$$\left(D_0^t D_0 \;+\; D_1^t D_1 \;+\; D_{11}^t D_{11}\right) \nabla_s G(p) \;=\; \nabla G(p), \tag{5. 42}$$

$$\left((1-\delta_t)^2 D_0^t D_0 \;+\; D_1^t D_1 \;+\; (\delta_t)^2 D_{11}^t D_{11}\right) \nabla_w G(p) \;=\; \nabla G(p), \tag{5. 43}$$

respectively.

6. Numerical Results

Consider the problem

$$p_t - p + \mid p \mid^2 p - p_{xx} = x(1 - t + x^2 t^3), \tag{6. 44}$$

with the condition

$$p(x, 0) = 0. \tag{6. 45}$$

Now Eq. (16) becomes

$$L_t p - p + \mid p \mid^2 p - L_x p = f(x, t), \tag{6. 46}$$

therefore,

$$p(x, t) = p(x, 0) + L_t^{-1} p_n - L_t^{-1} A_n + L_t^{-1} L_x p_n + L_t^{-1}(f(x, t)). \tag{6. 47}$$

The decomposition series solution $p(x,t)$ into $\sum_{n=0}^{\infty} p_n(x,t)$ gives the term by term

$$p_0 = p(x, 0) + L_t^{-1}(f(x, t)) = xt - \frac{1}{2} x t^2 + \frac{1}{4} x^3 t^4 \tag{6. 48}$$

$$p_1 = L_t^{-1} p_0 - L_t^{-1} A_0 + L_t^{-1} L_x p_0 = \frac{1}{2} x t^2 - \frac{1}{6} x t^3 - \frac{1}{4} x^3 t^4 - \frac{1}{10} x^3 t^5 + \frac{1}{5} x^4 t^5 + ..., \tag{6. 49}$$

and so on, on the similar pattern we can find the other components as well.
Now we solve given example by the Sobolev gradient method. Let $x \in [0, 1]$ with the initial condition $p(x, 0) \;=\; 0$. Writing equation in operator form

$$F(p) \;=\; D_0 \left((1-\delta_t)p + \delta_t z^3 - x(1 - t + x^2 t^3) - f\right) - \delta_t D_{11}(p), \tag{6. 50}$$

the gradient of the functional is given by

$$\nabla G(p) \;=\; \left[(1 - \delta_t + 3\delta_t p^2) D_0^t F(p) - \delta_t D_{11}^t F(p)\right]. \tag{6. 51}$$

TABLE 1. Comparison of results in L_2, H_2^2, $\hat{H}_2^2$ with $\delta_t = 0.2$ upto 5 time steps.

λ			iterations			CPUs			M
L_2	H_2^2	$\hat{H}_2^2$	L_2	H_2^2	$\hat{H}_2^2$	L_2	H_2^2	$\hat{H}_2^2$	-
5.0 $\times 10^{-7}$	3.0	1.7	806.735	393	114	78 86 819	0.789	0.281	51
$---$	3.0	1.7	$--$	385	99	$--$	3.593	0.914	101
$---$	3.0	1.7	$--$	371	101	$--$	20.220	5.592	201
$---$	3.0	1.7	$--$	381	100	$--$	138.383	37.628	401

To study the potency of our algorithm, we assume steepest descent in L_2, H_2^2, $\hat{H}_2^2$. To find the gradients in H_2^2, $\hat{H}_2^2$ we solve Eqs. (34) and (35) by some iterative methods such as conjugate gradient method.

To solve equation numerically, we discretize our domain into M nodes with internodal spacing δ. The initial state was set $f = 0.0$ on all nodes initially. The function p was then evolved. The amended value of p was taken as correct when the infinity norm of $\pi G(p))$ was less than 10^{-7}.

To see the efficiency of the algorithm in different spaces, total number of minimization steps and CPU time were recorded in Table 1.

From Table 1 the results in H_2^2 are far good than the results in L_2 but the best results are in $\hat{H}_2^2$.

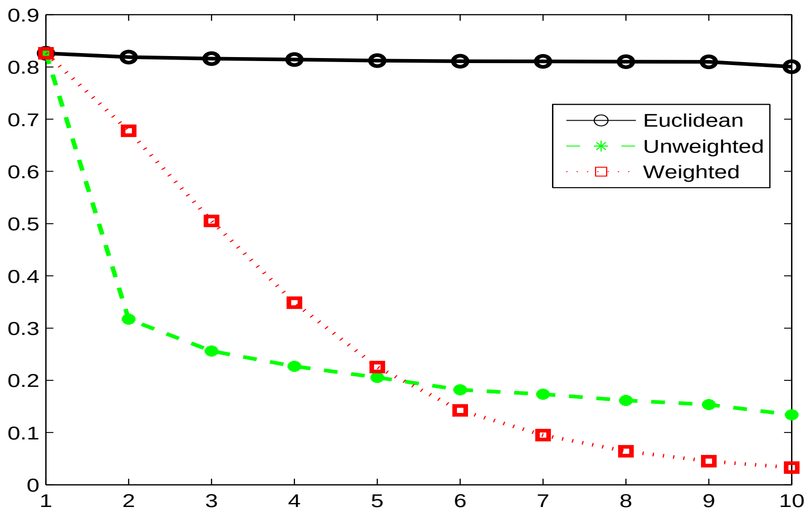

FIGURE 1. Graph of first 10 iterations in comparison with gradients in L_2, H_2^2, $\hat{H}_2^2$.

TABLE 2. Comparison of the approximate solution obtained by ADM and Sobolev gradient methods for time $t = 1$.

x	ADM	$\hat{H}_2^2$
0.0000	0.0000	0.0000
0.0938	0.0934	0.0937
0.2188	0.2118	0.2185
0.3125	0.3117	0.3121
0.4062	0.4039	0.4032
0.5000	0.4823	0.4956
0.6250	0.5963	0.6086
0.7188	0.6814	0.7012
0.8125	0.7709	0.7988
0.9062	0.8523	0.8791
1.0000	0.9543	0.9786
$\parallel$ E $\parallel_\infty$	0.0857	0.0214

In Figure 1 results of using steepest descent in L_2, H_2^2, $\hat{H}_2^2$ for the first ten iterations in comparison with infinity norm of the gradient vector, with an initial value of $u = 0$ is shown. It is quite evident from the graph that the $\hat{H}_2^2$ gradient is the best option for convergence.

To show the comparison with ADM, relative error can be subsequently as

$$\parallel E \parallel_\infty = \max_{i=1,..,M} \left| \frac{p(i) - p_{\text{exact}}}{p_{\text{exact}}} \right|, \tag{6. 52}$$

(where p_{exact} is the exact solution of the given GL equation.

To solve the given example with ADM we take fourth order approximation and we consider the gradient in $\hat{H}_2^2$ space when solving the given problem with descent methods. Table 2 shows the comparative study of Sobolev gradient method with ADM.

From the Table 2, we observe that the presented results are in excellent agreement with ADM.

7. CONCLUSION

In this paper, a comparison is given between the ADM and the Sobolev gradient methods for the solution of GL equation. The obtained results show that the Sobolev gradient method is robust and effective in terms of accuracy. If the domain of the problem becomes larger even than this method still converges. This is not true in the case of ADM, as the rate and region of convergence are potential shortcomings of the method. ADM converges very slowly for wider regions. The truncated solution is also very inaccurate in that region, therefore the application area of the method is very much limited. Also to find the series solution by ADM, the initial state of the system must be known. This is not in the case of Sobolev gradients. By the introduction of suitable weight functions in the construction of Sobolev space its performance can be further improved. Using simple optimization algorithm, for any arbitrary initial guess, this method finds the minimum of a functional.

The choice of underlying space and gradient plays a vital role in the construction of efficient algorithms. A number of different gradients can be defined from the same functional, which have different numerical properties. It is still an open problem as to how we can select a suitable space and define a gradient in it, such that it is best suited for the given problem except for linear problems [26].

REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, 1975.

[2] G. Adomian, *Nonlinear Stochastic operator equations*, Kluwer Acad. Publ. 1986.

[3] G. Adomian, *Solving frontiers problems of physics-The decomposition method*, Kluwer Acad. Publ. 1994.

[4] G. Adomian and R. E. Meyers, *The Ginzburg-Landau Equation*, Computer and Mathematics with Appl. **29**, No. 3 (1995) 3-4.

[5] A. Atangana and JF. Botha, *Analytical solution of the groundwater flow equation obtained via homotopy decomposition method*, J. Earth Sci. Clim Chang **3**, No. 115 (2012) doi:10.4172/2157-7617.1000115.

[6] A. Atangana and SB. Belhaouari, *Solving partial differential equation with space-and time-fractional derivatives via homotopy decomposition method*, Math. Probl. Eng. **9**, (2013) doi:10. 1155/2013/318590.

[7] M. Bayram et al. *Approximate solutions some nonlinear evolutions equations by using the reduced differential transform method*, Int. J. Appl. Math. Res. **1**, No. 3 (2012) 288302.

[8] C. Beasley, *Finite Element Solution to Nonlinear Partial Differential Equations*, PhD Thesis, University of North Texas, Denton, Tx. (1981).

[9] N. Bildik et al *Solution of different type of the partial differential equation by differential transform method and Adomians decomposition method*, Appl. Math. Comp. **172**, No. 1 (2006) 551567.

[10] N. Bildik and S. Deniz, *Implementation of Taylor collocation and Adomian decomposition method for systems of ordinary differential equations*, AIP Publ. **1648**, (2015) 370002.

[11] N. Bildik and A. Konuralp, *The use of variational iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations*, Int. J. Nonlinear Sci. Numer. Simul. **7**, No. 1 (2006) 6570.

[12] N. Bildik, M. Tosun and S. Deniz, *Euler matrix method for solving complex differential equations with variable coefficients in rectangular domains*, Int. J. Applied Phys. Math. **7**, No. 1 (2017) 69.

[13] N. Bildik and S. Deniz, *A Practical Method for Analytical Evaluation of Approximate Solutions of Fisher's Equations*, ITM Web of Conferences, **13**, EDP Sciences, 2017.

[14] N. Bildik and S. Deniz, *Comparative study between optimal homotopy asymptotic method and perturbation-iteration technique for different types of nonlinear equations*, Iran. J. Sci. Technol. Trans. Sci. **42**, No. 2 (2018) 647-654.

[15] B. Brown, M. Jais and I. Knowles, *A variational approach to an elastic inverse problem*, Inverse Problems **21**, (2005) 1953-1973.

[16] H. Bulut, M. Ergut and D. Evans, *The numerical solution of multidimensional partial differential equations by the decomposition method*, Int. J. Comp. Math. **80**, No. 9 (2003) 1189-1198.

[17] J. T. Edwards, J. Roberts and N. Ford, *A comparison of Adomian's decomposition method and Runge Kutta methods for approximate solution of some predator prey model equations*, University of Manchester, Numerical analysis Report No. 309, 1997

[18] J-H. He, *Variational iteration methoda kind of non-linear analytical technique: some examples*, Int. J. Non. Linear Mech. **34**, No. 4 (1999) 699708.

[19] J-H. He, *Homotopy perturbation method for bifurcation of nonlinear problems*, Int. J. Nonlinear Sci. Numer Simul. **6**, No. 2 (2005) 207208.

[20] J. Karatson and I. Farago, *Preconditioning operators and Sobolev gradients for nonlinear elliptic problems*, Comput. Math. Appl. **50**, (2005) 1077-1092.

[21] J. Karatson, *Constructive Sobolev gradient preconditioning for semilinear elliptic systems*, Electron J. Differential Equations **75**, (2004) 1-26.

[22] J. Karatson and L. Loczi, *Sobolev gradient preconditioning for the electrostatic potential equation*, Comput. Math. Appl. **50**, (2005) 1093-1104.

[23] H. Khalil, R. A. Khan, M. H. Al-Smadi and A. A. Freihat, *Approximation of Solution of Time Fractional Order Three-Dimensional Heat Conduction Problems with Jacobi Polynomials*, Punjab Univ. j. math. **47**, No. 1 (2015) 35-56.

[24] I. Knowles and A. Yan, *On the recovery of transport parameters in groundwater modelling*, J. Comp. Appl. Math. **171**, (2004) 277-290.

[25] W. T. Mahavier, *A numerical method utilizing weighted Sobolev descent to solve singluar differential equations*, Nonlinear World **4**, No. 4 (1997).

[26] W. T. Mahavier and J. Montgomery, *Single Iteration Sobolev descent for linear initial value Problems*, Missouri J. Math. Sci. **25**, No. 1 (2013) 15-26.

[27] J. W. Neuberger, *Sobolev Gradients and Differential Equations*, Springer Lecture Notes in Mathematics 1670 (Springer-Verlag, 1997).

[28] R. Nittka and M. Sauter, *Sobolev gradients for differential algebraic equations*, J. Differ. Equ. **42**, (2008) 1-31.

[29] T. Ozis and D. Agirseven, *He's homotopy perturbation method for solving heat-like and wave-like equations with variable coefficients*, Phys Lett A **372**, No. 38 (2008) 5944-5950.

[30] N. Raza, *Application of Sobolev Gradient Method to Solve Klein Gordon Equation*, Punjab Univ. j. math. **48**, No. 2 (2016) 135-145.

[31] N. Raza, S. Sial and S. Siddiqi, *Sobolev gradient approach for the time evolution related to Energy minimization of Ginzberg-Landau functionals*, J. Comp. Phys. **228**, (2009) 2566-2571.

[32] N. Raza, S. Sial and S.S. Siddiqi, *Approximating time evolution related to Ginzberg-Landau functionals via Sobolev gradient methods in a finite-element setting*, J. Comp. Phys. **229**, (2010) 1621-1625.

[33] A. U. Rehman, S. U. Rehman and J. Ahmad, *Numerical Approximation of Stiff Problems Using Efficient Nested Implicit RungeKutta Methods*, Punjab Univ. j. math. **51**, No. 2 (2019) 1-19.

[34] S. Sial, J. W. Neuberger, T. Lookman and A. Saxena, *Energy minimization using Sobolev gradients: Application to phase separation and ordering*, J. Comp. Phys. **189**, (2003) 88-97.

[35] S. Sial, J. Neuberger, T. Lookman and A. Saxena, *Energy minimization using Sobolev gradients finite element setting*, Proc. World Conference on 21st Century Mathematics, Lahore, Pakistan, March 2005 eds. Chaudhary and Bhatti.

[36] S. Sial, *Sobolev gradient algorithm for minimum energy states of s-wave superconductors: finite element setting*, Supercond. Sci. Technol. **18**, (2005) 675-677.

[37] M. Suleman and S. Riaz, *Unconditionally Stable Numerical Scheme to Study the Dynamics of Hepatitis B Disease*, Punjab Univ. j. math. **49**, No. 3 (2017) 99-118.

[38] A. M. Wazwaz, *A comparison between Adomian decomposition method and Taylor series method in the series solutions*, Applied Mathematics and Computation **97**, (1997) 37-44.