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Nanofluid Film Flow of Eyring Powell Fluid with Magneto Hydrodynamic Effect on Unsteady Porous Stretching Sheet

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Abstract. This article investigates two dimensional nanofluid film flow of Eyring Powell Fluid with variable heat transmission in the existence of uniform magnetic field (MHD) on an unsteady porous stretching sheet.The basic governing time dependent equations of momentum, heat transfer and mass transfer are modeled and reduced to a system of differential equations by employing appropriate similarity transformation with unsteady dimensionless parameters.The important influence of thermophoresis and Brownian motion have been taken in the nanofluids model.An optimal approach has been applied to get appropriate results from the modeled problem.The convergence of HAM (Homotopy Analysis Method) has been identified numerically.The discrepancy of the Nusslet number, skin friction, Sherwood number and their influence on the velocity, temperature and concentration profiles has been scrutinized.The influence of the unsteady parameter (A) over thin film is explored analytically for different values.Moreover, for comprehension the physical presentation of the embedded parameters like Film Thickness parameter (β), Magnetic parameter (M), Stretching parameter (γ), and Eyring Powell fluid parameters (k) have been plotted graphically and discussed.Prandtl number (Pr), Brownian motion parameter (Nb), Thermophoretic parameter (Nt), Schmidth number (Sc) have been represented by graph and discussed.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: MHD, Nanofluid, Unsteady porous stretching sheet, Eyring Powell fluid, HAM.

1. INTRODUCTION

In recent few years it has been scrutinized that the thin film flow analysis has significantly contributed in the area of industries, engineering and technology as well as in other emerging fields of science.Thin film flow problems have diverse applications in many fields, fluctuating from a particular situation of flow in human lungs to lubrication problems in engineering, this is probably one of the largest subclass of thin film flow problems.The practical applications of thin film flow is a provoking transaction between structural mechanics and fluid mechanics.Coating of wire and fiber is one of its substantial application.Extrusion of polymer and metal from die, crystal growing, food stuff processing, drawing of plastic sheets, plastic foam processing, manufacturing of plastic fluid, artificial fibers and fluidization of reactor are well known applications.In view of all these applications, it becomes an important issue for researchers to develop the study of liquid film on stretching surface.The flow of liquid film was first studied for viscous flow and further it is extended to non-Newtonian fluid.Eyring-Powell fluid is an integral part of nonNewtonian fluids.Many researchers investigated the effect of MHD and heat on Eyring-Powell fluid.The at hand amount of study in the form of nano-fluids is less then the least.Hayat et al.[17] derived Eyring Powell fluid model from kinetic theory of liquids instead of empirical relation.Sirohi et al.[41] has also reported some study on flows of Eyring-Powell fluid.Patel et al.[33] applied technique of satisfaction with asymptotic boundary condition for numerical solution on the flow of Eyring-Powell fluid.Crane [14] was the first one who deliberates the motion of viscous fluid in a linear stretching surface.Dandapat [15] studied the flow of viscoelastic fluids with heat transfer on a stretching sheet.Ushah and Sridharan [44] worked on the same problem and extended it to liquid film fluid with heat transmission analysis on horizontal sheet.Liu and Andersson [28] have used numerical techniques in their work to obtain solution and discussed parameters.Aziz et al.[10] has observed the effect of inner heat production in an unsteady stretching sheet due to flow in a thin liquid film on it.Thin film flow of non Newtonian fluids are in aboundance in many live walks of life.Therefore, it is one of the most common factors of the nature which is mostly used in field of industry, engineering and technology.Andersson [7] was the pioneer to study flow of thin liquid film of non Newtonian fluids by taking into account the Power Law model in an unsteady stretching sheet.After that most of the researchers [6,12,13,45] have studied Power Law of fluids applying different cases in unsteady stretching surface.Singh Megahe et al.[29] has observed Casson liquid thin film flow and temperature transmission in the presence of viscous dissipation and variable heat flux having slip velocity.Fareesa et al.[43] has studied flow of a Nanofluid films of Maxwell fluid with thermal radiation and magneto hydrodynamic properties on an unstable stretching sheet.Noor Saeed et al.[21] explored Brownian motion and thermophoresis effects on MHD mixed convective thin film second grade nanofluid flow with Hall effect and heat transfer past a stretching sheet.Shah et al.[38,39] studied the effects of Hall current on three-dimensional non-Newtonian nanofluids and micropolar nanofluids in a rotating frame.Hameed et al.[22] investigated the combined magnetohydrodynamic and electric field effect on an unsteady Maxwell nanofluid flow over a stretching surface under the influence of variable heat and thermal radiation.Muhammad et al.[30] has studied the rotating flow of magneto hydrodynamic carbon nanotubes over a stretching sheet with the impact of nonlinear thermal radiation and heat generation/absorption.Recently, Ishaq et al.[19] has investigated Entropy Generation on Nanofluid Thin Film Flow of EyringPowell Fluid with Thermal Radiation and MHD Effect on an Unsteady Porous Stretching Sheet.In the field of science and technology most of the mathematical problems are complex in their nature and the exact solution is almost very difficult or even impossible.Numerical and Analytical methods are used to find out the approximate solution of such problems.One of the popular and proficient method for the solution of such type of problems is HAM.Mathematical modeling of many phenomena, especially in heat transfer, usually leads to a nonlinear equation.Traditional approaches for solving such equations are time consuming and difficult affairs tasks.In this paper, based on the homotopy analysis method (HAM), a series solution for the problem of unsteady nonlinear convective-radiative equation is obtained.In HAM, one would be able to control the convergence of approximation series and adjust its convergence region, conveniently. Ability and efficiency of proposed approach are tested via some cases of above mentioned problem.It is found that homotopy-analysis approach provides a greatly accelerated convergence series solution for problem. Its main advantage is that it can be used to the nonlinear ordinary differential equations without discretization or linearization and is a substitute method.Liao [23-27] was the first one to investigate this technique for solving this type of problems and generally verified that HAM method converges rapidly to approximate solution.The current method also provides series solutions that includes single variable functions.The significance of this method is that it considers all physical parameters of the problem and provide the opportunity to explore its behavior conveniently.Due to its fast convergence, many researchers Rashidi [34], Abbasbandy [1,2], Hayat et al.[18], and Nadeem et al.[32] used this technique to solved highly nonlinear and coupled differential equations.Many researchers [3,4,5,8,9,11,16,20,31,35,36,37,40,42,46] have contributed in the same as well as in the related areas.The aim of present work is to analyze nanofluid liquid film of Eyring-Powell fluid and its flow in the existence of MHD.Keeping in view assumptions taken into the model problems and the similarity transformation method, the concerned PDEs are converted to non-linear ODEs and the obtained transformed equations are analytically solved using (HAM).

2. PROBLEM'S MATHEMATICAL FORMULATION:

Assume two dimensional incompressible nanofluid liquid film of Eyring-Powell fluid flow along with megnetohydrodynamics on an unsteady porous stretching sheet with simultaneous transfer of mass and heat.The coordinate axes are selected such that the slit is in the direction of x-axis and surface is perpendicular to y-axis respectively.The plate and its linear velocity are along positive x-axis and are assumed as:

$$U_0(x,t) = \frac{\alpha x}{1-\gamma t} \tag{2. 1}$$

$$U_0(x,t) = \frac{\alpha x}{1-\gamma t} \tag{2. 2}$$

which is stretching, where γ is the stretching parameter.The surface temperature of the nanofluid is

$$T_w(x,t) = T_0 - T_{ref}\left(\frac{\alpha x^2}{2v}\right) \times (1-\gamma t)^{-3/2} \tag{2. 3}$$

and similarly the volume concentration for the nanofluid is

$$C_w(x,t) = C_0 - C_{ref}\left(\frac{\alpha x^2}{2v}\right) \times (1-\gamma t)^{-3/2} \tag{2. 4}$$

The time dependent term

$$\frac{\alpha x^2}{v(1-\gamma t)}$$

is the local Reynold number, dependent on the stretching velocity $U_0(x,t)$. Here T_0 and C_0 are temperature and concentration at the slit respectively, C_{ref} and T_{ref} are the reference concentration and reference temperature such that $C_{ref} \in [0, C_0]$ and $T_{ref} \in [0, T_0]$. At the start, the slit is initiated along the Origin and after that an extrinsic force is acted to stretch the slit in the direction of positive horizontal axis at the rate $\frac{\alpha}{1-\gamma t}$ in the time $\gamma \in [0,1]$ with $U_0(x,t)$ initial velocity.

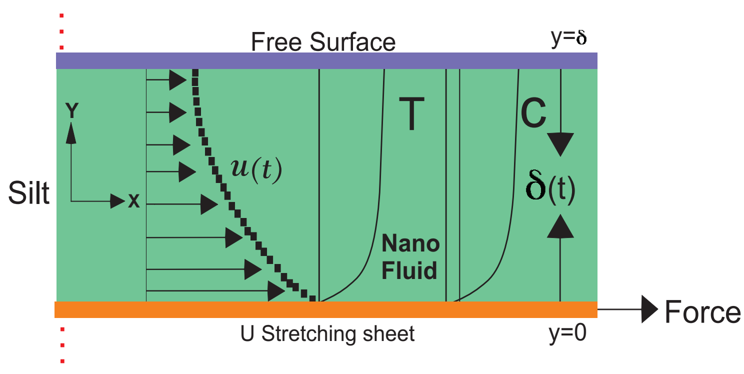

FIGURE 1. Geometry of the Physical Model.

The basic governing equations are

$$div\,\widehat{V} = 0 \tag{2. 5}$$

$$\rho\, a_i = -\nabla p + \nabla(T) + \widehat{J} \times \widehat{B} \tag{2. 6}$$

$$\left(\widehat{V}.\nabla\right) T = \alpha \nabla^2 T + \rho\left[D_B \nabla C.\nabla T + \frac{D_T}{T_0}\nabla T.\nabla T\right] \tag{2. 7}$$

$$(\widehat{V}.\nabla)C = D_B \nabla^2 C + \frac{D_T}{T_0}\nabla^2 T \tag{2. 8}$$

Cauchy stress tensor T can be expressed as

$$T = -pI + \tau \tag{2. 9}$$

The rheological model of Eyring-Powell fluid [3] is

$$\tau_{ij} = \mu \frac{\partial u_i}{\partial x_j} + \frac{1}{\beta} \sinh^{-1} \left[\frac{1}{c} \frac{\partial u_i}{\partial x_j} \right] \tag{2. 10}$$

Here β and c represents characteristics of the Eyring-Powell fluid, where p and I are the pressure and the identity tensor respectively. Where

$$\sinh^{-1} \left[\frac{1}{c} \frac{\partial u_i}{\partial x_j} \right] \approx \frac{1}{c} \frac{\partial u_i}{\partial x_j} - \frac{1}{6} \left[\frac{1}{c} \frac{\partial u_i}{\partial x_j} \right]^3, \left| \frac{1}{c} \frac{\partial u_i}{\partial x_j} \right| \prec 1. \tag{2. 11}$$

Considering the above assumptions, the leading equations for continuity, momentum, energy and concentration of two dimensional thin film flow are as under:

$$\frac{\partial \widehat{u}}{\partial x} + \frac{\partial \widehat{v}}{\partial y} = 0 \tag{2. 12}$$

$$\frac{\partial \widehat{u}}{\partial t} + \widehat{u} \frac{\partial \widehat{u}}{\partial x} + \widehat{v} \frac{\partial \widehat{u}}{\partial y} = \left(\upsilon + \frac{1}{\rho \beta C} \right) \frac{\partial^2 \widehat{u}}{\partial y^2} - \frac{1}{2 \rho \beta C^3} \left[\left(\frac{\partial \widehat{u}}{\partial y} \right)^2 \frac{\partial^2 \widehat{u}}{\partial y^2} \right] - \frac{\sigma \widehat{B}_0^2}{\rho} \widehat{u}(t) - \frac{\nu}{k^*} \widehat{u}(t) \tag{2. 13}$$

$$\frac{\partial T}{\partial t} + \widehat{u} \frac{\partial T}{\partial x} + \widehat{v} \frac{\partial T}{\partial y} = \frac{1}{\rho c_p} \frac{\partial}{\partial y} \left[K(T) \frac{\partial T}{\partial y} \right] + t \left[D_B \left(\frac{\partial C}{\partial y} \frac{\partial T}{\partial y} \right) + \frac{D_t}{T_0} \left(\frac{\partial T}{\partial y} \right)^2 \right] \tag{2. 14}$$

$$\frac{\partial C}{\partial t} + \widehat{u} \frac{\partial C}{\partial x} + \widehat{v} \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \left(\frac{D_T}{T_0} \right) \frac{\partial^2 T}{\partial y^2} \tag{2. 15}$$

Here u and v are the fluid velocity components, υ denotes coefficient of kinematic viscosity, ρ represents density where as σ andμ represent the electrical conductivity and dynamic viscosity respectively. In equation (2.14) T represents the temperature, α is thermal diffusivity, k^* represents porosity, c_p represents specific heat, thermal conductivity of fluid is represented by k_p, Brownian diffusion coefficient is denoted by D_B, $t = \frac{(\rho c_p)_p}{(\rho c_p)_f}$ where ρ_f denotes the base fluid density and ρ_b represents density of the particle, C represents coefficient of volumetric expansion.

The Boundary conditions for the state problem includes:

$$\widehat{u} = U_0, \quad \widehat{v} = 0, \quad T = T_w, \quad C = C_s, \quad at \quad y = 0, \tag{2. 16}$$

$$\frac{\partial \widehat{u}}{\partial y} = \frac{\partial T}{\partial y} = \frac{\partial C}{\partial y} = 0, \quad \widehat{v} = \frac{d\delta}{dx} = 0, \quad at \quad y = \delta(t), \tag{2. 17}$$

Where the thickness of liquid is $\delta(t)$. The similarity variables for non dimensionalization are as follows:

$$\eta = \sqrt{\frac{\alpha}{\upsilon(1 - \gamma t)}} y, \tag{2. 18}$$

$$\Psi(x, y, t) = x \sqrt{\frac{\upsilon \alpha}{1 - \gamma t}} f(\eta), \tag{2. 19}$$

$$T(x,y,t) = T_0 - T_{ref}\left(\frac{\alpha x^2}{2\upsilon}\right)(1-\gamma t)^{-3/2}\theta(\eta), \tag{2. 20}$$

$$C(x,y,t) = C_0 - C_{ref}\left(\frac{\alpha x^2}{2\upsilon}\right)(1-\gamma t)^{-3/2}\phi(\eta) \tag{2. 21}$$

Here Ψ represents stream function such that, $\left(\widehat{u},\widehat{v}\right) = \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x}\right)$. The prime indicates derivative w.r.t η and the thickness of non-dimensional nanofluid film is represented by β. where, $\beta = \left(\frac{\alpha}{\upsilon(1-\gamma t)}\right)\delta(t)$.

$$\frac{d\delta}{dt} = -\frac{\beta\gamma}{2}\left[\frac{\upsilon}{\alpha}\right]^{\frac{1}{2}}\left(1-\gamma t\right)^{\frac{-1}{2}} \tag{2. 22}$$

Inserting equations (2.18-2.21) in (2.12-2.15), where (2.12) identically holds and we get the following governing equations:

$$(1+k)f''' - (f')^2 + ff'' - A\left(f' + \tfrac{\eta}{2}f''\right) - \lambda(f'')^2 f''' - k^* f' - Mf' = 0, \tag{2. 23}$$

$$\left[1+\xi\theta\right]\theta'' + Pr\left[f\theta' - 2f'\theta - \frac{A}{2}\left(3\theta + \eta\theta'\right) + Nb\phi'\theta' + Nt(\theta')^2\right] = 0 \tag{2. 24}$$

$$\phi'' + Sc\left[f\phi' - 2f'\phi - \frac{A}{2}\left(3\phi + \eta\phi''\right)\right] + \frac{N_t}{N_b}\phi'' = 0 \tag{2. 25}$$

The corresponding non-dimensional boundary conditions are

$$f'(0) = 1, \quad f(0) = 0, \quad \theta(0) = \phi(0) = 1 \tag{2. 26}$$

$$f(\beta) = \frac{s\beta}{2}, \quad f''(\beta) = 0, \quad \theta'(\beta) = \phi'(\beta) = 0 \tag{2. 27}$$

The non-dimensional parameters after simplification can be defined as $Pr = \frac{\upsilon}{\alpha}$ is prandtl number, $M = \frac{\sigma B_0^2}{\rho\alpha}(1-\xi t)$ is magnetic parameter, $Sc = \frac{\upsilon}{D_B}$ Schmid number, $Nb = \frac{\tau D_B(C_\omega - C_0)}{\upsilon}$ is parameter of Brownian motion, $Nt = \frac{(\rho c_p)_p D_T T_0}{(\rho c_p)_f T_c}$ is Thermophoretic parameter, $A = \frac{\xi}{\alpha}$ is unsteadiness parameter, $\gamma = \frac{1}{2\rho\beta C^3}\left(\frac{\alpha}{1-\xi t}\right)^3\left(\frac{x}{r}\right)^3$ is Stretching parameter, $k^* = \frac{\upsilon}{\rho\kappa^*}\left(1-\xi t\right)$ is Porosity parameter, and $k = \frac{1}{\mu BCr}$ is Eyring Powell fluid parameter.

3. HOMOTOPIC SOLUTION:

The following initial guesses are nominated as under:

$$\widehat{f_0}(\eta) = \eta, \quad \widehat{\theta_0}(\eta) = 1, and \quad \widehat{\phi_0}(\eta) = 1. \tag{3. 28}$$

L_f, L_θ, and L_ϕ are representing linear operators.

$$L_f(\widehat{f}) = \widehat{f'''}, \quad L_\theta(\widehat{\theta}) = \widehat{\theta''}, \quad L_\phi(\widehat{\phi}) = \widehat{\phi''}. \tag{3. 29}$$

which have the subsequent applicability,

$$L_f(e_1 + e_2\eta + e_3\eta^2) = 0, \quad L_\theta(e_4 + e_5\eta) = 0, \quad L_\phi(e_6 + e_7\eta) = 0 \quad (3.\ 30)$$

The coefficients involve in the general solution are e_i where $1 \le i \le 7$. The corresponding nonlinear operators N_f, N_θ, N_ϕ, are carefully chosen of the form:

$$N_f\left[\widehat{f}(\eta;\xi)\right] = (1+k)\frac{\partial^3 \widehat{f}}{\partial \eta^3} - \left(\frac{\partial \widehat{f}}{\partial \eta}\right)^2 + f\frac{\partial^2 \widehat{f}}{\partial \eta^2} - A\left(\frac{\partial \widehat{f}}{\partial \eta} + \frac{\eta}{2}\frac{\partial^2 \widehat{f}}{\partial \eta^2}\right) - \lambda\left(\frac{\partial^2 \widehat{f}}{\partial \eta^2}\right)^2 \frac{\partial^3 \widehat{f}}{\partial \eta^3} - k^*\frac{\partial \widehat{f}}{\partial \eta} - M\frac{\partial \widehat{f}}{\partial \eta} \quad (3.\ 31)$$

$$N_\theta\left[\widehat{f}(\eta;\xi), \widehat{\theta}(\eta;\xi), \widehat{\phi}(\eta;\xi)\right] = \left(\frac{1+\xi\widehat{\theta}}{Pr}\right)\frac{\partial^2 \widehat{\theta}}{\partial \eta^2} + \widehat{f}\frac{\partial \widehat{\theta}}{\partial \eta} - 2\widehat{\theta}\frac{\partial \widehat{f}}{\partial \eta} - \frac{A}{2}\left(3\widehat{\theta} + \eta\frac{\partial \widehat{\theta}}{\partial \eta}\right) + Nt\left(\frac{\partial \widehat{\theta}}{\partial \eta}\right)^2 + Nb\frac{\partial \widehat{\theta}}{\partial \eta}\frac{\partial \widehat{\phi}}{\partial \eta} \quad (3.\ 32)$$

$$N_\phi\left[\widehat{f}(\eta;\xi), \widehat{\theta}(\eta;\xi), \widehat{\phi}(\eta;\xi)\right] = \frac{\partial^2 \widehat{\phi}}{\partial \eta^2} + Sc\left[\widehat{f}\frac{\partial \widehat{\phi}}{\partial \eta} - 2\widehat{\phi}\frac{\partial \widehat{f}}{\partial \eta} - \frac{A}{2}\left(3\widehat{\phi} + \eta\frac{\partial^2 \widehat{\phi}}{\partial \eta^2}\right)\right] + \frac{Nt}{Nb}\frac{\partial^2 \widehat{\theta}}{\partial \eta^2} \quad (3.\ 33)$$

The basic solution process by HAM is defined in [3.28-3.56], the 0^{th} Order system forms Eqs.(2.23-2.25) as:

$$(1-\zeta)L_f[\widehat{f}(\eta,\zeta) - \widehat{f}_0(\eta)] = ph_f N_f[\widehat{f}(\eta,\zeta)] \quad (3.\ 34)$$

$$(1-\zeta)L_\theta[\widehat{\theta}(\eta,\zeta) - \widehat{\theta}_0(\eta)] = ph_\theta N_\theta[\widehat{f}(\eta,\zeta), \widehat{\theta}(\eta,\zeta), \widehat{\phi}(\eta,\zeta)] \quad (3.\ 35)$$

$$(1-\zeta)L_\phi[\widehat{\phi}(\eta,\zeta) - \widehat{\phi}_0(\eta)] = \zeta h_\phi N_\phi[\widehat{f}(\eta,\zeta), \widehat{\theta}(\eta,\zeta), \widehat{\phi}(\eta,\zeta)] \quad (3.\ 36)$$

The corresponding boundary constraints are

$$\widehat{f}(\eta,\zeta)\Big|_{\eta=0} = 0, \qquad \widehat{f}(\eta,\zeta)\Big|_{\eta=\beta} = \frac{s\beta}{2} \quad (3.\ 37)$$

$$\frac{\partial \widehat{f}(\eta,\zeta)}{\partial \eta}\Bigg|_{\eta=0} = 1, \qquad \frac{\partial^2 \widehat{f}(\eta,\zeta)}{\partial \eta^2}\Bigg|_{\eta=\beta} = 0 \quad (3.\ 38)$$

$$\widehat{\theta}(\eta,\zeta)\Big|_{\eta=0} = 1, \qquad \frac{\partial \widehat{\theta}(\eta,\zeta)}{\partial \eta}\Bigg|_{\eta=\beta} = 0 \quad (3.\ 39)$$

$$\widehat{\phi}(\eta,\zeta)\Big|_{\eta=0} = 1, \qquad \frac{\partial \widehat{\phi}(\eta,\zeta)}{\partial \eta}\Bigg|_{\eta=\beta} = 0 \quad (3.\ 40)$$

where $\zeta \in [0,1]$ is the embedding constraint, h_f, h_θ, h_ϕ are used to regulate convergence. When $\zeta = 0$, $\zeta = 1$ we obtain;

$$\widehat{f}(\eta) = \widehat{f}(\eta, 1), \quad \widehat{\theta}(\eta) = \widehat{\theta}(\eta, 1), \quad \widehat{\phi}(\eta) = \widehat{\phi}(\eta, 1) \tag{3. 41}$$

Taylor's series approximation for $\zeta = 0$ is used to expand the velocity, temperature and concentration fields $\widehat{f}(\eta,\zeta), \widehat{\theta}(\eta,\zeta)$, and $\widehat{\phi}(\eta,\zeta)$

$$\widehat{f}(\eta,\zeta) = \widehat{f}_0(\eta) + \sum_{k=1}^{\infty} \widehat{f}_k(\eta)\zeta^k \tag{3. 42}$$

$$\widehat{\theta}(\eta,\zeta) = \widehat{\theta}_0(\eta) + \sum_{k=1}^{\infty} \widehat{\theta}_k(\eta)\zeta^k \tag{3. 43}$$

$$\widehat{\phi}(\eta,\zeta) = \widehat{\phi}_0(\eta) + \sum_{k=1}^{\infty} \widehat{\phi}_k(\eta)\zeta^k \tag{3. 44}$$

$$\widehat{f}_n(\eta) = \frac{1}{n!}\frac{\partial \widehat{f}(\eta,\zeta)}{\partial \eta}\bigg|_{\zeta=0}, \quad \widehat{\theta}_n(\eta) = \frac{1}{n!}\frac{\partial \widehat{\theta}(\eta,\zeta)}{\partial \eta}\bigg|_{\zeta=0}, \quad \widehat{\phi}_n(\eta) = \frac{1}{n!}\frac{\partial \widehat{\phi}(\eta,\zeta)}{\partial \eta}\bigg|_{\zeta=0} \tag{3. 45}$$

The secondary constraints h_f, h_θ and h_ϕ are selected in such a way that the series (3.42),(3.43) and (3.44) converges at $\zeta = 1$ so, switching $\zeta = 1$ in (3.42),(3.43) and (3.44) we obtain:

$$\widehat{f}(\eta) = \widehat{f}_0(\eta) + \sum_{n=1}^{\infty} \widehat{f}_n(\eta), \tag{3. 46}$$

$$\widehat{\theta}(\eta) = \widehat{\theta}_0(\eta) + \sum_{n=1}^{\infty} \widehat{\theta}_n(\eta), \tag{3. 47}$$

$$\widehat{\phi}(\eta) = \widehat{\phi}_0(\eta) + \sum_{n=1}^{\infty} \widehat{\phi}_n(\eta), \tag{3. 48}$$

The n^{th} order problem satisfies the following:

$$L_f\left[\widehat{f}_n(\eta) - \chi_n \widehat{f}_{n-1}(\eta)\right] = h_f R_k^f(\eta), \tag{3. 49}$$

$$L_\theta\left[\widehat{\theta}_n(\eta) - \chi_n \widehat{\theta}_{n-1}(\eta)\right] = h_\theta R_n^\theta(\eta), \tag{3. 50}$$

$$L_\phi\left[\widehat{\phi}_n(\eta) - \chi_n \widehat{\phi}_{n-1}(\eta)\right] = h_\phi R_n^\phi(\eta). \tag{3. 51}$$

The invariable boundary conditions are:

$$\widehat{f}_n(0) = \widehat{f}'_n(0) = \widehat{\theta}_n(0) = \widehat{\phi}_n(0) = 0, \qquad \widehat{f}''_n(\beta) = \widehat{\theta}'_n(\beta) = \widehat{\phi}'_n(\beta) = 0 \tag{3. 52}$$

Here

$$R_n^f(\eta) = (1+k)\widehat{f}'''_{n-1} - \sum_{k=0}^{n-1} \widehat{f}'_{n-1-k}\widehat{f}'_k + \sum_{k=0}^{n-1} \widehat{f}_{n-1-k}\widehat{f}''_k - A\left[\widehat{f}'_{n-1} + \frac{\eta}{2}\widehat{f}''_{n-1}\right] + \lambda \sum_{k=0}^{n-1}\left(\widehat{f}''_{n-1-k}\right)^2 \widehat{f}'''_k - M\widehat{f}'_{n-1} - k^*\widehat{f}'_{n-1}. \tag{3. 53}$$

$$R_n^\theta(\eta) = \left(1+\xi\theta\right)\widehat{\theta}''_{n-1} + \frac{1}{Pr}\xi\sum_{k=0}^{n-1}\widehat{\theta}_{n-1-k}\widehat{\theta}''_k + \sum_{k=0}^{n-1}\widehat{f}_{n-1-k}\widehat{\theta}'_k - 2\sum_{k=0}^{n-1}\widehat{f}'_{n-1-k}\widehat{\theta}_k - \frac{A}{2}\left(3\widehat{\theta}_{n-1} + \eta\widehat{\theta}'_{n-1}\right) + Nb\sum_{k=1}^{n-1}\widehat{\theta}'_{n-1-k}\widehat{\phi}'_k + Nt\sum_{k=1}^{n-1}\widehat{\theta}'_{n-1-k}\widehat{\theta}'_k. \tag{3. 54}$$

$$R_n^\phi(\eta) = \widehat{\phi}''_{n-1} + Sc\sum_{k=0}^{n-1}\widehat{f}_{n-1-k}\widehat{\phi}'_k - 2Sc\sum_{k=0}^{n-1}\widehat{f}'_{n-1-k}\widehat{\phi}_k - \frac{ASc}{2}\left(3\widehat{\phi}_{n-1} + \eta\widehat{\phi}'_{n-1}\right) + \frac{Nt}{Nb}\widehat{\theta}''_{n-1} \tag{3. 55}$$

Where

$$\chi_n = \begin{cases} 1, & \zeta > 1 \\ 0, & \zeta \leqslant 1 \end{cases} \tag{3. 56}$$

4. Convergence:

When we computed the series solutions of velocity, temperature and concentration functions using HAM, the assisting parameters $h_{f,\theta}$ and h_ϕ appears, which are responsible for adjusting the convergence of solutions. h-curve graphs of $f''(0)$,$\theta'(0)$ and $\phi'(0)$ for 7^{th} order Approximation are plotted to get the possible region of h curve in the Figures (2-3) for various values of embedded variables. The h-curves consecutively display the valid region. The convergence region of the h-curve in Figs. 2 and 3 is shown as $-0.2 \leq h \leq 0.0$ which is a valid region.

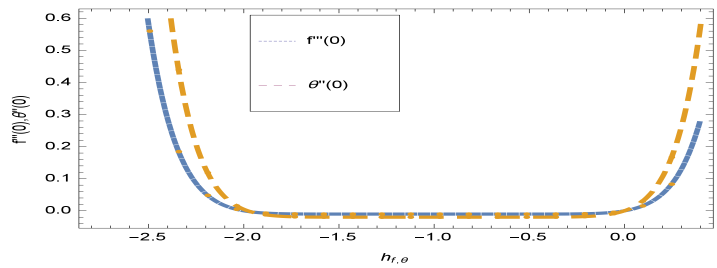

FIGURE 2. Combine h curve of function f and θ at 7^{th} order approximation, when $\gamma = Sc = A = \xi = \beta = k = 0.1$, $Nb = Nt = 0.3$, $M = Pr = 1$.

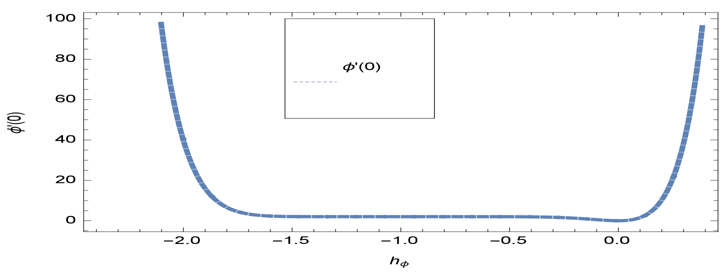

FIGURE 3. The h curve function of ϕ for 7^{th} order approximation, when $\gamma = Sc = A = \xi = \beta = k = 0.1$, $Nb = Nt = 0.3$, $M = Pr = 1$.

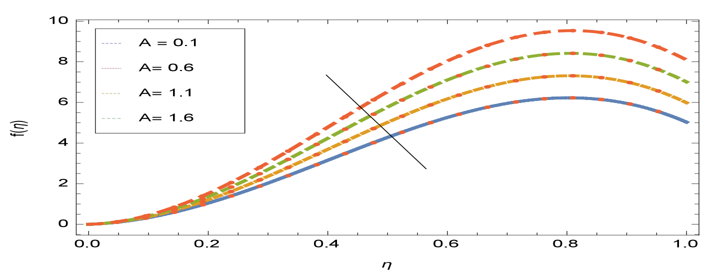

FIGURE 4. Influence of A on $f(\eta)$, where $\beta = 0.4$,$M = 1$,$\gamma = 0.7$,$\lambda = k = 0.6$.

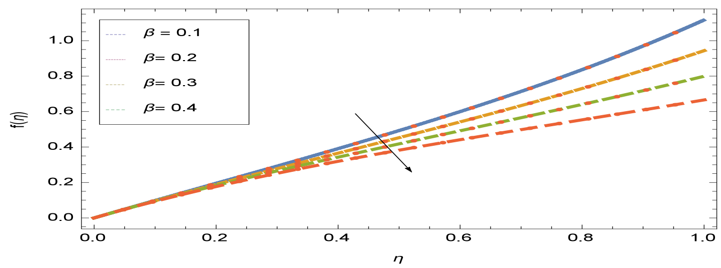

FIGURE 5. Influence of β on $f(\eta)$, where $A = 0.1$,$M = 1$,$\gamma = 0.7$,$\lambda = k = 0.6$.

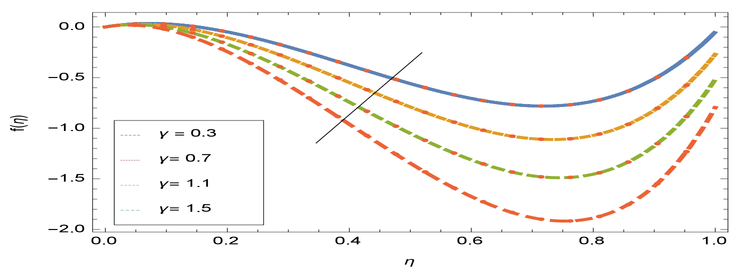

FIGURE 6. Influence of γ on $f(\eta)$, where $\beta = 0.4$,$M = 1$,$A = 0.9$,$\lambda = 0.5$,$k = 0.6$.

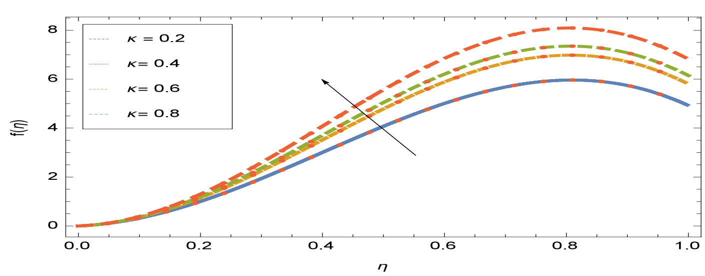

FIGURE 7. Influence of k on $f(\eta)$, where $\beta = 0.4$,$M = 1$,$A = 0.9$,$\lambda = \gamma = 0.5$.

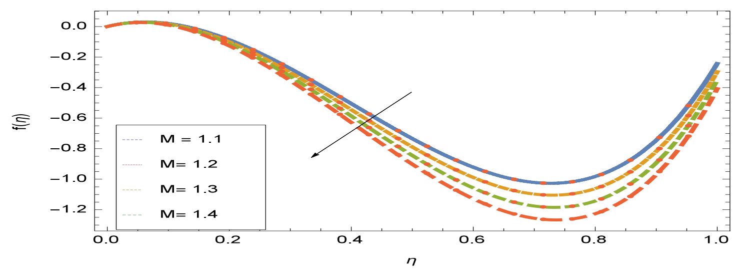

FIGURE 8. Influence of M on $f(\eta)$, where $\beta = 0.4$,$\gamma = 0.7$,$\lambda = 0.3$,$A = 0.9$,$k = 0.6$.

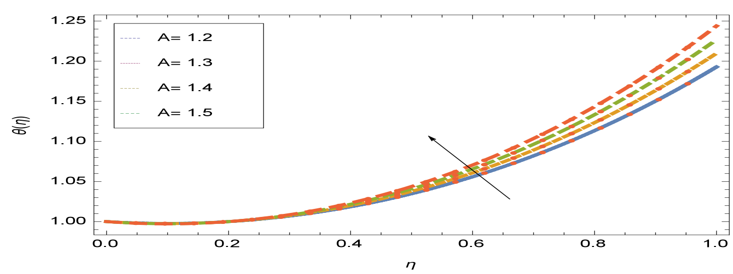

FIGURE 9. Influence of A on $\theta(\eta)$, where $\gamma = Sc = \xi = 0.6$, $\beta = 0.1$,$Nb = 0.3$, $M = 0.5$, $Nt = k = 0.4$, $Pr = 1$.

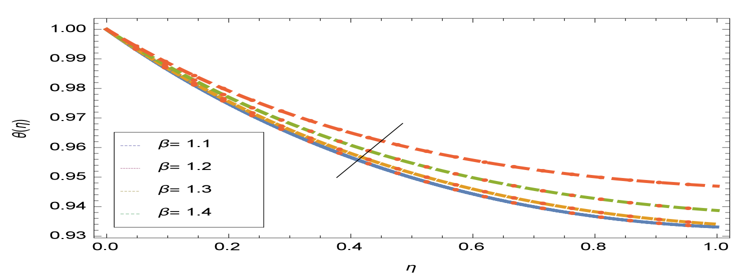

FIGURE 10. Influence of β on $\theta(\eta)$, where $\gamma = A = Sc = \xi = 0.6$,$M = 0.1$,$Nb = 0.3$, $Nt = k = 0.4$,$Pr = 1$.

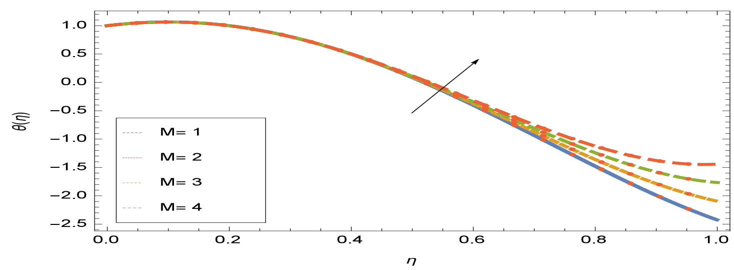

FIGURE 11. Influence of M on $\theta(\eta)$, where $\gamma = A = Sc = \xi = 0.6$,$\beta = 0.1$,$Nb = 0.3$, $Nt = k = 0.4$,$Pr = 1$.

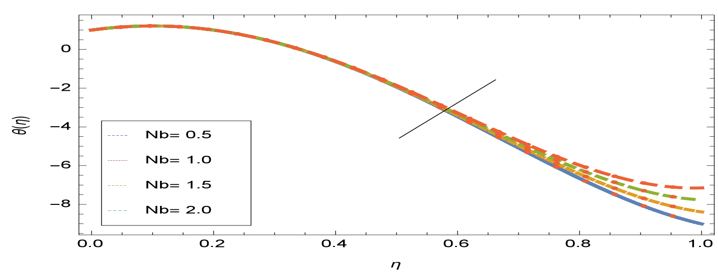

FIGURE 12. Influence of Nb on $\theta(\eta)$, where $\gamma = A = Sc = \xi = 0.6$,$\beta = 0.1$,$M = 0.5$,$Nt = Rd = k = 0.4$,$Pr = 1$.

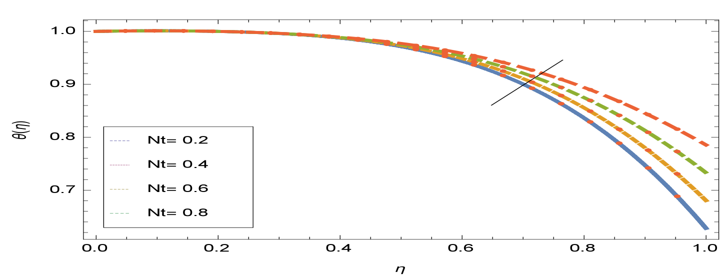

FIGURE 13. Influence of Nt on $\theta(\eta)$, where $\gamma = A = Sc = \xi = 0.6$,$\beta = 0.1$,$M = 0.5$,$Nb = 0.3$,$Rd = k = 0.4$,$Pr = 1$.

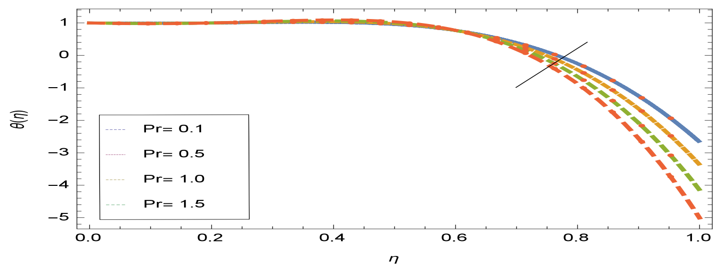

FIGURE 14. Influence of Pr on $\theta(\eta)$, where $\gamma = A = Sc = \xi = 0.6$,$\beta = 0.1$,$M = 5$,$Nb = 3$, $Nt = Rd = k = 0.4$.

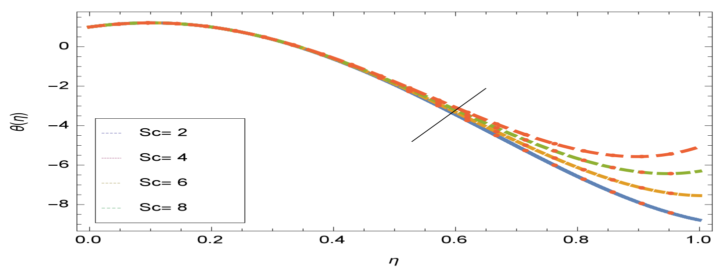

FIGURE 15. Influence of Sc on $\theta(\eta)$, where $\gamma = A = \xi = 0.6$,$\beta = 0.1$,$M = 0.2$,$Nb = 0.3$, $Nt = k = 0.4$, $Pr = 1$.

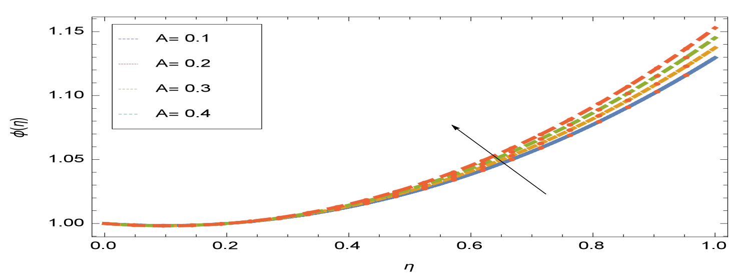

FIGURE 16. Influence of A on $\phi(\eta)$, where $\gamma = Sc = \xi = 0.6$,$\beta = 0.1$,$M = 0.5$,$Nb = 0.3$, $Nt = k = 0.4$, $Pr = 1$.

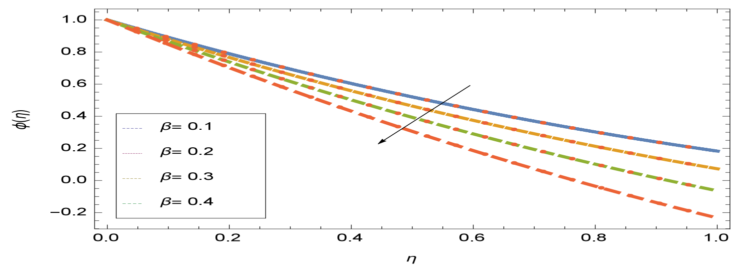

FIGURE 17. Influence of β on $\phi(\eta)$, where $\gamma = 1.9$,$Sc = 0.3$,$\xi = 1.1$,$M = 0.2$, $Nb = 0.5$, $A = 1.2$, $Nt = 0.5$, $k = Pr = 0.1$.

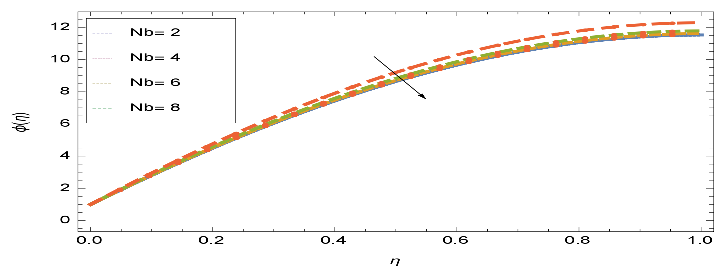

FIGURE 18. Influence of Nb on $\phi(\eta)$, where $\gamma = Sc = \xi = 0.6$,$\beta = 1$,$M = 0.1$,$A = 0.5$, $Nt = k = 0.4$, $Pr = 1$.

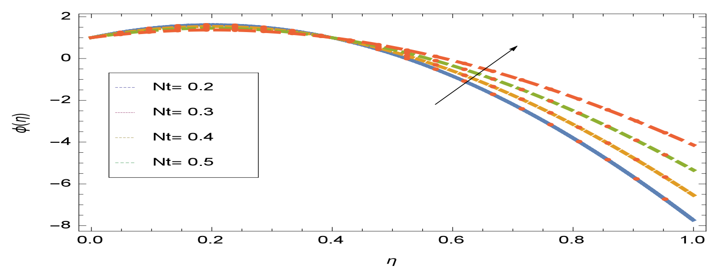

FIGURE 19. Influence of Nt on $\phi(\eta)$, where $\gamma = Sc = \xi = 0.6$, $\beta = 0.2$,$M = 0.1$, $Nb = A = 0.5$, $k = 0.5$, $Pr = 1$.

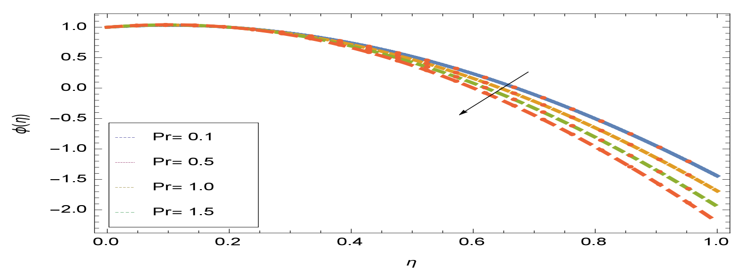

FIGURE 20. Influence of Pr on $\phi(\eta)$, where $\gamma = Sc = A = \xi = 0.6$,$\beta = 0.1$,$M = Nb = Nt = 0.5$, $k = 0.4$.

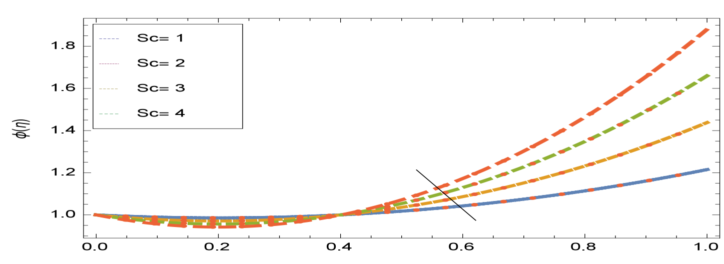

FIGURE 21. Influence of Sc on $\phi(\eta)$, where $\gamma = \xi = 0.6$, $\beta = 0.2$, $M = 0.1$,$Nb = A = 0.5$, $Nt = k = 0.4$, $Pr = 1$.

5. Tables Discussion

This section is about the discussion of tables. Table.1 displays numerical values of HAM solutions at different approximation using different values for various parameters. Clearly the table values shows that homotopy analysis technique is a quickly convergent technique. Table quantities such as film thickness β, skin friction co-efficient $f''(0)$, heat flux $Nu = -\theta'(0)$ and mass flux $Sh = -\phi'(0)$ for engineering interest are calculated from Tables 2,3 and 4. In Table.2 values of skin friction co-efficient $f''(0)$ and thin film thickness β are determined using increasing values of A. It is analyzed that skin friction co-efficient increases and the thin film thickness reduces randomly with increasing values of A. In table.3 the effect of M, Nt, A and Pr on wall temperature is calculated taking $A = 0.8$ the large value of M and Nt increase the wall temperature while the large value of A and Pr reduces the wall temperature. Table.4 examines the effect of embedding parameters Nb, β, Pr and Nt on the heat flux $Nu = -\theta'(0)$ and mass flux $Sh = -\phi'(0)$. It has been seen that the increasing values of Nb, β and Pr decreases mass flux while the increasing values of Nt increases mass flux. It has also been seen that the increasing values of embedded parameters randomly varies heat flux. The current results of $-\theta'(0)$ and $-\phi'(0)$ having a resemblance in appearance.

Table 1. Convergence of $f''(0)$, $\theta'(0)$ and $\phi'(0)$ by HAM Method when $Nt = Nb = 0.3$, $A = Sc = k = \xi = \gamma = \beta = 0.1$, $Pr = M = 1$.

Solution Approximation	$f''(0)$	$\theta'(0)$	$\phi'(0)$
1	-0.05401	-0.10070	-1.10075
4	-0.10218	-0.18890	-1.38506
7	-0.10813	-0.19903	-1.88867
10	-0.10888	-0.20281	-1.99293
13	-0.10894	-0.20154	-2.01113
14	-0.10896	-0.20475	-2.01406
17	-0.10897	-0.20478	-2.01451
20	-0.10897	-0.20479	-2.01458
25	-0.10897	-0.20479	-2.01458

Table 2. Results of skin friction coefficient $f''(0)$ and film thickness β for various values of A.

A	Present Results $f''(0)$	Present Results β
0.4	-4.33027	5.523451
0.6	-3.94882	4.002111
0.8	-2.64208	3.992358
1.0	-1.33999	3.113001
1.2	-0.92157	1.625391
1.4	-0.56897	1.896541
1.6	-0.34227	0.876512
1.8	-0.03027	0.266156

Table 3. The wall temperature for dissimilar values of M, A, Pr and Nt when $A = 0.8$,$Nb = 0.4$,$\beta = 0.9$,$Nb = 1.6$

M	Nt	A	Pr	Present (2017) Results $\theta(\beta)$
0	0.1	1.0	0.1	0.223456
1				0.432111
2	0.01			0.712351
	0.1	0.0		1.023001
		1.0	0.1	1.625341
			0.2	1.236540

Table 4. The Nusslet number $\phi'(0)$ and Sherwood numbers $\theta'(0)$ verses various value of embedded parameters when $A = 0.8$.

Nb	β	Pr	Nt	$-\theta'(0)$ Present Results	$-\phi'(0)$ Present Results
0.0	0.2	1.0	0.1	0.682385	6.68238
0.5				0.541422	4.94142
1.0				0.440569	5.44569
	0.2			0.321022	5.12101
	0.3			0.300420	5.70742
	0.4			0.291420	5.29140
		0.5		0.371420	5.37143
		1.5		0.182285	6.78223
		5.0		0.011422	7.01147
			0.4	0.612427	4.11207
			0.6	0.691428	4.69458
			0.8	0.500987	7.50097

6. DISCUSSION:

The present work focuses on the comprehension of the fluid film motion through modeled parameters.The graphical explanation of these parameters have been illustrated in figures [4-21] while Fig.2 is the combine h curve graph of velocity and temperature profiles and Fig.3 is the h curve graph of concentration profile.Both the graphs reflect valid region which gaurantees that homotopy analysis method is fast convergent technique.The influence of unsteady constraint A on the $\widehat{f}(\eta)$ field shown in Fig.4.The velocity profile $\widehat{f}(\eta)$ rises with the rise in A. Velocity increases with the unsteady constraint A.The influance of film thickness β has been displayed for unlike values of fluid velocity in Fig.5.It is observed that $\widehat{f}(\eta)$ falls over with greater values of β.The impact of stretching parameter γ on the $\widehat{f}(\eta)$ has been shown in Fig.6. It has been seen that $\widehat{f}(\eta)$ decreases with the growing values of stretching parameter.The effect of Erying fluid factor k over the $\widehat{f}(\eta)$ is exposed in Fig.7.It has been seen, when Erying fluid parameter k increases then it increases the nanofluid film motion, and this influance is clear at the stretching surface.The characteristics of magnetic factor M on fluid velocity and heat profile are shown in Figs.8 and 11.It is obvious from mathematical formulation that M is inversely varied with velocity flied $\widehat{f}(\eta)$.Increasing M decreases the velocity field.This effect of magnetic field is caused by the production of friction force to the movement known as Lorentz force which bring retardation to the flow of the fluid and hence decreases fluid velocity at the upper surface.It is clear from Fig.16 that all the fluids reflect the similar reaction to the unsteady parameter.The slope in the temperature distribution falls down with decreasing the width of thermal boundary layers.It means that unsteady parameter A has inverse effect on the temperature field. Fig.9. shows that heat profile decreases with the parameter A.Every fluid observes the same effect on temperature profile for parameter A.Actually, the fluid produces resistance to the flow of film and shows a tendency to decay the velocity of fluid flow having greater values of β and it is obvious in Fig.11. The fluid film size absorbs heat that causes falls down in heat distribution.The free surface temperature is increased with the Brownian motion constraint as illustrated in Fig.12.The reality is that the random movement of molecules of fluid produces collisions among the molecules.Increase in the value of Brownian motion parameter Nb, results increase in temperature of the fluid, consequently, it causes reduction in free surface nanoparticle volume fraction and is shown in Fig.18.The thermophoresis parameter Nt decreases as the temperature profile increases and it can be seen in Fig.13.The thermophoresis constraint is responsible for raise in surface temperature.The random motion of nano particles of fluid produces Brownian motion.This irregular motion of nano suspended particles is responsible for kinetic energy and it causes rise in temperature.Consequently, thermophoretic force is produced.This force produces intensity in the fluid to move away from the surface of the stretching sheet.Subsequently, the temperature inside the boundary layer rises as Nt increases.Physically, the Prandtl number is the ratio of kinematic viscidness to thermal diffusivity and is a dimensionless quantity. The value of Pr is increased if the value of thermal diffusivity is less than the momentum diffusivity.Therefore, the heat transmission at the exterior increases as the values of Pr increase while mass transmission reduces as the Prandtl number increases.The influance of Pr is shown in the Fig.14.It obviously shows that $\widehat{\theta}(\eta)$ reduces with large Pr number.The logic behind is that the large value of Prandtl number reduces thermal layer of the boundary.The consequences are more prominent for minor Prandtl quantity because of relatively greater width of thermal boundary layer.The heat distribution $\widehat{\theta}(\eta)$ increases with the change in the Schmidt number and is shown in Fig.15, and the non-dimensional concentration profile reduces with dissimilar measures of parameter Sc displayed in Fig.21.It is visible that a flow part increases in the horizontal direction by rising the Schmidt number.It clearly reflects that with rise in the Schmidt number, the flow part increases in the x-direction.The logic is that the Sc parameter is the ratio of momentum to the concentration diffusivities.The growth in Sc reduces width of fluid and causes fall down in $\widehat{\theta}(\eta)$.The viscidness dissipation effect on the nanoparticle volume fraction is insignificant for higher quantities of Schmidt numbers.The concentration of the fluid $\widehat{\phi}(\eta)$ increases as values of β grows as presented by Fig.17.The logic behind is that width of fluid film exhibit direct proportion with thermal conductivity as well as with viscosity.As thermophoresis parameter Nt increases, concentration field elevates.Just like surface temperature, thermophoresis factor also help in increasing the exterior of nanoparticle volume fraction and is illustrated in Fig.19.The increasing values of Nt decrease the surface mass transfer rate in both steady and unsteady cases, but shows high mass transfer rate on external edge in unsteady case as compared to steady one.Concentration profile shows the inverse relation with Pr number as shown in Fig.20.It means thinning of thermal boundary layer increases flow in the x-direction, which is clearly exibited in the graph.

7. Concluding Remarks:

This research work analyzes two dimensional nanofluid film flow of Eyring Powell Fluid with variable heat transmission over a porous stretching sheet in the existence of uniform magnetic flied (MHD). The observation of this work depends upon the influence of variable temperature and magnetic field on nanoliquid film flows. The influence of the skin fraction, Nusslet number and Sherwood number is shown numerically.
The key points of this work are as under:

- The increasing values of Pr increases the surface temperature, where opposite effect is found for unsteady parameter A that is the large values of A reduces the surface temperature.
- Non-dimensional velocity decline in variable viscosity and magnetic parameter.
- The temperature and concentration profile both are directly proportional with magnetic field.
- It is perceived that the large value of Magnetic parameter drops the velocity distribution of the nanofluid films.
- The larger values of Brownian motion parameter rises the profile of temperature.
- Thermal boundary layer thickness reduces with rise of Sc, Nusselt number rises with rise in Prandtl number.
- Porosity parameter decrease the motion of the liquid films.
- It is observed that the temperature profile falls with the large numbers of thermophoresis parameter Nt and increases for small values.
- Augmenting the nanoparticle concentration efficiently increases the friction feature of Eyring nanofluid.
- The increasing values of Nb reduces the mass flux, where Nt increases the mass flux. The higher values of Re reduces the mass flux, while it rises with rising values of Sc.
- The convergence of the HAM method with the variation of the physical parameters observed numerically.

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9. AUTHORCONTRIBUTIONS

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10. COMPETING INTERESTS

Authors have declared that no competing interests exist.

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