order. The resulting fractional-order equations can be rarely solved exactly or analytically. Consequently, approximate and numerical techniques are playing an important role in identifying the solutions behaviour of such fractional equations. Recently, considerable attention has been given to the establishment of techniques for the solution of the fractional differential equations using orthogonal functions. The main characteristic of this technique is that it reduces the solution of differential equations to the solution of a system of algebraic equations. Historically this approach originated from the use of Fourier [24], Walsh [11] and block-pulse functions [28] and was later extended to other classical orthogonal polynomials such as Chebyshev, Legendre, Hermite, and Laguerre polynomials [33]. In most of the presented works, the use of numerical techniques in conjunction with operational matrices for differentiation and integration operators of some orthogonal polynomials, for the solution of fractional differential equations on finite and infinite intervals, produced highly accurate solutions for such equations, see [6] for a recent review.

The fractional Riccati model can be obtained by using the fractional derivative operator on the classical Riccati equation. The mathematical theory of the classical Riccati differential equation along with some applications are considered in detail in the book [29]. This equation as an important model has appeared in a number of areas of science and engineering. Among others, we emphasize in control system theory [3], optimal filtering [20], and financial markets [7]. See also [14] for a recent survey on historical perspectives of the Riccati equation.

In the current work, we consider the fractional-order Riccati model of the form

$$\begin{cases} \mathcal{D}^{(q)} X(t) = a(t)\, X^2(t) + b(t)\, X(t) + c(t) \quad t > 0, \\ X(0) = X_0, \end{cases} \tag{1. 1}$$

where $\mathcal{D}^{(q)}$ is the standard Caputo fractional derivative operator and $0 < q \leqslant 1$. There has been significant interest in developing analytical as well as numerical schemes for the solution of the fractional Riccati differential equation. The most significant analytical schemes include Adomian's decomposition method [21], modified homotopy perturbation method [22, 17, 13], and He's variational iteration method [1, 19]. On the other hand, computational techniques such as polynomial approximations [35, 26], stochastic technique based on particle swarm optimization and simulated annealing [27], a combination of finite difference and Padé-variational iteration scheme [32], series solution [8], Legendre-wavelet operational matrix [4], predictor-corrector approaches [12], Haar wavelet method [18], fractional Chebyshev finite difference [16], and an iterative reproducing kernel Hilbert space method [30], have been developed in the past to solve the nonlinear equation (1. 1).

In this note, we take a further step towards proposing approximation methods as extension of the previous works [36],[15], and [34] for solving (1. 1). We use the fractional-order polynomials including the Bessel, Chelyshkov and shifted Legendre functions to approximate the solution of (1. 1) accurately. The main idea of the proposed technique based on using these (orthogonal) functions along with collocation points is that it converts the differential or integral operator involved in (1. 1) to an algebraic form, thus greatly reducing the computational effort.

The rest of this paper is divided into four sections: In Section 2, first definitions and mathematical preliminaries of fractional calculus are presented. Then in subsequent subsections a brief review of the properties of the Bessel, Chelyshkov, and (shifted) Legendre polynomials is outlined. Sections 3 is devoted to the presentation of the proposed collocation scheme applied to nonlinear Riccati initial value problem. The error analysis technique based on the residual function is developed for the present method. In computational Section 4, we apply the proposed method to the some test problems and report our numerical findings. We end the paper with few concluding remarks in Section 5.

2. Basic definitions

In this section, first some properties of the fractional calculus theory are presented. Afterwards, the definitions of fractional Bessel, Chelyshkov as well as (shifted) Legendre polynomials are recalled and some properties of them required for our subsequent sections are reviewed.

2.1. **Fractional calculus.**

Definition 2.2. *Suppose that $f(t)$ is m-times continuously differentiable. The fractional derivative $\mathcal{D}^{(q)}$ of $f(t)$ of order $q > 0$ in the Caputo's sense is defined as*

$$\mathcal{D}^{(q)} f(t) = \begin{cases} \mathcal{I}^{m-q} f^{(m)}(t) & \text{if } m-1 < q < m, \\ f^{(m)}(t), & \text{if } q = m, \quad m \in \mathbb{N}, \end{cases} \tag{2. 2}$$

where

$$\mathcal{I}^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}}\, ds, \quad t > 0.$$

The properties of the operator $\mathcal{D}^{(q)}$ can be found in [25]. We make use of the followings

$$\mathcal{D}^{(q)}(C) = 0 \quad (C \text{ is a constant}), \tag{2. 3}$$

$$\mathcal{D}^{(q)}\, t^{\gamma} = \begin{cases} \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-q)} t^{\gamma-q}, & \text{for } \gamma \in \mathbb{N}_0 \text{ and } \gamma \geqslant \lceil q \rceil, \text{ or } \gamma \notin \mathbb{N}_0 \text{ and } \gamma > \lfloor q \rfloor, \\ 0, & \text{for } \gamma \in \mathbb{N}_0 \text{ and } \gamma < \lceil q \rceil. \end{cases} \tag{2. 4}$$

We have used the ceiling function $\lceil q \rceil$ to denote the smallest integer greater than or equal to q, and the floor function $\lfloor q \rfloor$ to denote the largest integer less than or equal to q.

2.3. **Bessel polynomials.** Let N be a positive integer. For $n = 0, 1, \ldots, N$, the truncated polynomial series $J_{n,N}(t)$ defined by [5]

$$J_{n,N}(t) = \sum_{k=0}^{\lfloor \frac{N-n}{2} \rfloor} \frac{(-1)^k}{k!(n+k)!} \left(\frac{t}{2}\right)^{2k+n}, \quad 0 \leqslant t \leqslant 1. \tag{2. 5}$$

The fractional-order Bessel functions can be defined by introducing the change of variable $t \to t^{\alpha}$ based on the Bessel polynomials (2. 5). Let these polynomials denoted by $J^{\alpha}_{n,N}(t)$. These generalization are obtained as (cf. [36])

$$J^{\alpha}_{n,N}(t) = \sum_{k=0}^{\lfloor \frac{N-n}{2} \rfloor} \frac{(-1)^k}{k!(n+k)!} \left(\frac{t^{\alpha}}{\omega}\right)^{2k+n}, \quad 0 \leqslant t \leqslant T, \tag{2. 6}$$

where the real parameter $0 < \alpha < 1$ and $\omega = 2T$. Our aim is to find an approximate solution of model (1. 1) expressed in the truncated Bessel series form (2. 6)

$$X_{N,\alpha}(t) = \sum_{n=0}^{N} a_n \, J_{n,N}^{\alpha}(t), \quad 0 \leqslant t \leqslant T, \tag{2. 7}$$

where T is a given final time and the unknown coefficients a_n, $n = 0, 1, \dots, N$ are sought. To continue, we write $J_{n,N}^{\alpha}(t)$ in the matrix form as follows

$$\boldsymbol{J}_{\alpha}(t) = \boldsymbol{T}_{\alpha}(t)\, \boldsymbol{D}^t \Leftrightarrow \boldsymbol{J}_{\alpha}^t(t) = \boldsymbol{D}\, \boldsymbol{T}_{\alpha}^t(t), \tag{2. 8}$$

here, a superscript t denotes the matrix transpose operation and

$$\boldsymbol{J}_{\alpha}(t) = \begin{bmatrix} J_{0,N}^{\alpha}(t) & J_{1,N}^{\alpha}(t) & \dots & J_{N,N}^{\alpha}(t) \end{bmatrix}, \quad \boldsymbol{T}_{\alpha}(t) = \begin{bmatrix} 1 & t^{\alpha} & t^{2\alpha} & \dots & t^{N\alpha} \end{bmatrix}.$$

Also if N is odd, the $(N+1) \times (N+1)$ matrix $\boldsymbol{D}$ takes the form

$$\boldsymbol{D} = \begin{bmatrix} \frac{1}{0!\,0!\,\omega^0} & 0 & \frac{-1}{1!\,1!\,\omega^2} & \dots & \frac{(-1)^{\frac{N-1}{2}}}{(\frac{N-1}{2})!\,(\frac{N-1}{2})!\,\omega^{N-1}} & 0 \\ 0 & \frac{1}{0!\,1!\,\omega^1} & 0 & \dots & 0 & \frac{(-1)^{\frac{N-1}{2}}}{(\frac{N-1}{2})!\,(\frac{N+1}{2})!\,\omega^{N}} \\ 0 & 0 & \frac{1}{0!\,2!\,\omega^2} & \dots & \frac{(-1)^{\frac{N-3}{2}}}{(\frac{N-3}{2})!\,(\frac{N+1}{2})!\,\omega^{N-1}} & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{0!\,(N-1)!\,\omega^{N-1}} & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{0!\,N!\,\omega^N} \end{bmatrix},$$

while in the case of even N we have

$$\boldsymbol{D} = \begin{bmatrix} \frac{1}{0!\,0!\,\omega^0} & 0 & \frac{-1}{1!\,1!\,\omega^2} & \dots & 0 & \frac{(-1)^{\frac{N}{2}}}{(\frac{N}{2})!\,(\frac{N}{2})!\,\omega^{N}} \\ 0 & \frac{1}{0!\,1!\,\omega^1} & 0 & \dots & \frac{(-1)^{\frac{N-2}{2}}}{(\frac{N-2}{2})!\,(\frac{N}{2})!\,\omega^{N-1}} & 0 \\ 0 & 0 & \frac{1}{0!\,2!\,\omega^2} & \dots & 0 & \frac{(-1)^{\frac{N-2}{2}}}{(\frac{N-2}{2})!\,(\frac{N+2}{2})!\,\omega^{N}} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{0!\,(N-1)!\,\omega^{N-1}} & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{0!\,N!\,\omega^N} \end{bmatrix}.$$

By means of (2. 8) one can write the relation (2. 7) in the matrix form

$$X_{N,\alpha}(t) = \boldsymbol{T}_{\alpha}(t)\, \boldsymbol{D}^t\, \boldsymbol{A}, \tag{2. 9}$$

where the vector of unknown is $\boldsymbol{A} = [a_0 \quad a_1 \quad \dots \quad a_N]^t$.

2.4. **Chelyshkov polynomials.** The Chelyshkov polynomials were originally introduced by Chelyshkov [9, 10]. These polynomials are orthogonal over the interval $[0,1]$ with respect to the weight function $w(x)=1$, and are explicitly defined by

$$C_{n,N}(t)=\sum_{k=0}^{N-n}(-1)^k\binom{N-n}{k}\binom{N+n+k+1}{N-n}t^{n+k},\quad n=0,1,\dots,N. \tag{2. 10}$$

These polynomials satisfy the following orthogonality relation

$$\int_0^1 C_{n,N}(t)\,C_{m,N}(t)dt=\frac{\delta_{nm}}{n+m+1},$$

where δ_{nm} is the Kronecker delta. Moreover, they can be obtained through the Jacobi polynomials $P_m^{\alpha,\beta}(t)$, where $\alpha,\beta>-1$, and $m\geqslant 0$ as

$$C_{n,N}(t)=(-1)^{N-n}\,t^n\,P_{N-n}^{0,2n+1}(t).$$

Now, we construct the fractional-order version of (2. 10) by replacing $t\to t^\alpha$ as follows [34]

$$C_{n,N}^{\alpha}(t)=\sum_{k=n}^{N}(-1)^{k-n}\binom{N-n}{k-n}\binom{N+k+1}{N-n}\left(\frac{t^\alpha}{T}\right)^k,\quad n=0,1,\dots,N. \tag{2. 11}$$

It also is not a difficult task to show that the set of fractional polynomial functions $\{C_{0,N}^{\alpha},C_{1,N}^{\alpha},\dots\}$ is orthogonal on $[0,T]$ with respect to the weight function $w(t)\equiv t^{\alpha-1}$. This implies that

$$\int_0^T C_{n,N}^{\alpha}(t)\,C_{m,N}^{\alpha}(t)\,w(t)dt=\frac{T\delta_{nm}}{\alpha(2n+1)},\quad n,m\geqslant 0.$$

The Chelyshkov basis polynomials given by equation (2. 11) can be written in the matrix form[23, 34]

$$\boldsymbol{C}_\alpha(t)=\begin{bmatrix}C_{0,N}^{\alpha}(t) & C_{1,N}^{\alpha}(t) & \dots & C_{N,N}^{\alpha}(t)\end{bmatrix}=\boldsymbol{T}_\alpha(t)\,\mathfrak{D}, \tag{2. 12}$$

where $\mathfrak{D}$ is an $(N+1)\times(N+1)$ matrix. Using $\tau=1/T$, if N is odd, the matrix $\mathfrak{D}$ becomes

$$\mathfrak{D}=\begin{bmatrix}
\binom{N}{0}\binom{N+1}{N} & 0 & \dots & 0 & 0\\
-\tau\binom{N}{1}\binom{N+2}{N} & \tau\binom{N-1}{0}\binom{N+2}{N-1} & \dots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
\tau^{N-1}\binom{N}{N-1}\binom{2N}{N} & -\tau^{N-1}\binom{N-1}{N-2}\binom{2N}{N-1} & \dots & \tau^{N-1}\binom{1}{0}\binom{2N}{1} & 0\\
-\tau^{N}\binom{N}{N}\binom{2N+1}{N} & \tau^{N}\binom{N-1}{N-1}\binom{2N+1}{N-1} & \dots & \tau^{N}\binom{1}{1}\binom{2N+1}{1} & \tau^N
\end{bmatrix},$$

and if N is even we have

$$\mathfrak{D}=\begin{bmatrix} \binom{N}{0}\binom{N+1}{N} & 0 & \dots & 0 & 0\\ -\tau\binom{N}{1}\binom{N+2}{N} & \tau\binom{N-1}{0}\binom{N+2}{N-1} & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -\tau^{N-1}\binom{N}{N-1}\binom{2N}{N} & \tau^{N-1}\binom{N-1}{N-2}\binom{2N}{N-1} & \dots & \tau^{N-1}\binom{1}{0}\binom{2N}{1} & 0\\ \tau^{N}\binom{N}{N}\binom{2N+1}{N} & -\tau^{N}\binom{N-1}{N-1}\binom{2N+1}{N-1} & \dots & -\tau^{N}\binom{1}{1}\binom{2N+1}{1} & \tau^{N} \end{bmatrix},$$

Our goal for model (1. 1) is to approximate $X(t)$ as the truncated Chelyshkov series form as $X_{N,\alpha}(t)=\sum_{n=0}^{N} a_n\, C_{n,N}^{\alpha}(t)$. Using (2. 12) one may rewrite $X_{N,\alpha}(t)$ as follows

$$X_{N,\alpha}(t)=\boldsymbol{T}_{\alpha}(t)\,\mathfrak{D}\,\boldsymbol{A}. \tag{2. 13}$$

2.5. **Legendre polynomials.** The orthogonal Legendre polynomials are originally defined on $[-1,1]$. Using the change of variable $x=(\frac{2t}{T}-1)$ one can obtain the shifted Legendre polynomials defined in $[0,T]$ and satisfies in the following recurrence relation [2]

$$\begin{cases} P_{n+1}(t)=\dfrac{2n+1}{n+1}(\dfrac{2t}{T}-1)\,P_n(t)-\dfrac{n}{n+1}P_{n-1}(t), \quad n=1,2,\dots,\\ P_0(t)=1, \quad P_1(t)=\dfrac{2t}{T}-1. \end{cases} \tag{2. 14}$$

The analytical form of $P_n(t)$ is explicitly defined for $n=0,1,\dots$

$$P_n(t)=\sum_{k=0}^{n} p_{n,k}\,t^k, \quad p_{n,k}=(-1)^{n+k}\frac{(n+k)!}{(n-k)!\,T^k\,(k!)^2}, \quad k=0,1,\dots,n. \tag{2. 15}$$

Based on the shifted Legendre polynomials (2. 15) one generates an orthogonal set of fractional-order Legendre functions by setting $t\to t^{\alpha}$ for $0<\alpha\leqslant 1$, see [15]. They take the form

$$P_n^{\alpha}(t)=\sum_{k=0}^{n} p_{n,k}\,t^{k\alpha}, \quad n=0,1,\dots. \tag{2. 16}$$

It is proved in [15] that the set of fractional polynomial functions $\{P_0^{\alpha},P_1^{\alpha},\dots\}$ is orthogonal on $[0,T]$ with respect to the weight function $w(t)\equiv t^{\alpha-1}$; i.e.

$$\int_0^T P_n^{\alpha}(t)\,P_m^{\alpha}(t)w(t)dt=\frac{T}{\alpha(2n+1)}\delta_{nm}, \quad n,m\geqslant 0.$$

The main important properties of the fractional-order Legendre functions can be found in [15] and [31]

Now, let us approximate the solution $X(t)$ of (1. 1) in terms of fractional-order Legendre functions. Thus one gets $X_{N,\alpha}(t)=\sum_{n=0}^{N} a_n\,P_n^{\alpha}(t)$ or equivalently

$$X_{N,\alpha}(t)=\boldsymbol{P}_{\alpha}(t)\,\boldsymbol{A}, \quad \boldsymbol{P}_{\alpha}(t)=[P_0^{\alpha}(t) \quad P_1^{\alpha}(t) \quad \dots \quad P_N^{\alpha}(t)]. \tag{2. 17}$$

In a similar way as the Bessel and Chelyshkov functions, we write $P_n^{\alpha}(t)$ in the matrix form as follows

$$\boldsymbol{P}_{\alpha}(t)=\boldsymbol{T}_{\alpha}(t)\,\mathbb{D}^t \Leftrightarrow \boldsymbol{P}^t_{\alpha}(t)=\mathbb{D}\,\boldsymbol{T}^t_{\alpha}(t), \tag{2. 18}$$

where the monomial basis vector $\boldsymbol{T}_{\alpha}(t)$ is previously defined in (2. 8). Moreover, the matrix $\mathbb{D}$ in this case is a lower triangular matrix whose entries are obtained via (2. 15) and has the form

$$\mathbb{D}=\begin{bmatrix} p_{0,0} & 0 & 0 & \dots & \dots & \dots & 0\\ p_{1,0} & p_{1,1} & 0 & 0 & \dots & \dots & 0\\ p_{2,0} & p_{2,1} & p_{2,2} & 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ p_{N-1,0} & p_{N-1,1} & p_{N-1,2} & \cdots & p_{N-1,N-2} & p_{N-1,N-1} & 0\\ p_{N,0} & p_{N,1} & p_{N,2} & \cdots & p_{N,N-2} & p_{N,N-1} & p_{N,N} \end{bmatrix}.$$

Therefore, an equivalent form of (2. 17) can be written as

$$X_{N,\alpha}(t)=\boldsymbol{T}_{\alpha}(t)\,\mathbb{D}^t\,\boldsymbol{A}. \tag{2. 19}$$

Ultimately, to obtain a solution in the form (2. 9), (2. 13), or (2. 19) of the problem (1. 1) on the interval $0 < t \leqslant T$, we will use the collocation points defined by

$$t_j=\frac{T}{N}j, \quad j=0,1,\dots,N. \tag{2. 20}$$

3. DESCRIPTION OF THE METHOD

Now, suppose that we approximate the solution $X(t)$ of the nonlinear Riccati equation (1. 1) in terms of $(N+1)$-terms Bessel, Chelyshkov or Legendre polynomials series denoted by $X_{N,\alpha}(t)$ on the interval $[0,T]$. As previously stated, in the vector form one may write

$$X(t)\cong X_{N,\alpha}(t)=\boldsymbol{T}_{\alpha}(t)\,\boldsymbol{Q}\,\boldsymbol{A}, \tag{3. 21}$$

where the matrix $\boldsymbol{Q}$ is either the matrix $\boldsymbol{D}^t, \mathfrak{D}$ or $\mathbb{D}^t$ depending on which polynomial basis function is selected in the approximation. These matrices are previously defined in (2. 8), (2. 12) and (2. 18) respectively. Inserting the collocation points (2. 20) into (3. 21), we arrive at a system of matrix equations

$$X_{N,\alpha}(t_j)=\boldsymbol{T}_{\alpha}(t_j)\,\boldsymbol{Q}\,\boldsymbol{A}, \quad j=0,1,\dots,N.$$

These equation can be written in a single and compact representation as follows

$$\boldsymbol{X}=\boldsymbol{T}\,\boldsymbol{Q}\,\boldsymbol{A}, \tag{3. 22}$$

where

$$\boldsymbol{X}=\begin{bmatrix} X_{N,\alpha}(t_0)\\ X_{N,\alpha}(t_1)\\ \vdots\\ X_{N,\alpha}(t_N)\end{bmatrix}, \quad \boldsymbol{T}=\begin{bmatrix} \boldsymbol{T}_{\alpha}(t_0)\\ \boldsymbol{T}_{\alpha}(t_1)\\ \vdots\\ \boldsymbol{T}_{\alpha}(t_N)\end{bmatrix}.$$

By taking the fractional derivative of order q from the both sides of (3. 21), we get

$$\mathcal{D}^{(q)}X_{N,\alpha}(t)=\mathcal{D}^{(q)}\,\boldsymbol{T}_{\alpha}(t)\,\boldsymbol{Q}\,\boldsymbol{A}. \tag{3. 23}$$

One can easily compute $\mathcal{D}^{(q)}\boldsymbol{T}_\alpha(t)$ by means of the property (2. 3) and (2. 4) as follows

$$\boldsymbol{T}_\alpha^{(q)}(t) = \mathcal{D}^{(q)}\boldsymbol{T}_\alpha(t) = [0 \quad \mathcal{D}^{(q)}t^\alpha \quad \dots \quad \mathcal{D}^{(q)}t^{\alpha N}].$$

In order to obtain a system of matrix equations for the fractional derivative, we put the collocation points (2. 20) into (3. 23) to obtain

$$\mathcal{D}^{(q)}X_{N,\alpha}(t_j) = \boldsymbol{T}_\alpha^{(q)}(t_j)\,\boldsymbol{Q}\,\boldsymbol{A}, \quad j = 0, 1\dots, N,$$

which can be written in the matrix form

$$\boldsymbol{X}^{(q)} = \boldsymbol{T}^{(q)}\,\boldsymbol{Q}\,\boldsymbol{A}, \tag{3. 24}$$

where

$$\boldsymbol{X}^{(q)} = \begin{bmatrix} \mathcal{D}^{(q)}X_{N,\alpha}(t_0) \\ \mathcal{D}^{(q)}X_{N,\alpha}(t_1) \\ \vdots \\ \mathcal{D}^{(q)}X_{N,\alpha}(t_N) \end{bmatrix}, \quad \boldsymbol{T}^{(q)} = \begin{bmatrix} \boldsymbol{T}_\alpha^{(q)}(t_0) \\ \boldsymbol{T}_\alpha^{(q)}(t_1) \\ \vdots \\ \boldsymbol{T}_\alpha^{(q)}(t_N) \end{bmatrix}.$$

To proceed, we need to approximate the nonlinear term $X^2(t)$. By substituting the collocation points into $X^2_{N,\alpha}(t)$ we arrive at the following matrix representation

$$\boldsymbol{X}^2 = \begin{bmatrix} X^2_{N,\alpha}(t_0) \\ X^2_{N,\alpha}(t_1) \\ \vdots \\ X^2_{N,\alpha}(t_N) \end{bmatrix} = \begin{bmatrix} X_{N,\alpha}(t_0) & 0 & \dots & 0 \\ 0 & X_{N,\alpha}(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X_{N,\alpha}(t_N) \end{bmatrix} \begin{bmatrix} X_{N,\alpha}(t_0) \\ X_{N,\alpha}(t_1) \\ \vdots \\ X_{N,\alpha}(t_N) \end{bmatrix} = \widetilde{\boldsymbol{X}}\,\boldsymbol{X}. \tag{3. 25}$$

Moreover, the matrix $\widetilde{\boldsymbol{X}}$ can be written as a product of three block diagonal matrices as

$$\widetilde{\boldsymbol{X}} = \widetilde{\boldsymbol{T}}\,\widetilde{\boldsymbol{Q}}\,\widetilde{\boldsymbol{A}}, \tag{3. 26}$$

where

$$\widetilde{\boldsymbol{T}} = \begin{bmatrix} \boldsymbol{T}_\alpha(t_0) & 0 & \dots & 0 \\ 0 & \boldsymbol{T}_\alpha(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T_\alpha(t_N) \end{bmatrix}, \widetilde{\boldsymbol{Q}} = \begin{bmatrix} \boldsymbol{Q} & 0 & \dots & 0 \\ 0 & \boldsymbol{Q} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{Q} \end{bmatrix}, \widetilde{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & 0 & \dots & 0 \\ 0 & \boldsymbol{A} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{A} \end{bmatrix}.$$

Now, we are able to compute the Bessel, Chelyshkov, and Legendre solutions of (1. 1). The collocation procedure is based on calculating these polynomial coefficients by means of collocation points defined in (2. 20). To proceed, inserting the collocation points into the fractional Riccati differential equation to get the system

$$\mathcal{D}^{(q)}\,X(t_j) = a(t_j)\,X^2(t_j) + b(t_j)\,X(t_j) + c(t_j), \quad j = 0, 1, \dots, N.$$

In the matrix form we may write the above equations as

$$\boldsymbol{X}^{(q)} - \boldsymbol{R}\,\boldsymbol{X}^2 - \boldsymbol{B}\,\boldsymbol{X} = \boldsymbol{C}, \tag{3. 27}$$

where the coefficient matrices $\boldsymbol{R}, \boldsymbol{B}$ of size $(N+1) \times (N+1)$ and the vector $\boldsymbol{C}$ of size $(N+1) \times 1$ have the following forms

$$\boldsymbol{R} = \begin{bmatrix} a(t_0) & 0 & \dots & 0 \\ 0 & a(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a(t_N) \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} b(t_0) & 0 & \dots & 0 \\ 0 & b(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b(t_N) \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} c(t_0) \\ c(t_1) \\ \vdots \\ c(t_N) \end{bmatrix}.$$

By putting the relations (3. 22), (3. 24), and (3. 25) into (3. 27), the fundamental matrix equation is obtained

$$\boldsymbol{W}\boldsymbol{A} = \boldsymbol{C}, \qquad \boldsymbol{W} := \boldsymbol{T}^{(q)}\boldsymbol{Q} - \boldsymbol{R}\widetilde{\boldsymbol{T}}\widetilde{\boldsymbol{Q}}\widetilde{\boldsymbol{A}}\boldsymbol{T}\boldsymbol{Q} - \boldsymbol{B}\boldsymbol{T}\boldsymbol{Q}. \tag{3. 28}$$

Obviously, (3. 28) is a nonlinear matrix equation with a_n, $n = 0, 1, \dots, N$, being the unknowns Bessel, Chelyshkov, or Legendre coefficients. To take into account the initial condition $X(0) = X_0$, we tend $t \to 0$ in (3. 21) to get the following matrix representation

$$\bar{X}_0\boldsymbol{A} = \bar{X}_0, \qquad \bar{X}_0 := \boldsymbol{T}_\alpha(0)\boldsymbol{Q} = [x_{00} \quad x_{01} \quad \dots \quad x_{0N}]^t.$$

Consequently, by replacing the first row of the augmented matrix $[\boldsymbol{W}; \boldsymbol{C}]$ by the row matrix $[\bar{X}_0; X_0]$, we arrive at the nonlinear algebraic system

$$\widehat{\boldsymbol{W}}\boldsymbol{A} = \widehat{\boldsymbol{C}}.$$

Thus, the unknown Bessel, Chelyshkov, or Legendre coefficients in (3. 21) will be calculated via solving this nonlinear system of equations. This task can be performed using for instance the Newton's iterative method.

3.1. **Accuracy of solutions.** In general, the exact explicit solution of the Riccati differential equation is only known for $q = 1$. Therefore, we need to measure the accuracy of the proposed collocation scheme when $0 < q < 1$. Since the truncated Bessel, Chelyshkov, and Legendre series (2. 6), (2. 10), and (2. 15) are approximate solutions of (1. 1), we expect that the residual obtained by inserting the computed approximated solutions $X_{N,\alpha}(t)$ into the differential equation becomes approximately small. This implies that for $t = t_s \in [0, T]$, $s = 0, 1, \dots$

$$E_{N,\alpha}(t_s) = \mathcal{D}^{(q)} X_{N,\alpha}(t_s) - a(t_s)\, X^2_{N,\alpha}(t_s) - b(t_s)\, X_{N,\alpha}(t_s) - c(t_s) \cong 0, \tag{3. 29}$$

and $E_{N,\alpha}(t_s) \leqslant 10^{-k_s}$ (k_s is positive integer). If $\max 10^{-k_s} \leqslant 10^{-k}$ (k positive integer) is prescribed, then the truncation limit N is increased until the difference $E_{N,\alpha}(t_s)$ at each of the points becomes smaller than the prescribed 10^{-k}, see [35, 36]. Here, we note that the qth-order fractional derivative of the approximate solution (3. 29) is computed by using the property (2. 4). As the error function is clearly zero at the collocation points (2. 20), we expect that $E_{N,\alpha}(t)$ tend to zero as N increased. This says that the smallness of the residual error function means that the approximate solutions are close to the exact solution.

4. Test problems

To illustrate the effectiveness of the proposed polynomials collocation methods, two test examples are solved in this section.

Test problem 4.1. *We consider the nonlinear Riccati differential equation* [4, 21, 22, 26]

$$\mathcal{D}^{(q)} X(t) = 1 - X^2(t), \quad 0 < q \leqslant 1, \tag{4. 30}$$

with the initial condition given by $X_0 = 0$*. The exact analytical solution when* $q = 1$ *is*

$$X(t) = \frac{e^{2t} - 1}{e^{2t} + 1}.$$

To begin the computations, we take $q = 1$ in (4. 30) and set $\alpha = 1$ as the order of basis functions. The approximate solutions $X_N(t)$ of this model problem using Bessel, Chelyshkov, and Legendre basis functions for $N = 5$ in the interval $0 \leqslant t \leqslant 1$ are obtained as follows, respectively:

$$X_5^B(t) = -0.0425208333486894\, t^5 + 0.263393561049807\, t^4 - 0.511947952786299\, t^3$$
$$+ 0.0628989648274941\, t^2 + 0.989484942139681\, t,$$
$$X_5^C(t) = -0.0425208131343566\, t^5 + 0.263393514919765\, t^4 - 0.511947912623583\, t^3$$
$$+ 0.0628989482627761\, t^2 + 0.989484945229446\, t - 3.2012155000427 \times 10^{-20},$$
$$X_5^L(t) = -0.0425208332627086\, t^5 + 0.263393560841792\, t^4 - 0.51194795259138\, t^3$$
$$+ 0.0628989647337824\, t^2 + 0.989484942163854\, t - 2.25514051876985 \times 10^{-17}.$$

In Table 1, we report the numerical results correspond to $X_N(1)$ obtained by the Bessel, Chelyshkov, and Legendre-collocation procedures using different choice of the number of basis functions $N = 2, 4, \ldots, 16$ and $q = 1$. All calculations are reported with 15 decimal places of accuracy. Note that the exact value up to 32 digits is

$$X(1) = 0.76159415595576485102924380043987.$$

It can be seen from Table 1 that by increasing the number of basis functions N, more

TABLE 1. Comparison of numerical approximations for test problem (4. 30) using Bessel, Chelyshkov, and Legendre-collocation methods for $q = 1$ and different number of N.

N	$X_N^B(1)$	$X_N^C(1)$	$X_N^L(1)$
2	**0.7**71769772367609	**0.7**71769772265604	**0.7**71769772170532
4	**0.761**772997737468	**0.761**772998309460	**0.761**772997642195
6	**0.7615**48003560260	**0.7615**48003548052	**0.7615**48003548681
8	**0.76159**7630707855	**0.76159**7568464948	**0.76159**7568465396
10	**0.760**452998932030	**0.76159**3991848573	**0.76159**3991789546
12	**0.760**598176921084	**0.761594**160240097	**0.761594**160785043
14	**0.7**38266020081461	0.924036516298884	**0.761594155**975480
16	**0.7**42191938572435	0.412541261646161	**0.761594155**947834

accurate results are obtained using these three basis functions. The best results in the Bessel-collocation method is obtained up to $N = 8$ while in the Chelyshkov-collocation scheme is achieved up to $N = 12$. However, a more reliable result is obtained via the Legendre-collocation method.

In the next experiment, we set $q, \alpha = 1/2$. In this case, we first consider the approximate solutions $X_8(t)$ obtained via (3. 28) of the model (1. 1) for different polynomials in the interval $[0, 1]$. These polynomials of fractional order $\alpha = 1/2$ are obtained as follows

$$X^B_{8,\frac{1}{2}}(t) = 0.223384938281968\,t + 2.07746816156096\,t^2 - 0.109555476113683\,t^3 - 0.0574991063918645\,t^4 + 1.10511540696442\,t^{1/2} - 1.89648537368994\,t^{3/2} - 0.867368413136722\,t^{5/2} + 0.223654174001963\,t^{7/2},$$

$$X^C_{8,\frac{1}{2}}(t) = 0.224032864807523\,t + 2.08206405196646\,t^2 - 0.106092430634112\,t^3 - 0.0573500972260452\,t^4 + 1.10503683117717\,t^{1/2} - 1.89881407585456\,t^{3/2} - 0.872646352740204\,t^{5/2} + 0.222483411690925\,t^{7/2} + 2.13713937852323 \times 10^{-21},$$

$$X^L_{8,\frac{1}{2}}(t) = 0.224039710764838\,t + 2.08212578315359\,t^2 - 0.106024811403993\,t^3 - 0.0573443996836568\,t^4 + 1.10503606899795\,t^{1/2} - 1.89884146690467\,t^{3/2} - 0.872729959985898\,t^{5/2} + 0.222453277371297\,t^{7/2} + 8.20613989629439 \times 10^{-17}.$$

Note that using the integer-order versions of theses polynomials, i.e., $q = 1/2$ and $\alpha = 1$ yield

$$X^B_8(t) = -22.6148147092024\,t^8 + 108.448175575276\,t^7 - 221.79963969032\,t^6 + 252.94388627954\,t^5 - 176.7204056547\,t^4 + 78.552429771116\,t^3 - 22.6768503632544\,t^2 + 4.56352530569598\,t,$$

$$X^C_8(t) = -22.6148147225516\,t^8 + 108.448175652689\,t^7 - 221.799639848434\,t^6 + 252.943886440175\,t^5 - 176.720405744409\,t^4 + 78.5524297988918\,t^3 - 22.6768503677063\,t^2 + 4.56352530598814\,t - 4.94320058517937 \times 10^{-23},$$

$$X^L_8(t) = -22.6148147053515\,t^8 + 108.448175585942\,t^7 - 221.799639741515\,t^6 + 252.943886348675\,t^5 - 176.720405699346\,t^4 + 78.5524297861592\,t^3 - 22.676850365792\,t^2 + 4.5635253058676\,t + 2.68882138776405 \times 10^{-17}.$$

The above approximated solutions $X_8(t)$ and $X_{8,\frac{1}{2}}(t)$ are compared in Fig. 1. It can be seen Fig. 1, left plot, that a more accurate result is achieved by using the fractional basis functions rather than non-fractional ones. To further justify this fact, we plot the estimated errors obtained by the relations (3. 29). The graphs of $E_{8,\alpha}(t)$ for $\alpha = 1$, and $\alpha = 1/2$ on the interval $[0, 1]$ correspond to $q = 1/2$ are visualized in Fig. 1, right plot. Referring to Fig. 1, it is clearly seen that using a fixed value of $q = 1/2$ almost the same performance is observed with each value of $\alpha = 1$ and $\alpha = 1/2$.

Due to the fact that the exact solution for fractional order case is not available for the test problem (4. 30), we made a comparison between the approximate solutions given by

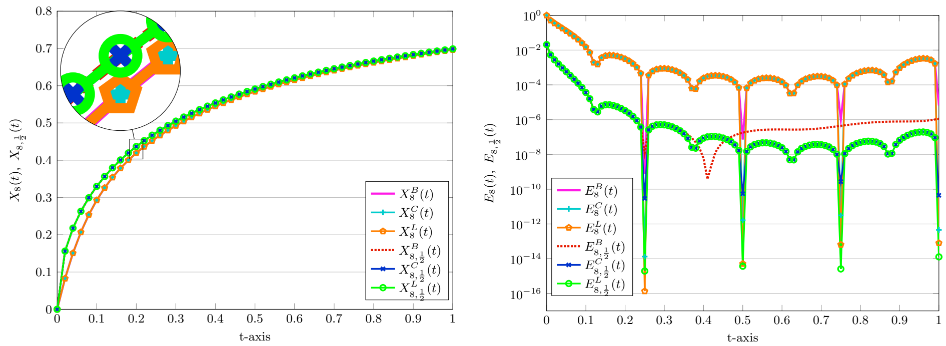

FIGURE 1. comparison of numerical solutions (left) and the corresponding error functions using Bessel, Chelyshkov, and Legendre functions (right) with $q = 1/2$, $\alpha = 1, 1/2$, and $N = 8$.

our proposed approaches and reported numerical results of other approaches. First, in Table 2 we use $N = 8$ and $q, \alpha = 1/2$ and report our numerical results obtained by the three collocation schemes at points $t = j/10$ for $j = 1, 2, \ldots, 10$. A comparison in this table is made with the fractional Bernoulli polynomials approach from [26]. Furthermore, in the next Table 3 we compare our results with existing computational procedures available in the standard literature such as a predictor-corrector approach based on Adams-Bashforth-Moultton Method (ABFMM) [12] with $h = 0.001$, Legendre wavelet operational matrix method (LWM) [4], stochastic solver based on swarm intelligence optimization (PSO) algorithm [27], fractional variational iteration method (FVIM) [19], and a modification of He's homotopy perturbation method (HPM) [22]. From the presented results in Table 2 and 3 we can identify that guarantee of convergence of the proposed collocation schemes is very high compared to other schemes.

In Fig. 1 we have seen that using the fractional order polynomials lead to a more accurate result compared to the corresponding integer ones. Next, to see the effect of using various values of $\alpha \geqslant q$, we fix $N = 7$ and $q = 1/2$. Hence, we vary α starting from $\alpha = q$ to $\alpha = 1$ and see the behaviour of error functions $E_{7,\alpha}(t)$ defined in (3. 29) on the interval $[0, 1]$. The results are shown in Fig. 2 while using the Legendre basis functions. As one can see from Fig. 2 that the smallest error is achieved when $\alpha = q$ and the largest error is obtained if one uses $\alpha = 1$. Therefore, in the next experiments we only consider this case. To be more precise, the numerical approximations utilizing the same q and α both equal to $3/4$ with $N = 8$ evaluated at various points $t = j/10$ for $j = 1, 2, \ldots, 10$ are given in Table 4. To validate our results and as in Table 2, the last column is devoted to the results obtained via Bernoulli polynomials [26]. A more complete comparison is done in Table 5 for these values of q and α.

Finally, we make a comparison between Bessel, Chelyshkov, and Legendre-collocation methods in terms of error functions defined in (3. 29) for a fixed $N = 9$ but employ different values of q equals to $\alpha = 1/3, 2/3$. Table 6 demonstrates the numerical values of these error functions at the points $t = 0, 1/10, 2/10, \ldots, 1$. The comparison between

TABLE 2. Comparison of numerical approximations in fractional Bessel, Chelyshkov, and Legendre-collocation methods for $N = 8$ and $q, \alpha = 1/2$ in test problem (4. 30).

t	Bessel	Chelyshkov	Legendre	Bernoulli
0.1	**0.3**29821778995450	**0.3**29821685183483	**0.3**29820449061496	0.330101
0.2	**0.436**687309406414	**0.436**687277316919	**0.436**686638490342	0.436844
0.3	**0.504**786726382638	**0.504**786701429466	**0.504**786267243779	0.504894
0.4	**0.5537**06012096569	**0.5537**05997020231	**0.5537**05676894624	0.553776
0.5	**0.5911**35046252850	**0.5911**35041914654	**0.5911**34793189228	0.591188
0.6	**0.62**0965973154147	**0.62**0965968681406	**0.62**0965767331645	0.621017
0.7	**0.6454**45930748345	**0.6454**45917428470	**0.6454**45751307430	0.645494
0.8	**0.66**5985381080626	**0.66**5985363965563	**0.66**5985224802811	0.666018
0.9	**0.6835**23576703947	**0.6835**23569692847	**0.6835**23448508664	0.683542
1.0	**0.6987**14311477103	**0.6987**14308834321	**0.6987**14202309494	0.698768

TABLE 3. Comparison of numerical results for test problem (4. 30) for $N = 8$ and $q, \alpha = 1/2$.

t	Bes/Chel/Leg	ABFMM	LWM	PSO	FVIM	MHP
0.1	0.329820	0.330108	0.273600	0.289667	0.086513	0.273875
0.2	0.436687	0.436839	0.386358	0.386489	0.161584	0.454125
0.3	0.504786	0.504889	0.441104	0.441120	0.238256	0.573932
0.4	0.553706	0.553782	0.482304	0.482348	0.321523	0.644422
0.5	0.591135	0.591194	0.520664	0.516379	0.413682	0.674137
0.6	0.620966	0.621014	0.533287	0.544872	0.515445	0.671987
0.7	0.645446	0.645485	0.558743	0.568545	0.626403	0.648003
0.8	0.665985	0.666019	0.587812	0.587895	0.745278	0.613306
0.9	0.683523	0.683552	0.596234	0.603344	0.870074	0.579641
1.0	0.698714	0.698739	0.610642	0.615268	0.998176	0.558557

$E_{9,\alpha}(t)$ using the same q as α shows that the performance of the fractional Legendre-collocation is a slightly better rather than the Bessel and Chelyshkov-collocation schemes.

Test problem 4.2. *As a second test example, we consider the following differential* [17]

$$\mathcal{D}^{(q)}\, X(t) = -X(t) + X^2(t), \quad 0 < q \leqslant 1, \tag{4. 31}$$

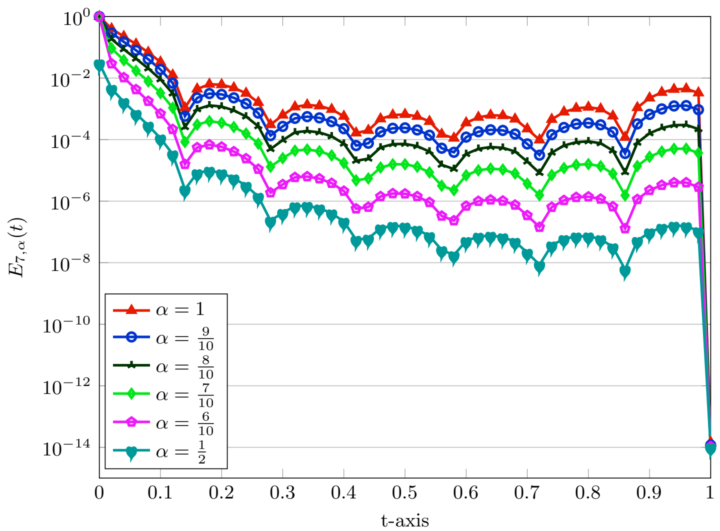

FIGURE 2. Comparison of error functions using Legendre-collocation method for $q = 1/2$, $N = 7$ and various values of α.

TABLE 4. Comparison of numerical approximations in fractional Bessel, Chelyshkov, and Legendre-collocation methods for $N = 8$ and $q, \alpha = 3/4$ in test problem (4. 30).

t	Bessel	Chelyshkov	Legendre	Bernoulli [26]
0.1	**0.1901**33343182368	**0.1901**33342156594	**0.1901**33340742126	0.190102
0.2	**0.3099**98925381054	**0.3099**98924666591	**0.3099**98923662126	0.309975
0.3	**0.4046**33663354412	**0.4046**33662791810	**0.4046**33661990896	0.404615
0.4	**0.4816**47038110234	**0.4816**47037650133	**0.4816**47036997255	0.481633
0.5	**0.545**101987075590	**0.545**101986702863	**0.545**101986166129	0.545090
0.6	**0.5977**93180460225	**0.5977**93180157799	**0.5977**93179713331	0.597781
0.7	**0.64182**9114200803	**0.64182**9113951715	**0.64182**9113586379	0.641821
0.8	**0.67885**6897919783	**0.67885**6897708743	**0.67885**6897423021	0.678851
0.9	**0.7101**81243784475	**0.7101**81243598305	**0.7101**81243534913	0.710173
1.0	**0.73684**2125584433	**0.73684**2125449495	**0.73684**2126390755	0.736843

with the initial condition given by $X_0 = 1/2$. The exact analytical solution for $q = 1$ is

$$X(t) = \frac{e^{-t}}{e^{-t} + 1}.$$

If one tends $t \to \infty$ then $X(t) \to 0$.

TABLE 5. Comparison of numerical results for test problem (4. 30) for $N = 8$ and $q, \alpha = 3/4$.

t	Bes/Chel/Leg	ABFMM	LWM	PSO	FVIM	MHP
0.1	0.190133	0.190101	0.165056	0.165087	0.190102	0.184795
0.2	0.309999	0.309975	0.276332	0.276350	0.310033	0.313795
0.3	0.404634	0.404615	0.356115	0.356196	0.405062	0.414562
0.4	0.481647	0.481632	0.416817	0.416916	0.483479	0.492889
0.5	0.545102	0.545089	0.465480	0.465520	0.550470	0.462117
0.6	0.597793	0.597783	0.505894	0.506004	0.610344	0.597393
0.7	0.641829	0.641820	0.540606	0.540629	0.666961	0.631772
0.8	0.678857	0.678849	0.569998	0.570632	0.723760	0.660412
0.9	0.710181	0.710175	0.596600	0.596636	0.783638	0.687960
1.0	0.736842	0.736837	0.618824	0.618873	0.848783	0.718260

TABLE 6. Comparison of error functions in Bessel, Chelyshkov, Legendre-collocation methods for test problem (4. 30) with $N = 9$ and different $q, \alpha = 1/3, 2/3$.

	Bessel		Chelyshkov		Legendre	
t	$q,\alpha=\frac{1}{3}$	$q,\alpha=\frac{2}{3}$	$q,\alpha=\frac{1}{3}$	$q,\alpha=\frac{2}{3}$	$q,\alpha=\frac{1}{3}$	$q,\alpha=\frac{2}{3}$
0.0	2.49006_{-02}	2.65171_{-03}	1.49519_{+00}	2.76535_{-03}	2.85781_{-02}	2.73314_{-03}
0.1	1.04354_{-07}	3.93847_{-06}	1.06758_{-04}	4.26056_{-06}	6.92585_{-07}	4.16496_{-06}
0.2	1.26538_{-08}	3.39413_{-07}	8.64592_{-06}	4.01378_{-07}	3.42935_{-08}	3.92513_{-07}
0.3	2.06293_{-08}	4.50240_{-08}	2.07218_{-06}	7.55739_{-08}	4.40458_{-09}	6.98377_{-08}
0.4	1.36572_{-07}	7.07511_{-08}	3.33390_{-07}	1.93124_{-08}	1.01481_{-09}	1.89277_{-08}
0.5	2.25715_{-07}	1.26052_{-07}	7.93911_{-07}	7.87744_{-09}	3.62729_{-10}	7.06264_{-09}
0.6	3.71601_{-07}	2.33313_{-07}	7.85713_{-08}	7.41092_{-09}	1.87083_{-10}	3.31482_{-09}
0.7	5.07439_{-07}	4.00319_{-07}	1.42057_{-07}	3.45699_{-09}	1.33715_{-10}	2.27165_{-09}
0.8	5.27986_{-07}	7.08723_{-07}	8.93911_{-07}	4.45683_{-09}	1.28061_{-10}	1.65876_{-09}
0.9	5.65955_{-07}	1.11019_{-06}	6.13107_{-07}	3.98801_{-08}	1.49493_{-10}	1.25466_{-08}
1.0	1.24536_{-06}	1.77359_{-06}	1.34617_{-07}	1.43248_{-08}	2.42745_{-13}	5.04853_{-08}

We first calculate the approximate solutions $X_N(t)$ of (4. 31) using Bessel, Chelyshkov, and Legendre functions for $N = 8$ in the interval $[0, 1]$. Using $q, \alpha = 1$, these solutions at some points $t \in [0, 1]$ are presented in Table 7. In the last column of this table the results of a new homotopy perturbation method (NHPM) [17], which depends only on two components of the homotopy series are reported for comparison. Note that we only evaluate the 21 terms analytical solution obtained by the NHPM at the corresponding points. Obviously, the results of the proposed collocation schemes are in excellent agreement with the exact solutions.

TABLE 7. Comparison of error functions in Bessel, Chelyshkov, Legendre-collocation methods for test problem 4. 30 with $N = 8$ and different $q, \alpha = 1$.

t	Exact	Bessel	Chelyshkov	Legendre	NHPM
0.0	0.500000000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
0.1	0.475020812521060	0.4750208175	0.4750208125	0.4750208125	0.4210088076
0.2	0.450166002687522	0.4501660075	0.4501660027	0.4501660027	0.3885663037
0.3	0.425557483188341	0.4255574880	0.4255574832	0.4255574832	0.3640779382
0.4	0.401312339887548	0.4013123445	0.4013123399	0.4013123399	0.3439273776
0.5	0.377540668798145	0.3775406731	0.3775406688	0.3775406688	0.3267242818
0.6	0.354343693774205	0.3543436980	0.3543436938	0.3543436938	0.3117493761
0.7	0.331812227831834	0.3318122323	0.3318122278	0.3318122278	0.2985586586
0.8	0.310025518872388	0.3100255231	0.3100255189	0.3100255189	0.2868388286
0.9	0.289050497374996	0.2890505012	0.2890504974	0.2890504974	0.2763429687
1.0	0.268941421369995	0.2689414255	0.2689414215	0.2689414215	0.2668580880

In the next simulation we consider a large interval $[0, 5]$ and plot the approximate solutions obtained via various collocation schemes with different values of q equals to $\alpha = 1/4, 1/2, /3/4$, and $\alpha = 1$. Figure 3 shows the numerical solutions obtained by using the Bessel-collocation method using $N = 6$. It can be seen from this figure that the solution correspond to $q = 1$ are very close to the exact solution and are not distinguishable. We emphasize that using the Chelyshkov and Legendre functions the behaviour of solutions are very similar and therefore we omit them for clarity. To confirm this fact and to see the difference between different polynomials functions more closely, we present the approximate analytical solutions correspond to $q, \alpha = 1/4$ using the Bessel, Chelyshkov, and Legendre polynomials as follow

$$X^B_{6,\frac{1}{4}}(t) = 0.12385747459\, t^{3/4} - 0.0146104086107\, t^{1/2} - 0.274438507289\, t^{1/4}$$
$$- 0.00292497911335\, t^{3/2} - 0.0816353287686\, t + 0.0243906029891\, t^{5/4} + 0.5,$$

$$X^C_{6,\frac{1}{4}}(t) = 0.123928236755\, t^{3/4} - 0.0146563501743\, t^{1/2} - 0.274426515138\, t^{1/4}$$
$$- 0.00292822252168\, t^{3/2} - 0.0816899188965\, t + 0.0244116558345\, t^{5/4} + 0.5,$$

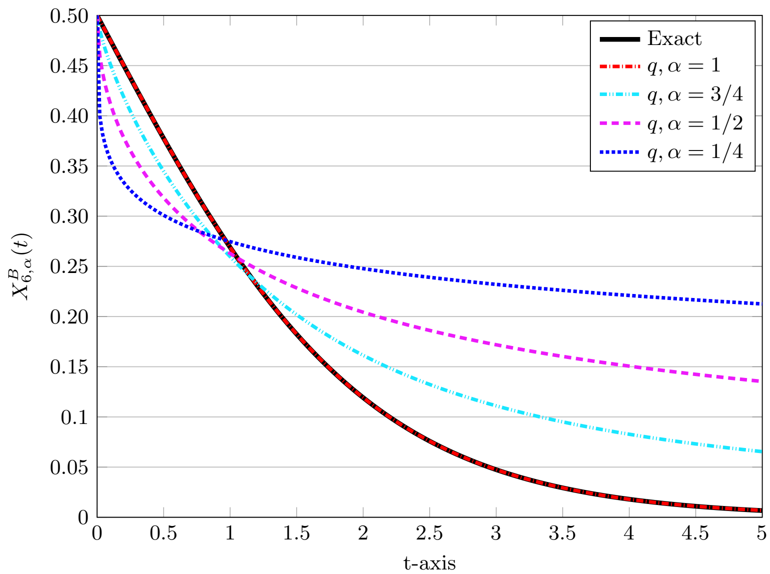

FIGURE 3. The approximated Bessel series solution using $q, \alpha = 1/4, 1/2, 3/4, 1$ for $N = 6$.

and

$$X^L_{6,\frac{1}{4}}(t) = 0.0902540962280\, t^{3/4} + 0.00692825008568\, t^{1/2} - 0.279991322087\, t^{1/4} - 0.00132142729136\, t^{3/2} - 0.0553675802012\, t + 0.0141229781199\, t^{5/4} + 0.500000011116336.$$

In the next experiments, we report the results obtained by the three collocation schemes more closely at the points $t = j/2$ for $j = 0, 1, \ldots, 10$. We use $N = 8$ for the Bessel, Chelyshkov and Legendre-collocation procedures. In all schemes we consider q equals to $\alpha = 1/2$. For each method the corresponding residual error function evaluated at these points via relation (3. 29) is presented. Obviously, a slightly more accurate result in terms of residual is obtained via the Legendre-collocation scheme compared to two other methods. However, for each point the numerical solutions for all three schemes are the same up to four to five digits.

5. CONCLUSIONS

In this note, a collocation method based upon well-known (orthogonal) polynomials is developed for numerical solutions of fractional-order Riccati differential equation arising in optimal control theory. Using the fractional version of the Bessel, Chelyshkov, and Legendre functions along with the collocation points we convert the differential equation into an algebraic system of nonlinear equations. Numerical test problems are given to demonstrate efficiency and accuracy of the proposed method. Moreover, the performance

TABLE 8. Comparison of numerical and error functions in Bessel, Chelyshkov, Legendre-collocation methods for test problem (4. 31) with $N = 8$ and $q, \alpha = 1/2$.

	Bessel		Chelyshkov		Legendre	
t	$q, \alpha = \frac{1}{2}$	$E_{8,\frac{1}{2}}(t)$	$q, \alpha = \frac{1}{2}$	$E_{8,\frac{1}{2}}(t)$	$q, \alpha = \frac{1}{2}$	$E_{8,\frac{1}{2}}(t)$
0.0	0.5000000000	1.04559_{-2}	0.5000000000	8.50992_{-3}	0.5000000000	8.46430_{-03}
0.5	0.3189674611	1.66283_{-5}	0.3189046014	1.21728_{-5}	0.3189031198	1.20616_{-05}
1.0	0.2631436497	1.56105_{-6}	0.2631121635	1.01634_{-6}	0.2631114186	1.00995_{-06}
1.5	0.2286775558	4.83678_{-8}	0.2286568533	1.47289_{-7}	0.2286563610	1.45046_{-07}
2.0	0.2044747325	3.06997_{-7}	0.2044598605	2.35118_{-8}	0.2044595049	2.34672_{-08}
2.5	0.1862714944	1.14639_{-7}	0.1862600018	7.77197_{-9}	0.1862597260	5.15070_{-11}
3.0	0.1719553961	1.56326_{-7}	0.1719461331	1.07596_{-8}	0.1719459114	6.08827_{-09}
3.5	0.1603306465	4.70888_{-7}	0.1603232439	2.55769_{-8}	0.1603230700	8.56536_{-09}
4.0	0.1506602009	9.01897_{-7}	0.1506543363	2.38558_{-8}	0.1506542044	1.22332_{-08}
4.5	0.1424611579	3.22795_{-7}	0.1424561468	3.84763_{-8}	0.1424560387	1.55650_{-08}
5.0	0.1354015090	4.91217_{-7}	0.1353967963	2.10580_{-8}	0.1353966875	1.47722_{-10}

of these three basis functions has assessed and a comparison between them and other well-established computational methods is made when applied to the Riccati model problem. Furthermore, the reliability of the proposed technique is checked through defining the residual error functions. Referring to graphs and tables we conclude that using fractional basis functions and in particular taking $\alpha = q$ produces a more accurate result rather than the corresponding integer-order basis functions. This shows the applicability of this approach for engineering problems that have fractional solutions.

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