5 \quad h_2(d) = 0.7$$
$$h_3(c) = 0.5 \quad h_3(d) = 0.5$$

If $\tau = \{0_x, h_1, h_2, 1_x\}$. Then, h_3 is not a fuzzy infra-α-open (fuzzy α- open) and not a fuzzy supra-pre-open (pre-open) set but it is a fuzzy infra-semiopen set.

Theorem 3.19.

(a): *The union of any number of a fuzzy infra-semiopen set is a fuzzy infra-semiopen set.*

(b): *The intersection of any number of a fuzzy infra-semiclosed set is a fuzzy infra-semiclosed set.*

Proof.

(a): Let $\{\eta_\rho\}$ be family of a fuzzy infra-semiopen set. Then, for each ρ, $\eta_\rho \leq Cl^*(Int\eta_\rho)$ and $\vee\eta_\rho \leq \vee(Cl^*(Int\eta_\rho)) \leq Cl^*Int(\vee\eta_\rho)$. Hence $\vee\eta_\rho$ is a fuzzy infra-semiopen set.

(b): It is obvious. □

4. Fuzzy infra-semicontinuous Function

In this section, the new types of fuzzy functions, namely, fuzzy infra-semicontinuous, fuzzy infra-irresolute, fuzzy infra-semiopen and fuzzy infra-semiclosed functions are studied and established their properties. The composition of fuzzy infra-semicontinuous functions are investigated.

Moreover, we introduce new concept. Fuzzy infra-semiconnected and study the effect of a fuzzy infra-semicontinuous function on it.

Definition 4.1. *A function $f : X \to Y$ is said to be:*

- *Fuzzy infra-semicontinuous if $f^{-1}(\eta) \in IFSC(X)$ $(IFSO(X))$, for every fuzzy closed (open) set $\eta \in Y$.*
- *Fuzzy infra-irresolute if $f^{-1}(\eta) \in IFPC(X)$ $(IFPO(X))$, for every fuzzy infra-semiclosed (infra-semiopen) set $\eta \in Y$.*

Theorem 4.2. *The next properties are equipollent for $f : X \to Y$:*

(i): *f is fuzzy infra-semicontinuous.*
(ii): *for every fuzzy singleton $t \in X$ and every fuzzy open set $\omega \in Y$ such that $f(t) \leq \omega$, there exists a fuzzy infra-semiopen set, $\eta \in X$ such that $t \leq \eta$ and $\eta \leq f^{-1}(\omega)$.*
(iii): *for each fuzzy singleton $t \in X$ and each fuzzy open set $\omega \in Y$ such that $f(t) \leq \omega$, there exists a fuzzy infra-semiopen set, $\eta \in X$ such that $t \leq \eta$ and $f(\eta) \leq \omega$.*
(iv): *$f^{-1}(\omega) \in IFSC(X)$, $\forall\, \omega \in FC(Y)$.*
(v): *$f(Iscl(\eta)) \leq Cl(f(\eta))$, $\forall\, \eta \in X$.*
(vi): *$IsCl(f^{-1}(\omega)) \leq f^{-1}(Cl(\omega))$, $\forall\, \omega \in Y$.*
(vii): *$f^{-1}(Int(\omega)) \leq IsCl(f^{-1}(\omega))$, $\forall\, \omega \in Y$.*

Proof.

$(i) \Rightarrow (ii)$.**:** Suppose that $\omega \in FO(Y)$ and a fuzzy singleton $t \in X$ such that $f(t) \subseteq \omega$, then there exists $n \in FO(Y)$ such that $f(t) \leq n \leq \omega$. Because f is a fuzzy infra-semicontinuous, $\eta = f^{-1}(n)$ is fuzzy infra-semiopen set and we get $t \leq \eta = f^{-1}(n) \leq f^{-1}(\omega)$.

$(ii) \Rightarrow (iii)$.**:** Suppose that $\omega \in FO(Y)$ and t be fuzzy singleton in X such that $f(t) \subseteq \omega$, then there exists a fuzzy infra-semiopen set, η such that $t \leq \eta$ and $\eta \leq f^{-1}(\omega)$. So, we get ($t \leq \eta$) and $f(\eta) \leq f^{-1}((f(\omega))) \leq \omega$.

$(iii) \Rightarrow (i)$.**:** Let's take $t \leq f^{-1}(\omega)$ and $\omega \in FO(Y)$. Thus, $f(t) \leq f(f^{-1}(\omega)) \leq \omega$. Then, there exists a fuzzy infra-semiopen set, η such that ($t \leq \eta$) and $f(\eta) \leq \omega$. Then, we get $t \leq \eta \leq f^{-1}(f(\eta)) \leq f^{-1}(\omega)$. From Theorem 3.13, we get $f^{-1}(\omega) \in X$ is fuzzy infra-semiopen set. Then, f is a fuzzy infra-semicontinuous..

$(i) \Rightarrow (iv)$.**:** Let $\omega \in FC(Y)$. Thus, $\omega^c \in FO(Y)$. Then, $f^{-1}(\omega^c) \in IFSO(X)$. Hence, $f^{-1}(\omega) \in IFSC(X)$.

$(iv) \Rightarrow (v)$.**:** Let $\eta \in X$, then $\eta \leq f^{-1}(f(\eta)) \leq f^{-1}(Clf(\eta))$ then, we get $IsCl(\eta) \leq f^{-1}(Clf(\eta)) = IsCl(f^{-1}(Clf(\eta)))$.
Hence, $f(Iscl(\eta)) \leq Cl(f(\eta))$.

$(v) \Rightarrow (vi)$.**:** Let $\omega \in Y$, then $f^{-1}(\omega)$ in X, by (iv), we get $f(Iscl(f^{-1}(\omega))) \leq Cl(f(f^{-1}(\omega))) \leq Cl\omega$, then $Iscl(f^{-1}(\omega)) \leq f^{-1}(Cl\omega)$.

$(vi) \Rightarrow (vii)$.**:** Let $\omega \in Y$, then $\omega^c \in Y$. By (vi), we have $Iscl(f^{-1}(\omega^c)) \leq f^{-1}(Cl\omega^c)$. Then, $f^{-1}(Int\omega) \leq IsInt(f^{-1}(\omega))$.

$(vi) \Rightarrow (i)$.**:** Let $\omega \in Y$, by (vi), we have $f^{-1}(\omega) = f^{-1}(Int\omega) \leq IsInt(f^{-1}(\omega))$. Hence, $f^{-1}(\omega) \in IFSO(X)$. Then, f is a fuzzy infra-semicontinuous function. □

Using the same arguments as Theorem 4.2, Propositions 3.8 and Propositions 3.9, one can prove the following theorem.

Theorem 4.3. *The next properties are equipollent for $f : X \to Y$:*

(i): *f is fuzzy infra-semicontinuous.*
(ii): *$f^{-1}(\omega) \in IFSC(X)$, $\forall\, \omega \in FC(Y)$.*
(iii): *$Int^*Cl(\omega) \leq (f^{-1}(Cl\omega))$, $\forall\, \omega \in Y$.*
(iv): *$f(Cl^*Int(\eta)) \leq Cl\ f(\eta)$, $\forall\, \eta \in X$.*

Remark 4.4. *The composition of two fuzzy infra-semicontinuous functions may not be a fuzzy infra-semicontinuous function as we can see by the next example.*

Example 4.5. *Suppose that $X = Y = Z = \{c, d\}$ and h_1, h_2, h_3, h_4, h_5 and h_6 be fuzzy sets of X:*

$$\begin{array}{ll} h_1(c) = 0.3 & h_1(d) = 0.3 \\ h_2(c) = 0.5 & h_2(d) = 0.5 \\ h_3(c) = 0.3 & h_3(d) = 0.5 \\ h_4(c) = 0.5 & h_4(d) = 0.8 \\ h_5(c) = 0.4 & h_5(d) = 0.5 \\ h_6(c) = 0.4 & h_6(d) = 0.4 \end{array}$$

If $\tau_x = \{0_x, h_1, h_2, 1_x\}$, $\tau_y = \{0_x, h_3, h_4, 1_x\}$ and $\tau_z = \{0_x, h_5, 1_x\}$ and the function $g : (X, \tau_x) \to (Y, \tau_y)$ and $f : (Y, \tau_y) \to (Z, \tau_z)$ be the identity fuzzy functions. We can see that g and f are fuzzy infra-semicontinuous functions. But (fog) is not a fuzzy infra-semicontinuous function.

Theorem 4.6. *If $g : X \to Y$ be a fuzzy infra-semicontinuous and $f : Y \to Z$ be is a fuzzy continuous function such that $(fog) : X \to Z$, then (fog) is fuzzy infra-semicontinuous function.*

Proof. Let $\omega \in Z$ be a fuzzy open set. Thus, $((fog)^{-1}(\omega)) = (g^{-1}(f^{-1}(\omega)))$. Then, $f^{-1}(g^{-1}(v)) \in X$ is a fuzzy infra-semiopen set. □

The relation among various kinds of fuzzy continuous functions are illustrated by the following "Implication Figure 2".

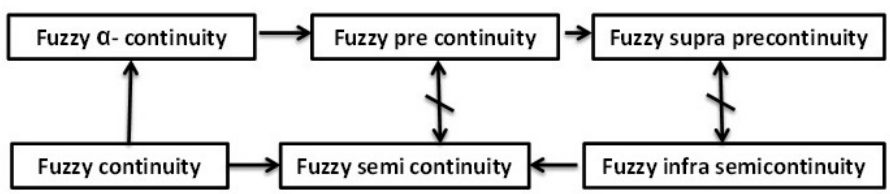

FIGURE 2

Remark 4.7. *In general, the reverse of the above implication (Figure 2) is not true as shown by the next examples.:*

Example 4.8. *Suppose that $X = Y = Z = \{c, d\}$ and $h_1,\ h_2$ and h_3 be fuzzy sets of X:*

$$\begin{array}{ll} h_1(c) = 0.3 & h_1(d) = 0.5 \\ h_2(c) = 0.4 & h_2(d) = 0.7 \\ h_3(c) = 0.5 & h_3(d) = 0.5 \end{array}$$

If $\tau_x = \{0_x, h_1, h_2, 1_x\}$ and $\tau_y = \{0_x, h_3, 1_x\}$ and the function $g : (X, \tau_x) \to (Y, \tau_y)$ be the fuzzy identity functions. Then, g is not a fuzzy infra-semicontinuous (pre-continuous) function which is a fuzzy supra-pre-continuous and fuzzy semicontinuous.

Example 4.9. *Suppose that $X = Y = \{c, d\}$ and $h_1,\ h_2$ and h_3 be fuzzy sets of X:*

$$\begin{array}{ll} h_1(c) = 0.4 & h_1(d) = 0.5 \\ h_2(c) = 0.5 & h_2(d) = 0.7 \\ h_3(c) = 0.5 & h_3(d) = 0.5 \end{array}$$

If $\tau_x = \{0_x, h_1, h_2, 1_x\}$, $\tau_y = \{0_x, h_3, 1_x\}$ and $f : (X, \tau_x) \to (Y, \tau_y)$ be the fuzzy identity function. We can see f is not a fuzzy infra-α-continuous (fuzzy α-continuous) function and not a fuzzy supra-precontinuous (fuzzy pre-continuous) function which is a fuzzy infra-semicontinuous function.

Definition 4.10. *A fuzzy function $f : X \to Y$ called:*

- *Fuzzy infra-semiopen if $f(\eta) \in IFSO(Y)$, $\forall\, \eta \in FO(X)$.*
- *Fuzzy infra-semiclosed if $f(\eta) \in IFSC(Y)$, $\forall\, \eta \in FC(X)$.*

The following example explain the relation between fuzzy infra-semiopen (infra-semiclosed) and fuzzy semiopen (semiclosed).

Example 4.11. *Suppose that $X = Y = Z = \{c, d\}$ and $h_1,\ h_2$ and h_3 be fuzzy sets of X:*

$$h_1(c) = 0.3 \quad h_1(d) = 0.5$$
$$h_2(c) = 0.4 \quad h_2(d) = 0.7$$
$$h_3(c) = 0.5 \quad h_3(d) = 0.5$$

If $\tau_x = \{0_x, h_3, 1_x\}$ and $\tau_y = \{0_x, h_1, h_2, 1_x\}$ and the function $g : (X, \tau_x) \to (Y, \tau_y)$ be the fuzzy identity functions. Then, g is not a fuzzy infra-semiopen (infra-semiclosed) which is a fuzzy semiopen (semiclosed).

Theorem 4.12. *The following statements are equipollent for a function $f : X \to Y$:*

(i): *f is a fuzzy infra-semiopen.*
(ii): $f(Int\,\eta) \leq IsInt\,(f(\eta))$, $\forall\, \eta \in X$.
(iii): $Int\,(f^{-1}(\omega)) \leq f^{-1}(IsInt(\omega))$, $\forall\, \omega \in Y$.
(iv): $f^{-1}(IsCl(\omega)) \leq Cl\,(f^{-1}(\omega))$, $\forall\, \omega \in Y$.
(v): $f(Int\,\eta) \leq Int^{*}Cl\,(f(\eta))$, $\forall\, \eta \in X$.

Proof.

$(i) \Rightarrow (ii)$.**:** Let f be a fuzzy infra-semiopen function and $\eta \in X$, $f(Int(\eta)) \leq f(\eta)$. we have $IsInt(f(Int(\eta))) \leq IsIntf(\eta)$.
Then, $f(Int(\eta)) \leq IsIntf(\eta)$.

$(ii) \Rightarrow (iii)$.**:** Let $\omega \in Y$, then $f^{-1}(\omega) \in X$. We put $f^{-1}(\omega) = \eta$ in (ii), we get $f(Int(f^{-1}(\omega))) \leq IsInt(f(f^{-1}(\omega))) \leq IsInt(\omega)$. Then, $Int(f^{-1}(\omega)) \leq f^{-1}(IsInt(\omega))$.

$(iii) \Rightarrow (iv)$.**:** Let $\omega \in Y$ and $\omega^c \in Y$. In (iii) we put $\omega^c = \omega$, then we get $Int(f^{-1}(\omega^c)) \leq f^{-1}(IsInt(\omega^c))$. Then, $(Cl(f^{-1}(\omega)))^c \leq (f^{-1}(IsCl(\omega)))^c$. Hence $f^{-1}(IsCl(\omega)) \leq Cl(f^{-1}(\omega))$.

$(iv) \Rightarrow (v)$.**:** Let $\eta \in X$, then $(f(\eta))^c \in Y$. Using (iv), we get $f^{-1}(IsCl((f(\eta))^c)) \leq Cl(f^{-1}((f(\eta))^c))$. This implies that $(f^{-1}(IsInt(f(\eta))))^c \leq (Int(f^{-1}(f(\eta))))^c$, then $Int(\eta) \leq f^{-1}(IsInt(f(\eta)))$ and $f(Int(\eta)) \leq IsInt(f(\eta)) \leq Int^{*}Cl(IsInt(f(\eta)))$.
Hence, $f(Int\eta) \leq Int^{*}Cl(f(\eta))$.

$(v) \Rightarrow (i)$.**:** Let $\eta \in X$. By using (v), we have $f(\eta) \leq Int^{*}Cl(f(\eta))$, then f is a fuzzy infra-semiopen function. □

Corollary 4.13. *The following statements are equipollent for a function $f : X \to Y$:*

(i): *f is fuzzy infra-semiclosed.*
(ii): $f(IsCl\omega) \leq Cl(f(\omega))$, $\forall\, \omega \in Y$.
(iii): $Int(f^{-1}(\omega)) \leq f^{-1}(IsInt(\omega))$, $\forall\, \omega \in Y$.
(iv): $Int(f^{-1}(\omega)) \leq f^{-1}(Int^{*}Cl(\omega))$, $\forall\, \omega \in Y$.

Now, the concept of fuzzy infra-semiconnected is introduced as follows:

Definition 4.14. *A fuzzy set $\eta \in X$ is said to be fuzzy infra-semiconnected iff η can not be expressed as the Union of two fuzzy infra-semiseparated sets..*

Theorem 4.15. *Let $g : Y \to Z$ be a surjective fuzzy infra-semicontinuous function. If δ is a fuzzy infra-semiconnected subset in Y hence, $f(\delta)$ is fuzzy connected in Z.*

Proof. Suppose to the contrary that $f(\delta) \in Y$ is not a fuzzy infra-semiconnected. Hence, there exists fuzzy infra-semiseparated subsets η and ω in Y such that $f(\delta) = \eta \cup \omega$.
Because f is a surjective fuzzy infra-semicontinuous function, $f^{-1}(\eta)$ and $f^{-1}(\omega) \in FISO(X)$ and $\delta = f^{-1}(f(\delta)) = f^{-1}(\eta \cup \omega) = f^{-1}(\eta) \cup f^{-1}(\omega)$.
It is clear that $f^{-1}(\eta)$ and $f^{-1}(\omega)$ are fuzzy infra-semi separated in X, then δ is not fuzzy infra-semiconnected in X, contrary to the hypothesis!!!!
Thus $f(\delta)$ is fuzzy infra-semiconnected. □

5. Conclusion

In the present paper, we extend the notions of (infra-semiopen) $semi * open$ sets and $closure*$ and $interior*$ operators to fuzzy topology space. Further, the concepts of fuzzy infra-semicontinuous, fuzzy infra-irresolute, fuzzy infra-semiopen and fuzzy infra-semiclosed function are introduced and the relations with other weak forms of fuzzy continuous functions (fuzzy open sets) are discussed. The concept of fuzzy infra-semiconnected is introduced and some of interesting results about these new concepts are investigated.
Moreover, This study will be useful and open new windows in the study of infra-semiopen set, infra-semicontinuities, infra-semiseparation axioms and infra-semicompactness in fuzzy topological spaces as applications of these new concepts.

6. Acknowledgement

I would like to thank the referees for their useful comments and suggestions.

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