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A family of $2n$-point approximating subdivision schemes based on least squares method

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Abstract. In this paper, a procedure to construct a family of $2n$-point approximating subdivision schemes is presented for an integer $n \geqslant 4$. Firstly, a least squares technique has been used to fit a polynomial of degree seven to data. Secondly, a family of $2n$-point subdivision scheme is constructed. In particular, some important features of first three members of the family of $2n$-point schemes are also discussed. Geometric performances of some members of the family are shown with the help of several examples.

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1. Introduction

The method of subdivision scheme is one of the important technique to fit a curve to set of data points. The one kind of subdivision scheme is approximating scheme and the other one is interpolating. In [4], Dyn et al. studied a family of 4-point schemes with tension parameter. In [5], Dubuc and Deslauriers presented a family of $2N$-point schemes of different arity. After that, Mustafa and Rehman [14] offered the common families of (2b+4)-point n-ary scheme. Mustafa et al. [15] presented generalized and unified families of interpolating subdivision schemes. In 2012, Hormann [9] presented detail notes on the analysis of subdivision schemes. In 2018, Asghar and Mustafa [2] presented univariate stationary and non-stationary subdivision schemes and their analysis. The analysis of the subdivision schemes has also been presented by Han in [10, 11, 12, 13]. All above subdivision schemes have been introduced by using different techniques.

Dyn et al. [6] and Mustafa et al. [16] introduced subdivision schemes based on least squares method with kernel weight and iterative re-weighted least squares techniques to fit noisy data with outliers respectively. The least squares techniques have also been used by [1, 17, 18, 19, 20]. In this paper, we also use least squares regression to introduce a family of schemes.

The rest of the structure of this paper is: In Section 2, a family of $2n$-point approximating subdivision schemes is introduced. In Section 3, we present the analysis of the first three members of the family of $2n$-point subdivision schemes. Applications of the proposed schemes are presented in Section 4.

2. A FAMILY OF $2n$-POINT APPROXIMATING SCHEMES

In this section, firstly we find the most suitable fitted polynomial to the data by using least squares methods. Secondly, we introduce a family of $2n$-point subdivision schemes for fitting curves. In our procedure, we use the observations (r, f_r) for $r = -n+1, \ldots n$, where $n \geq 4$, to find the best fitted polynomial of degree 7

$$f(r) = \eta_0 + \eta_1 r + \eta_2 r^2 + \eta_3 r^3 + \eta_4 r^4 + \eta_5 r^5 + \eta_6 r^6 + \eta_7 r^7. \qquad (2.\ 1)$$

Now the task is to find unknowns in (2. 1) to get minimum value of the sum of squares of residuals S, which is defined below

$$S = \sum_{r=-n+1}^{n} [f_r - (\eta_0 + \eta_1 r + \eta_2 r^2 + \eta_3 r^3 + \eta_4 r^4 + \eta_5 r^5 + \eta_6 r^6 + \eta_7 r^7)]^2. \qquad (2.\ 2)$$

Now take the derivatives of S with respect to η_0 - η_7 and set them to zero to get minimum value of S. In this way, we get a system of eight normal linear equations with eight unknowns. The solution of this system will give the values of these eight unknowns. These unknowns are given in (4. 16)-(4. 24) of Appendix A.

Now by putting the value of $r = \frac{1}{4}$ in (2. 1) and then substituting the values of η_0 - η_7 from (4. 16)-(4. 24) of Appendix A, we get the first refinement rule of the $2n$-point scheme,

$$f\left(\frac{1}{4}\right) = \frac{1}{28} \sum_{r=-n+1}^{n} \rho_n \alpha_{r,n} f_r, \qquad (2.\ 3)$$

where

$$\rho_n = \frac{35}{65536}[n(256n^{14} - 8960n^{12} + 119392n^{10} - 766480n^8 + 2475473n^6 - 3822910n^4 + 2400129n^2 - 396900)]^{-1}, \qquad (2.\ 4)$$

and

$$\begin{aligned}\alpha_{r,n} &= 14204142810r^5n^2 - 8094202690n^4 + 2968245315n^2 - 3296011338r\\ &-7473803337r^5 - 14057472r^6n^8 + 936359424r^5n^6 - 141672960r^3n^8\\ &+6757792713r^2 + 625641120rn^8 - 52715520r^5n^8 + 6857605475n^6\\ &+1146880n^{14} + 22708224r^4n^{10} - 2520756280n^8 + 2034248832r^4n^6\\ &-9266839791r^4n^2 + 2580480rn^{12} + 1533268737r^6 - 12431058255r^3\\ &+1854721440r^7n^2 + 52715520r^7n^6 + 2305224567rn^4 - 11570812533\\ &\times r^2n^4 - 908454690r^7 - 401111040r^4n^8 - 649059840r^7n^4 - 10321920\\ &\times r^2n^{12} + 129153024r^6n^6 - 69318144rn^{10} - 2243781306rn^6 -\\ &5912498592r^5n^4 + 19598758146r^3n^2 - 36915200n^{12} - 1417739400r^4n^4\\ &+206414208r^6n^4 - 1431702744r^2n^2 - 2352759168r^2n^8 + 8765596728r^2\\ &\times n^6 - 8643698910r^3n^4 + 7975771650r^4 + 445406080n^{10} + 262047744r^2\\ &\times n^{10} + 1645631361rn^2 + 1653065568r^3n^6 + 5677056r^3n^{10} - 2284102392\\ &\times r^6n^2 + 379470420. \qquad (2.\ 5)\end{aligned}$$

Similarly, by putting the value of $r = \frac{3}{4}$ in (2. 1) and then substituting the values of η_0 - η_7 in (2. 1), we get the 2nd refinement rule of the $2n$-point scheme,

$$f\left(\frac{3}{4}\right) = \frac{1}{28}\sum_{r=-n+1}^{n}\rho_n\beta_{r,n}f_r, \qquad (2.\ 6)$$

where

$$\begin{aligned}\beta_{r,n} &= -39448678698r^5n^2 - 33776373190n^4 + 27288854145n^2 + 29699101170r +\\ &17351739405r^5 - 14057472r^6n^8 - 2818303488r^5n^6 + 2554421760r^3n^8 -\\ &53497302957r^2 + 6457262688rn^8 + 137060352r^5n^8 + 18184963265n^6 +\\ &1146880n^{14} + 22708224r^4n^{10} - 4857431320n^8 + 10498384512r^4n^6 +\\ &92407588779r^4n^2 + 18063360rn^{12} - 4825914093r^6 + 56396544435r^3 -\\ &1854721440r^7n^2 - 52715520r^7n^6 + 85305881421rn^4 - 115667375103r^2n^4 +\\ &908454690r^7 - 875550720r^4n^8 + 649059840r^7n^4 - 10321920r^2n^{12} +\\ &498161664r^6n^6 - 562641408rn^{10} - 34209328086rn^6 + 18304269984r^5n^4 -\\ &143806029642r^3n^2 - 44656640n^{12} - 50601113640r^4n^4 - 4337004672r^6n^4 +\\ &148492575408r^2n^2 - 5922461568r^2n^8 + 38338201944r^2n^6 + 92028452670r^3\\ &\times n^4 - 38190128130r^4 + 666520960n^{10} + 415328256r^2n^{10} - 90810290925rn^2\\ &-23581758816r^3n^6 - 7463024100 - 96509952r^3n^{10} + 10698947688r^6n^2, \qquad (2.\ 7)\end{aligned}$$

and ρ_n is defined in (2. 4). Now by using the the notations $f^{k+1}_{2i,2n} = f\left(\frac{1}{4}\right)$, $f^{k+1}_{2i+1,2n} = f\left(\frac{3}{4}\right)$ and $f^k_{i+r,2n} = f_r$, we get the $2n$-point binary subdivision scheme defined below

$$\begin{cases} f^{k+1}_{2i,2n} = \frac{1}{2^8}\sum_{r=-n+1}^{n} \rho_n \alpha_{r,n} f^k_{i+r,2n}, \\ f^{k+1}_{2i+1,2n} = \frac{1}{2^8}\sum_{r=-n+1}^{n} \rho_n \beta_{r,n} f^k_{i+r,2n}, \end{cases} \tag{2. 8}$$

where $f^{k+1}_{2i,2n}$ and $f^{k+1}_{2i+1,2n}$ represent even and odd refinement rules of the $2n$-point binary scheme respectively.

For $n = 4$, (2. 8) give the 8-point dual approximating subdivision scheme define below

$$\begin{cases} f^{k+1}_{2i,8} = -\frac{495}{262144} f^k_{i-3,8} + \frac{5005}{262144} f^k_{i-2,8} - \frac{27027}{262144} f^k_{i-1,8} + \frac{225225}{262144} f^k_{i,8} \\ \quad + \frac{75075}{262144} f^k_{i+1,8} - \frac{19305}{262144} f^k_{i+2,8} + \frac{4095}{262144} f^k_{i+3,8} - \frac{429}{262144} f^k_{i+4,8}, \\ f^{k+1}_{2i+1,8} = -\frac{429}{262144} f^k_{i-3,8} + \frac{4095}{262144} f^k_{i-2,8} - \frac{19305}{262144} f^k_{i-1,8} + \frac{75075}{262144} f^k_{i,8} \\ \quad + \frac{225225}{262144} f^k_{i+1,8} - \frac{27027}{262144} f^k_{i+2,8} + \frac{5005}{262144} f^k_{i+3,8} - \frac{495}{262144} f^k_{i+4,8}. \end{cases} \tag{2. 9}$$

where $f^{k+1}_{i,2n}$ and $f^k_{i,2n}$ are control points of $2n$-point scheme at $(k+1)$-th and k-th level of iterations.

In the same manner, if we substitute $n = 5$ and $n = 6$ in (2. 8), we get 10-point and 12-point dual approximating schemes define in (2. 10) and (2. 11) respectively.

$$\begin{cases} f^{k+1}_{2i,10} = A_1 f^k_{i-4,10} + A_2 f^k_{i-3,10} + A_3 f^k_{i-2,10} + A_4 f^k_{i-1,10} + A_5 f^k_{i,10} \\ \quad + A_6 f^k_{i+1,10} + A_7 f^k_{i+2,10} + A_8 f^k_{i+3,10} + A_9 f^k_{i+4,10} + A_{10} f^k_{i+5,10}, \\ f^{k+1}_{2i+1,10} = A_{10} f^k_{i-4,10} + A_9 f^k_{i-3,10} + A_8 f^k_{i-2,10} + A_7 f^k_{i-1,10} + A_6 f^k_{i,10} \\ \quad + A_5 f^k_{i+1,10} + A_4 f^k_{i+2,10} + A_3 f^k_{i+3,10} + A_2 f^k_{i+4,10} + A_1 f^k_{i+5,10}, \end{cases} \tag{2. 10}$$

where $A_1 = -\frac{11633029}{1274544128}$, $A_2 = \frac{414723467}{6372720640}$, $A_3 = -\frac{46093791}{245104640}$, $A_4 = \frac{760211473}{3186360320}$, $A_5 = \frac{445407151}{796590080}$, $A_6 = \frac{59570609}{159318016}$, $A_7 = \frac{30834499}{3186360320}$, $A_8 = -\frac{23405827}{289669120}$, $A_9 = \frac{239919809}{6372720640}$ and $A_{10} = -\frac{38564243}{6372720640}$.

$$\begin{cases} f^{k+1}_{2i,12} = B_1 f^k_{i-5,12} + B_2 f^k_{i-4,12} + B_3 f^k_{i-3,12} + B_4 f^k_{i-2,12} + B_5 f^k_{i-1,12} \\ \quad + B_6 f^k_{i,12} + B_7 f^k_{i+1,12} + B_8 f^k_{i+2,12} + B_9 f^k_{i+3,12} + B_{10} f^k_{i+4,12} \\ \quad + B_{11} f^k_{i+5,12} + B_{12} f^k_{i+6,12}, \\ f^{k+1}_{2i+1,12} = B_{12} f^k_{i-5,12} + B_{11} f^k_{i-4,12} + B_{10} f^k_{i-3,12} + B_9 f^k_{i-2,12} + B_8 f^k_{i-1,12} \\ \quad + B_7 f^k_{i,12} + B_6 f^k_{i+1,12} + B_5 f^k_{i+2,12} + B_4 f^k_{i+3,12} + B_3 f^k_{i+4,12} \\ \quad + B_2 f^k_{i+5,12} + B_1 f^k_{i+6,12}, \end{cases} \tag{2. 11}$$

where $B_1 = -\frac{105897143}{6604455936}$, $B_2 = \frac{6178576919}{72649015296}$, $B_3 = -\frac{10229053873}{72649015296}$, $B_4 = -\frac{1289378429}{72649015296}$, $B_5 = \frac{5051545633}{18162253824}$, $B_6 = \frac{7967274805}{18162253824}$, $B_7 = \frac{482662915}{1397096448}$, $B_8 = \frac{1968681907}{18162253824}$, $B_9 = -\frac{429163981}{6604455936}$, $B_{10} = -\frac{3943530931}{72649015296}$, $B_{11} = \frac{3536467133}{72649015296}$ and $B_{12} = -\frac{69715829}{6604455936}$.

Similarly for other values of n, we get other subdivision schemes with complexity $2n$.

3. Analysis of the family of $2n$-point schemes

In this section, we give the analysis of the proposed 8-point, 10-point and 12-point approximating subdivision schemes. We use the methodologies of [7] and [3] for the analysis of proposed schemes.

Property 3.1. *The family of $2n$-point approximating subdivision schemes satisfies the necessary condition for convergence.*

Proof. The family of schemes (2. 8) satisfies necessary condition for convergence if sums of the even and odd mask coefficients are equal to one. From (2. 8), we calculate that

$$\frac{1}{28}\sum_{r=-n+1}^{n}\rho_n\alpha_{r,n} = \frac{1}{28}\sum_{r=-n+1}^{n}\rho_n\beta_{r,n} = 1.$$

This completes the proof. □

We denote the Laurent polynomial and mask of the proposed $2n$-point schemes by $a_{2n}(z)$ and a_{2n} respectively.

Remark 3.2. *The 8-point approximating subdivision scheme (2. 9) is same as the 8-point dual scheme constructed by the Lagrange's interpolatory polynomial and presented in* [8]. *This scheme is also presented in* [10]. *The L_2 smoothness of this 8-point scheme is 4.5 and therefore by Sobolev Imbedding Theorem given in* [21], *the scheme is C^3-continuous. In other words, for one-dimensional scheme, if its L_2 smoothness is t, then its L_∞ smoothness is at least $t - 0.5$ and is at most t. Hence, the scheme is $C^{4-\epsilon}$ for all $\epsilon > 0$.*

Property 3.3. *The proposed 10-point scheme (2. 10) corresponding to the Laurent polynomial $a_{10}(z)$ is C^2-continuous.*

Proof. The Laurent polynomial of subdivision scheme (2. 10) is

$$a_{10}(z) = \left(\frac{1+z}{2z}\right) b_{10}(z),$$

where

$$\begin{aligned} b_{10}(z) = \ & \frac{1}{3186360320}[-38564243z^{-9} - 19600902z^{-8} + 259520711z^{-7} \\ & +155202756z^{-6} - 670130950z^{-5} - 528307616z^{-4} + 589976614z^{-3} \\ & +930446332z^{-2} + 1452378028z^{-1} + 2110879180 + 1452378028z \\ & +930446332z^{2} + 589976614z^{3} - 528307616z^{4} - 670130950z^{5} + \\ & 155202756z^{6} + 259520711z^{7} - 19600902z^{8} - 38564243z^{9}]. \end{aligned}$$

Let $S_{b_{10}}$ be the subdivision scheme corresponding to the Laurent polynomial $b_{10}(z)$. Then we find

$$c_{10}(z) = \left(\frac{1}{1+z}\right) b_{10}(z),$$

which is the Laurent polynomial of the difference scheme of the scheme $S_{b_{10}}$

$$\begin{aligned} c_{10}(z) \;=\; & \frac{1}{3186360320}[-38564243z^{-9} + 18963341z^{-8} + 240557370z^{-7} - 85354614 \\ & \times z^{-6} - 584776336z^{-5} + 56468720z^{-4} + 533507894z^{-3} + 396938438z^{-2} \\ & +1055439590z^{-1} + 1055439590 + 396938438z + 533507894z^{2} + 56468720z^{3} \\ & -584776336z^{4} - 85354614z^{5} + 240557370z^{6} + 18963341z^{7} - 38564243z^{8}]. \end{aligned}$$

The norm of difference scheme $S_{c_{10}}$ corresponding to the Laurent polynomial $c_{10}(z)$ is

$$\|S_{c_{10}}\|_\infty \;=\; \max\left\{\frac{1505285273}{1593180160}, \frac{1505285273}{1593180160}\right\} < 1.$$

Therefore by [7], the schemes $S_{c_{10}}$, $S_{b_{10}}$ and $S_{a_{10}}$ are contractive, convergent and C^1-continuous respectively. Again the Laurent polynomial of scheme (2. 10) can be factorize as

$$a_{10}(z) \;=\; \left(\frac{1+z}{2z}\right)^2 d_{10}(z)$$

where

$$\begin{aligned} d_{10}(z) \;=\; & \frac{1}{1593180160}[-38564243z^{-8} + 18963341z^{-7} + 240557370z^{-6} - 85354614 \\ & \times z^{-5} - 584776336z^{-4} + 56468720z^{-3} + 533507894z^{-2} + 396938438z^{-1} + \\ & 1055439590 + 1055439590z + 396938438z^{2} + 533507894z^{3} + 56468720z^{4} \\ & -584776336z^{5} - 85354614z^{6} + 240557370z^{7} + 18963341z^{8} - 38564243z^{9}]. \end{aligned}$$

Let $S_{d_{10}}$ be the scheme corresponding to $d_{10}(z)$ then we get the following polynomial $e_{10}(z)$ (Laurent polynomial of the difference scheme of the scheme $S_{d_{10}}$) by polynomial $d_{10}(z)$, i.e.

$$e_{10}(z) \;=\; \left(\frac{1}{1+z}\right) d_{10}(z).$$

Which implies that

$$\begin{aligned} e_{10}(z) \;=\; & \frac{1}{1593180160}[-38564243z^{-8} + 57527584z^{-7} + 183029786z^{-6} - 268384400 \\ & \times z^{-5} - 316391936z^{-4} + 372860656z^{-3} + 160647238z^{-2} + 236291200z^{-1} \\ & +819148390 + 236291200z + 160647238z^{2} + 372860656z^{3} - 316391936z^{4} \\ & -268384400z^{5} + 183029786z^{6} + 57527584z^{7} - 38564243z^{8}]. \end{aligned}$$

The norm of difference scheme $S_{e_{10}}$ corresponding to the Laurent polynomial $e_{10}(z)$ is

$$\|S_{e_{10}}\|_\infty \;= \max\left\{\tfrac{554103699}{398295040}, \tfrac{5844149}{4978688}\right\} > 1.$$

Therefore by using [7], we calculate $e_{10}^2(z)$, i.e.

$$e_{10}^2(z) = e_{10}(z)e_{10}(z^2).$$

$$\|S_{e^2_{10}}\|_\infty = \max\left\{\frac{26606284863239269}{31727787777720320}, \frac{275237708664251}{495746684026880}, \frac{603267010679043099}{634555755554406400}, \frac{275237708664251}{495746684026880}\right\} < 1.$$

Hence by [7] the schemes $S_{e^2_{10}}$, $S_{d_{10}}$ and $S_{a_{10}}$ are contractive, convergent and C^2-continuous respectively. The Laurent polynomial corresponding to the scheme (2. 10) can be factorize as

$$a_{10}(z) = \left(\frac{1+z}{2z}\right)^3 f_{10}(z),$$

where

$$f_{10}(z) = \frac{1}{796590080}[-38564243z^{-7} + 57527584z^{-6} + 183029786z^{-5} - 268384400 \times z^{-4} - 316391936z^{-3} + 372860656z^{-2} + 160647238z^{-1} + 236291200 + 819148390z + 236291200z^2 + 160647238z^3 + 372860656z^4 - 316391936z^5 - 268384400z^6 + 183029786z^7 + 57527584z^8 - 38564243z^9].$$

Now we calculate

$$g_{10}(z) = \left(\frac{1}{1+z}\right) f_{10}(z),$$

which gives

$$g_{10}(z) = \frac{1}{796590080}[-38564243z^{-7} + 96091827z^{-6} + 86937959z^{-5} - 355322359 \times z^{-4} + 38930423z^{-3} + 333930233z^{-2} - 173282995z^{-1} + 409574195 + 409574195z - 173282995z^2 + 333930233z^3 + 38930423z^4 - 355322359z^5 + 86937959z^6 + 96091827z^7 - 38564243z^8].$$

Now the norm of difference scheme $S_{g_{10}}$ corresponding to the Laurent polynomial $g_{10}(z)$ is

$$\|S_{g_{10}}\|_\infty = \max\left\{\tfrac{766317117}{398295040}, \tfrac{766317117}{398295040}\right\} = 1.92399 > 1.$$

Hence by using [7], we calculate $g^2_{10}(z)$, i.e.

$$g^2_{10}(z) = g_{10}(z)g_{10}(z^2).$$

Which gives

$$\|S_{g^2_{10}}\|_\infty = \max\left\{\frac{19150586310494659}{9331702287564800}, \frac{19150586310494659}{9331702287564800}, \frac{12019153184701607}{4957466840268800}, \frac{12019153184701607}{4957466840268800}\right\} = 2.42445.$$

Again by using [7], we calculate $g^3_{10}(z)$, i.e.

$$g^3_{10}(z) = g_{10}(z)g_{10}(z^2)g_{10}(z^4).$$

Which gives

$$\begin{aligned}\|S_{g_{10}^3}\|_\infty &= \max\left\{\frac{4461552628674598867092292 3}{15796275627548282454016000},\frac{44615526286745988670922923}{15796275627548282454016000}\right.\\&,\frac{42564658424384377012816265 9}{12637020502038625963212800 0},\frac{425646584243843770128162659}{126370205020386259632128000}\\&,\frac{364977927604507610473078109}{126370205020386259632128000},\frac{364977927604507610473078109}{126370205020386259632128000}\\&\left.,\frac{833160932909087941469896093}{252740410040772519264256000},\frac{833160932909087941469896093}{252740410040772519264256000}\right\}\\&= 3.36825.\end{aligned}$$

Now by using same method, we calculate $\|S_{g_{10}^4}\|_\infty = 4.42648$, $\|S_{g_{10}^5}\|_\infty = 5.87366$, $\|S_{g_{10}^6}\|_\infty = 7.82012$, $\|S_{g_{10}^7}\|_\infty = 10.3532$, $\|S_{g_{10}^8}\|_\infty = 13.752$, $\|S_{g_{10}^9}\|_\infty = 18.2318$ and $\|S_{g_{10}^{10}}\|_\infty = 24.2006$ where $g_{10}^L(z) = g_{10}(z)g_{10}(z^2)\dots g_{10}(z^{2^{L-1}})$. Hence $\|S_{g_{10}^L}\|_\infty \to \infty$ as $L \to \infty$. Which means that $\|S_{g_{10}^L}\|_\infty$ does not converge to a positive real number less than one. Therefore, by [7] the schemes $S_{g_{10}^L}$, $S_{f_{10}}$ and $S_{a_{10}}$ are not contractive, not convergent and not C^3-continuous respectively. This completes the proof. □

Remark 3.4. *The L_2 smoothness of the proposed 10-point scheme is 2.6015 and hence it is at least $C^{2.1015}$ but not $C^{2.6015}$.*

Property 3.5. *The proposed 12-point scheme (2. 11) corresponding to the Laurent polynomial $a_{12}(z)$ is C^2-continuous.*

Proof. The proof of this property is similar to that of Property 3.3. □

Remark 3.6. *The L_2 smoothness of the proposed 12-point scheme is 3.116305 and hence its L_∞ smoothness is at least $C^{2.61163}$ but not $C^{3.11635}$.*

Property 3.7. *The 8-point, 10-point and 12-point schemes defined in (2. 9), (2. 10) and (2. 11) respectively generate polynomials up to degree 8.*

Proof. Since Laurent polynomial of scheme (2. 9) can be written as

$$a_8(z) = \left(\frac{1+z}{2}\right)^{8+1} b(z), \tag{3. 12}$$

where

$$b(z) = \frac{z^{-8}}{512}[-429 + 3366z - 10755z^2 + 16660z^3 - 10755z^4 + 3366z^5 - 429z^6].$$

Similarly, the Laurent polynomial of 10-point scheme (2. 10) can be written as

$$a_{10}(z) = \left(\frac{1+z}{2}\right)^{8+1} c(z), \tag{3. 13}$$

where

$$c(z) = -\frac{z^{-10}}{12446720}[38564243 - 288913042z + 971984821z^2 - 2001113756z^3 +2943099352z^4 - 3352136676z^5 + 2943099352z^6 - 2001113756z^7 +971984821z^8 - 288913042z^9 + 38564243z^{10}].$$

In the same manner, the Laurent polynomial of 12-point scheme (2. 11) can also be written as

$$a_{12}(z) = \left(\frac{1+z}{2}\right)^{8+1} d(z), \quad (3.\ 14)$$

where

$$d(z) = -\frac{z^{-12}}{141892608}[766874119 - 5736998498z + 20489051065z^2 - 48465516572z^3 +87809076577z^4 - 130138656434z^5 + 162769297535z^6 - 175270040800z^7 +162769297535z^8 - 130138656434z^9 + 87809076577z^{10} - 48465516572z^{11} +20489051065z^{12} - 5736998498z^{13} + 766874119z^{14}].$$

Hence the Laurent polynomial of all these schemes give factor $(1 + z)^9$. So by [3], these schemes generate polynomials up to degree 8. This completes the proof. □

Property 3.8. *The* 8*-point,* 10*-point and* 12*-point schemes defined in (2. 9), (2. 10) and (2. 11) respectively are dual subdivision schemes.*

Proof. By [3], a subdivision scheme is a dual subdivision scheme if its Laurent polynomial satisfy the following condition

$$za(z) = a(z^{-1}).$$

The Laurent polynomials of schemes (2. 9), (2. 10) and (2. 11) which are presented in (3. 12)-(3. 14) hold $za_8(z) = a_8(z^{-1})$, $za_{10}(z) = a_{10}(z^{-1})$ and $za_{12}(z) = a_{12}(z^{-1})$ respectively. Hence these schemes are dual subdivision schemes and therefore these schemes have dual parameterizations. □

Property 3.9. *The* 8*-point,* 10*-point and* 12*-point schemes defined in (2. 9), (2. 10) and (2. 11) respectively reproduces polynomials up to degree 7.*

Proof. It is easy to calculate that $a_8^{(1)}(1) = a_{10}^{(1)}(1) = a_{12}^{(1)}(1) = 1$, where $a_8^{(1)}(1)$ is the derivative of $a_8(z)$ with respect to z and evaluated it at $z = 1$. Hence by [3], $\tau_8 = \tau_{10} = \tau_{12} = \frac{1}{2}$. Also

$$a_m^{(\kappa)}(1) = 2\prod_{l=0}^{\kappa-1}(\tau_m - l) \text{ and } a_m^{(\kappa)}(-1) = 0.$$

where $\kappa = 0, 1, ..., 7$ and $m = 8, 10, 12$.

Here $a_m^{(\kappa)}(1)$ and $a_m^{(\kappa)}(-1)$ denote the κ-th derivative of $a_m(z)$ with respect to z and evaluated at $z = 1$ and $z = -1$ respectively.

So by [3], proof is completed. □

Remark 3.10. *Since by Property 3, the* 8*-point,* 10*-point and* 12*-point subdivision schemes reproduce polynomials up to degree 7. Therefore by* [7] *the approximation order of these schemes is 8.*

Property 3.11. *The limit stencil of the 8-point subdivision scheme is*

$$[0,\ 0.000020,\ 0.000381,\ -0.012352,\ 0.099205,\ -0.461460,\ 2.245036,\ 2.245036,$$
$$-0.461460,\ 0.099205,\ -0.012352,\ 0.000381,\ 0.000020,\ 0].$$

Proof. Since the matrix representation of the scheme for i = -3, -2 , -1 , 0 , 1 , 2 , 3 is

$$f^{k+1} \quad = \quad Sf^k,$$

where

$$f^{k+1} \quad = \quad (f^{k+1}_{-6,8}\ f^{k+1}_{-5,8}\ f^{k+1}_{-4,8}\ f^{k+1}_{-3,8}\ f^{k+1}_{-2,8}\ f^{k+1}_{-1,8}\ f^{k+1}_{0,8}\ f^{k+1}_{1,8}\ f^{k+1}_{2,8}\ f^{k+1}_{3,8}\ f^{k+1}_{4,8}\ f^{k+1}_{5,8}\ f^{k+1}_{6,8}\ f^{k+1}_{7,8})^T,$$

$$f^k = (f^k_{-6,8}\ f^k_{-5,8}\ f^k_{-4,8}\ f^k_{-3,8}\ f^k_{-2,8}\ f^k_{-1,8}\ f^k_{0,8}\ f^k_{1,8}\ f^k_{2,8}\ f^k_{3,8}\ f^k_{4,8}\ f^k_{5,8}\ f^k_{6,8}\ f^k_{7,8})^T$$

and

$$S \quad = \quad \frac{1}{\xi}\begin{pmatrix}
\xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_8 & 0 & 0 & 0 & 0 & 0 & 0 \\
\xi_8 & \xi_7 & \xi_6 & \xi_5 & \xi_4 & \xi_3 & \xi_2 & \xi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_8 & 0 & 0 & 0 & 0 & 0 \\
0 & \xi_8 & \xi_7 & \xi_6 & \xi_5 & \xi_4 & \xi_3 & \xi_2 & \xi_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_8 & 0 & 0 & 0 & 0 \\
0 & 0 & \xi_8 & \xi_7 & \xi_6 & \xi_5 & \xi_4 & \xi_3 & \xi_2 & \xi_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_8 & 0 & 0 & 0 \\
0 & 0 & 0 & \xi_8 & \xi_7 & \xi_6 & \xi_5 & \xi_4 & \xi_3 & \xi_2 & \xi_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_8 & 0 & 0 \\
0 & 0 & 0 & 0 & \xi_8 & \xi_7 & \xi_6 & \xi_5 & \xi_4 & \xi_3 & \xi_2 & \xi_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi & \xi_8 & 0 \\
0 & 0 & 0 & 0 & 0 & \xi_8 & \xi_7 & \xi_6 & \xi_5 & \xi_4 & \xi_3 & \xi_2 & \xi_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_8 \\
0 & 0 & 0 & 0 & 0 & 0 & \xi_8 & \xi_7 & \xi_6 & \xi_5 & \xi_4 & \xi_3 & \xi_2 & \xi_1
\end{pmatrix},$$

with ξ=262144, ξ_1=-495, ξ_2=5005,ξ_3=-27027, ξ_4=225225, ξ_5=75075, ξ_6=-19305, ξ_7=4095, ξ_8=-429.

The following are the eigenvalues of the matrix S

$$\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = 0.25, \lambda_4 = 0.125, \lambda_5 = -0.037317, \lambda_6 = -0.028919,$$
$$\lambda_7 = 0.0625, \lambda_8 = 0.052656, \lambda_9 = 0.03125, \lambda_{10} = 0.015625, \lambda_{11} = 0.003906,$$
$$\lambda_{12} = 0.007812, \lambda_{13} = 0.010767, \lambda_{14} = 0.009993.$$

The eigenvectors corresponding to the eigenvalues are
$\gamma_1 = (0.267261, 0.267261, 0.267261, 0.267261, 0.267261, 0.267261, 0.267261,$
$0.267261, 0.267261, 0.267261, 0.267261, 0.267261, 0.267261, 0.267261)^T$,

$\gamma_2 = (-0.430946, -0.364646, -0.298347, -0.232048, -0.165748, -0.099449,$
$-0.033150, 0.033150, 0.099449, 0.165748, 0.232048, 0.298347, 0.364646, 0.430946)^T$,

$\gamma_3 = (-0.519712, -0.372102, -0.249093, -0.150686, -0.076881, -0.027677,$
$-0.003075, -0.003075, -0.027677, -0.076881, -0.150686, -0.249093, -0.372102,$
$-0.519712)^T$,

$\gamma_4 = (-0.576412, -0.349205, -0.191263, -0.089991, -0.032795, -0.007084,$
$-0.000262, 0.000262, 0.007084, 0.032795, 0.089991, 0.191263, 0.349205, 0.576412)^T$,

$\gamma_5 = (-0.653307, -0.268453, -0.014924, 0.029730, 0.004067, -0.000348,$
$-0.000083, -0.000083, -0.000348, 0.004067, 0.029730, -0.014924, -0.268453,$
$-0.653307)^T$,

$\gamma_6 = (0.632820, 0.311404, 0.046431, -0.019797, -0.004340, 0.000168, 0.000025,$
$-0.000025, -0.000168, 0.004340, 0.019797, -0.046431, -0.311404, -0.632820)^T$,

$\gamma_7 = (-0.614735, -0.315127, -0.141216, -0.051678, -0.013452, -0.001743,$
$-0.000022, -0.000022, -0.001743, -0.013452, -0.051678, -0.141216, -0.315127,$
$-0.614735)^T$,

$\gamma_8 = (0.622278, 0.306024, 0.130340, 0.044871, 0.010739, 0.001230, 0.000020,$
$-0.000020, -0.001230, -0.010739, -0.044871, -0.130340, -0.306024, -0.622278)^T$,

$\gamma_9 = (-0.641353, -0.278191, -0.101998, -0.029032, -0.005398, -0.000420,$
$-0.000002, 0.000002, 0.000420, 0.005398, 0.029032, 0.101998, 0.278191, 0.641353)^T$,

$\gamma_{10} = (-0.660117, -0.242280, -0.072680, -0.016090, -0.002137, -0.000100,$
$0, 0, -0.000100, -0.002137, -0.016090, -0.072680, -0.242280, -0.660117)^T$,

$\gamma_{11} = (-0.682982, -0.179480, -0.036048, -0.004833, -0.000334, -0.000012,$
$-0.000007, -0.000007, -0.000012, -0.000334, -0.004833, -0.036048, -0.17948,$
$-0.682982)^T$,

$\gamma_{12} = (0.673456, 0.209148, 0.051334, 0.008839, 0.000838, 0.000023, 0.000000,$
$-0.000000, -0.000023, -0.000838, -0.008839, -0.051334, -0.209148, -0.673456)^T$,

$\gamma_{13} = (0.664326, 0.234554, 0.059388, 0.011278, 0.001277, 0.000050, 0.000004,$
$0.000004, 0.000050, 0.001277, 0.011278, 0.059388, 0.234554, 0.664326)^T$,

$\gamma_{14} = (0.667459, 0.225926, 0.057781, 0.010759, 0.001162, 0.000040, -0.000000,$
$0.000000, -0.000040, -0.001162, -0.010759, -0.057781, -0.225926, -0.667459)^T$.

Since the matrix S can be diagonalize as $S = Q \wedge Q^{-1}$, therefore $S^k = Q \wedge^k Q^{-1}$, where Q is the matrix whose columns are the eigenvectors of the matrix S whereas the diagonals entries of the diagonal matrix $\wedge$ are the eigenvalues of the matrix S. Since $\wedge^k$ is a diagonal matrix therefore the diagonal entries other than first entry approaches to zero when $k \to \infty$. Since $f^{k+1} = Sf^k = ... = S^k f^0$ then $f^{k+1} = (Q \wedge^k Q^{-1})f^0$. This implies $f^\infty = Q(\lim_{k\to\infty} \wedge^k)Q^{-1}f^0$. So

$$\begin{pmatrix} f^\infty_{-6,8} \\ f^\infty_{-5,8} \\ f^\infty_{-4,8} \\ f^\infty_{-3,8} \\ f^\infty_{-2,8} \\ f^\infty_{-1,8} \\ f^\infty_{0,8} \\ f^\infty_{1,8} \\ f^\infty_{2,8} \\ f^\infty_{3,8} \\ f^\infty_{4,8} \\ f^\infty_{5,8} \\ f^\infty_{6,8} \\ f^\infty_{7,8} \end{pmatrix} = \begin{pmatrix} 0 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & 0 \\ 0 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & 0 \\ 0 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & 0 \\ 0 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & 0 \\ 0 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & 0 \\ 0 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & 0 \\ 0 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & 0 \\ 0 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & 0 \\ 0 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & 0 \\ 0 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & 0 \\ 0 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & 0 \\ 0 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & 0 \\ 0 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & 0 \\ 0 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} f^0_{-6,8} & f^0_{-5,8} & f^0_{-4,8} & f^0_{-3,8} & f^0_{-2,8} & f^0_{-1,8} & f^0_{0,8} & f^0_{1,8} & f^0_{2,8} & f^0_{3,8} & f^0_{4,8} & f^0_{5,8} & f^0_{6,8} & f^0_{7,8} \end{pmatrix}^T.$$

Hence the limit stencils is
$[0, \varsigma_2 = 0.000020, \varsigma_3 = 0.000381, \varsigma_4 = -0.012352, \varsigma_5 = 0.099205, \varsigma_6 = -0.461460,$
$\varsigma_7 = 2.245036, \varsigma_8 = 2.245036, \varsigma_9 = -0.461460, \varsigma_{10} = 0.099205, \varsigma_{11} = -0.012352,$
$\varsigma_{12} = 0.000381, \varsigma_{13} = 0.000020, 0]$.
This means, when we take the entries of limit stencil as coefficients of the initial points $f^0_{-6,8}, f^0_{-5,8}$, ,..., $f^0_{7,8}$, then by adding up these points, we get the limit position of the point $f^0_{0,8}$. □

Similarly, we can compute the limit stencils of other schemes.

4. Applications

Here we take different types of data then we fit curves by 8-point, 10-point, and 12-point approximating subdivision schemes. Results are depicted in Figures 1-3. We observe that the 8-point scheme preserve shape of the initial data as limit curves pass very close to the initial data. While 10-point and 12-point schemes do not pass very close to the initial data. Hence by Figure 3, it is easy to see that 12-point scheme is a good choice for noisy data. Moreover, for the 8-point subdivision scheme the points on the boundary are calculated by substituting $f^k_{-m,8} = 2f^k_{0,8} - f^k_{m,8}$ and $f^k_{n+m,8} = 2f^k_{n,8} - f^k_{n-m,8}$, where for the given $(n+1)$-points $f^k_{0,8}$ and $f^k_{n,8}$ are the first and last boundary points at level k respectively and m is any positive integer. We rewrite the 8-point scheme as

$$f^{k+1}_{2i,8} = \sum_{j=1}^{8} \xi_j f^k_{i+j-4,8}, \quad f^{k+1}_{2i+1,8} = \sum_{j=1}^{8} \xi_{9-j} f^k_{i+j-4,8}, \tag{4. 15}$$

where $\xi_1 = -\frac{495}{262144}$, $\xi_2 = \frac{5005}{262144}$, $\xi_3 = -\frac{27027}{262144}$, $\xi_4 = \frac{225225}{262144}$, $\xi_5 = \frac{75075}{262144}$, $\xi_6 = -\frac{19305}{262144}$, $\xi_7 = \frac{4095}{262144}$ and $\xi_8 = -\frac{429}{262144}$.

For $i=0$ (4. 15) gives (refinement rules to modify the first point $f^k_{0,8}$ of level k)

$$\begin{aligned} f^{k+1}_{0,8} &= \xi_1 f^k_{-3,8}+\xi_2 f^k_{-2,8}+\xi_3 f^k_{-1,8}+\xi_4 f^k_{0,8}+\xi_5 f^k_{1,8}+\xi_6 f^k_{2,8}+\xi_7 f^k_{3,8}+\xi_8 f^k_{4,8} \\ &= \xi_1(2f^k_{0,8}-f^k_{3,8})+\xi_2(2f^k_{0,8}-f^k_{2,8})+\xi_3(2f^k_{0,8}-f^k_{1,8})+\xi_4 f^k_{0,8}+\xi_5 f^k_{1,8} \\ &+\xi_6 f^k_{2,8}+\xi_7 f^k_{3,8}+\xi_8 f^k_{4,8} \\ &= (2\xi_1+2\xi_2+2\xi_3+\xi_4)f^k_{0,8}+(\xi_5-\xi_3)f^k_{1,8}+(\xi_6-\xi_2)f^k_{2,8}+(\xi_7-\xi_1) \\ &\times f^k_{3,8}+\xi_8 f^k_{4,8}, \end{aligned}$$

$$\begin{aligned} f^{k+1}_{1,8} &= \xi_8 f^k_{-3,8}+\xi_7 f^k_{-2,8}+\xi_6 f^k_{-1,8}+\xi_5 f^k_{0,8}+\xi_4 f^k_{1,8}+\xi_3 f^k_{2,8}+\xi_2 f^k_{3,8}+\xi_1 f^k_{4,8} \\ &= \xi_8(2f^k_{0,8}-f^k_{3,8})+\xi_7(2f^k_{0,8}-f^k_{2,8})+\xi_6(2f^k_{0,8}-f^k_{1,8})+\xi_5 f^k_{0,8}+\xi_4 f^k_{1,8} \\ &+\xi_3 f^k_{2,8}+\xi_2 f^k_{3,8}+\xi_1 f^k_{4,8} \\ &= (2\xi_8+2\xi_7+2\xi_6+\xi_5)f^k_{0,8}+(\xi_4-\xi_6)f^k_{1,8}+(\xi_3-\xi_7)f^k_{2,8}+(\xi_2-\xi_8) \\ &\times f^k_{3,8}+\xi_1 f^k_{4,8}. \end{aligned}$$

The refinement rules to modify second point $f^k_{1,8}$ of level k by two points at level $(k+1)$ are

$$\begin{aligned} f^{k+1}_{2,8} &= (2\xi_1+2\xi_2+\xi_3)f^k_{0,8}+(\xi_4-\xi_2)f^k_{1,8}+(\xi_5-\xi_1)f^k_{2,8}+\xi_6 f^k_{3,8}+\xi_7 f^k_{4,8}+\xi_8 f^k_{5,8}, \\ f^{k+1}_{3,8} &= (2\xi_8+2\xi_7+\xi_6)f^k_{0,8}+(\xi_5-\xi_7)f^k_{1,8}+(\xi_4-\xi_8)f^k_{2,8}+\xi_3 f^k_{3,8}+\xi_2 f^k_{4,8}+\xi_1 f^k_{5,8}. \end{aligned}$$

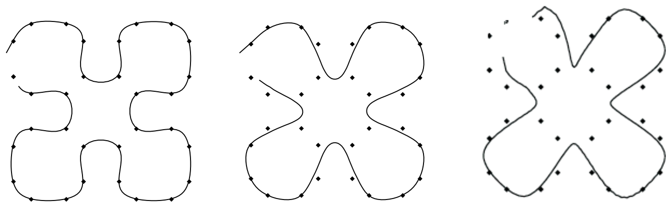
For $i=2$, the refinement rules to modify third point of level k are

$$\begin{aligned} f^{k+1}_{4,8} &= (2\xi_1+\xi_2)f^k_{0,8}+(\xi_3-\xi_1)f^k_{1,8}+\xi_4 f^k_{2,8}+\xi_5 f^k_{3,8}+\xi_6 f^k_{4,8}+\xi_7 f^k_{5,8}+\xi_8 f^k_{6,8}, \\ f^{k+1}_{5,8} &= (2\xi_8+\xi_7)f^k_{0,8}+(\xi_6-\xi_8)f^k_{1,8}+\xi_5 f^k_{2,8}+\xi_4 f^k_{3,8}+\xi_3 f^k_{4,8}+\xi_2 f^k_{5,8}+\xi_1 f^k_{6,8}. \end{aligned}$$

For $i=n-3$, the refinement rules to modify $(n-3)$-th point of level k are

$$\begin{aligned} f^{k+1}_{2n-6,8} &= \xi_1 f^k_{n-6,8}+\xi_2 f^k_{n-5,8}+\xi_3 f^k_{n-4,8}+\xi_4 f^k_{n-3,8}+\xi_5 f^k_{n-2,8}+\xi_6 f^k_{n-1,8} \\ &+\xi_7 f^k_{n,8}+\xi_8 f^k_{n+1,8} \\ &= \xi_1 f^k_{n-6,8}+\xi_2 f^k_{n-5,8}+\xi_3 f^k_{n-4,8}+\xi_4 f^k_{n-3,8}+\xi_5 f^k_{n-2,8}+\xi_6 f^k_{n-1,8} \\ &+\xi_7 f^k_{n,8}+\xi_8(2f^k_{n,8}-f^k_{n-1,8}) \\ &= \xi_1 f^k_{n-6,8}+\xi_2 f^k_{n-5,8}+\xi_3 f^k_{n-4,8}+\xi_4 f^k_{n-3,8}+\xi_5 f^k_{n-2,8}+(\xi_6-\xi_8) \\ &\times f^k_{n-1,8}+(\xi_7+2\xi_8)f^k_{n,8}, \end{aligned}$$

$$\begin{aligned} f^{k+1}_{2n-5,8} &= \xi_8 f^k_{n-6,8}+\xi_7 f^k_{n-5,8}+\xi_6 f^k_{n-4,8}+\xi_5 f^k_{n-3,8}+\xi_4 f^k_{n-2,8}+\xi_3 f^k_{n-1,8} \\ &+\xi_2 f^k_{n,8}+\xi_1 f^k_{n+1,8}, \\ &= \xi_8 f^k_{n-6,8}+\xi_7 f^k_{n-5,8}+\xi_6 f^k_{n-4,8}+\xi_5 f^k_{n-3,8}+\xi_4 f^k_{n-2,8}+\xi_3 f^k_{n-1,8} \\ &+\xi_2 f^k_{n,8}+\xi_1(2f^k_{n,8}-f^k_{n-1,8}) \\ &= \xi_8 f^k_{n-6,8}+\xi_7 f^k_{n-5,8}+\xi_6 f^k_{n-4,8}+\xi_5 f^k_{n-3,8}+\xi_4 f^k_{n-2,8}+(\xi_3-\xi_1) \\ &\times f^k_{n-1,8}+(\xi_2+2\xi_1)f^k_{n,8}. \end{aligned}$$



(a) 8-point scheme (b) 10-point scheme (c) 12-point scheme

FIGURE 1. *Solid lines show limit curves whereas diamond symbols show the control points.*

For $i = n-2$, the refinement rules to modify $(n-2)$-th point of level k are

$$\begin{aligned} f^{k+1}_{2n-4,8} &= \xi_1 f^k_{n-5,8} + \xi_2 f^k_{n-4,8} + \xi_3 f^k_{n-3,8} + (\xi_4 - \xi_8) f^k_{n-2,8} + (\xi_5 - \xi_7) f^k_{n-1,8} \\ &\quad +(\xi_6 + 2\xi_7 + 2\xi_8) f^k_{n,8}, \\ f^{k+1}_{2n-3,8} &= \xi_8 f^k_{n-5,8} + \xi_7 f^k_{n-4,8} + \xi_6 f^k_{n-3,8} + (\xi_5 - \xi_1) f^k_{n-2,8} + (\xi_4 - \xi_2) f^k_{n-1,8} \\ &\quad +(\xi_3 + 2\xi_2 + 2\xi_1) f^k_{n,8}. \end{aligned}$$

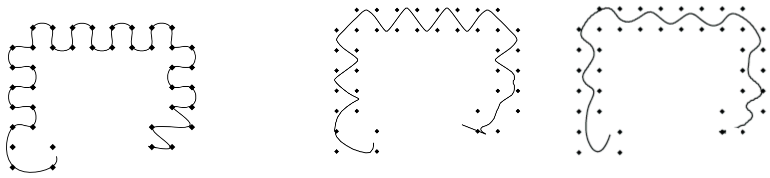
For $i = n-1$, the refinement rules to modify $(n-1)$-th point of level k are

$$\begin{aligned} f^{k+1}_{2n-2,8} &= \xi_1 f^k_{n-4,8} + (\xi_2 - \xi_8) f^k_{n-3,8} + (\xi_3 - \xi_7) f^k_{n-2,8} + (\xi_4 - \xi_6) f^k_{n-1,8} \\ &\quad +(\xi_5 + 2\xi_6 + 2\xi_7 + 2\xi_8) f^k_{n,8}, \\ f^{k+1}_{2n-1,8} &= \xi_8 f^k_{n-4,8} + (\xi_7 - \xi_1) f^k_{n-3,8} + (\xi_6 - \xi_2) f^k_{n-2,8} + (\xi_5 - \xi_3) f^k_{n-1,8} \\ &\quad +(\xi_4 + 2\xi_3 + 2\xi_2 + 2\xi_1) f^k_{n,8}. \end{aligned}$$

For $i = n$, the refinement rules to modify n-th point of level k are

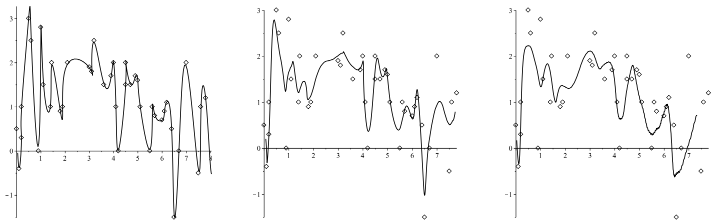
$$\begin{aligned} f^{k+1}_{2n,8} &= -\xi_8 f^k_{n-4,8} + (\xi_1 - \xi_7) f^k_{n-3,8} + (\xi_2 - \xi_6) f^k_{n-2,8} + (\xi_3 - \xi_5) f^k_{n-1,8} \\ &\quad +(\xi_4 + 2\xi_5 + 2\xi_6 + 2\xi_7 + 2\xi_8) f^k_{n,8}, \\ f^{k+1}_{2n+1,8} &= -\xi_1 f^k_{n-4,8} + (\xi_8 - \xi_2) f^k_{n-3,8} + (\xi_7 - \xi_3) f^k_{n-2,8} + (\xi_6 - \xi_4) f^k_{n-1,8} \\ &\quad +(\xi_5 + 2\xi_4 + 2\xi_3 + 2\xi_2 + 2\xi_1) f^k_{n,8}. \end{aligned}$$

The above refinement rules are used for refining the boundary points $f^k_{j,8} : j = 0, 1, 2, n-3, n-2, n-1, n$. These rules eliminate the involvement of the points $f^k_{j,8} : j = -3, -2, -1, n+1, n+2, n+3, n+4$ which was involved by (4. 15) but are not given.



(a) 8-point scheme (b) 10-point scheme (c) 12-point scheme

FIGURE 2. *Solid lines show limit curves whereas diamond symbols show the control points.*



(a) 8-point scheme (b) 10-point scheme (c) 12-point scheme

FIGURE 3. *Solid lines show limit curves whereas diamond symbols show the control points.*

APPENDIX A

$$\eta_0 = \frac{35(n^2-4)(n^2-1)(n^2-9)}{4\alpha} \sum_{r=-n+1}^{n} [6435r^7 - (858n^2+12012)r^6 - (9009n^2 -54054)r^5 + (1386n^4+4389n^2-64680)r^4 + (3465n^4-41580n^2+93555)r^3 +(630n^6+4410n^4+15603n^2-59388)r^2 - (315n^6-5670n^4+26271n^2- 27396)r + (70n^8-1155n^6+4179n^4+326n^2-5040)]f_r, \quad (4.\ 16)$$

$$\eta_1 = -\frac{9}{4\alpha}\sum_{r=-n+1}^{n}[(50050n^6-925925n^4+4389385n^2-4671810)r^7-(105105n^6 -1891890n^4-8765757n^2+9141132)r^6-(90090n^8-2102100n^6 +15807792n^4-45471426n^2+39243204)r^5+(121275n^8-2748900n^6 +20301435n^4-57747690n^2+49221480)r^4+(48510n^{10}-1334025n^8 +12984510n^6-55885830n^4+104415465n^2-67920930)r^3-(33075n^{10} -904050n^8+8834700n^6-37975770n^4+70186473n^2-45194268)r^2 -(7350n^{12}-227850n^{10}+2621745n^8-14207900n^6+37612603n^4 -45784004n^2+19889496)r+(1225n^{12}-39200n^{10}+470890n^8 -2661400n^6+7291445n^4-8898400n^2+3835440)]f_r, \quad (4.\ 17)$$

$$\eta_2 = -\frac{21}{4\alpha}\sum_{r=-n+1}^{n}[(289575n^4-2702700n^2+4549545)r^7-(30030n^6+345345n^4 -5246241n^2+9375366)r^6-(405405n^6-6216210n^4+29072043n^2 -38216178)r^5+(44550n^8-6257790n^4+36100020n^2-50191680)r^4 +(155925n^8-3326400n^6+24123330n^4-68690160n^2+66143385)r^3 -(17010n^{10}-184275n^8-674730n^6+13678560n^4-45423399n^2 +45426234)r^2-(14175n^{10}-387450n^8+3786300n^6-16275330n^4 +30079917n^2-19368972)r+(1050n^{12}-22050n^{10}+128345n^8+65100n^6 -2395365n^4+5786200n^2-3563280)]f_r, \quad (4.\ 18)$$

$$\eta_3 = \frac{3465}{4\alpha}\sum_{r=-n+1}^{n}[(1170n^4-15210n^2+36543)r^7-(3003n^4-36036n^2+81081)r^6 -(2002n^6-37037n^4+197197n^2-312312)r^5+(3465n^6-57750n^4 +287595n^2-436590)r^4+(990n^8-23100n^6+177870n^4-543840n^2 +558327)r^3-(945n^8-20160n^6+146202n^4-416304n^2+400869)r^2 -(126n^{10}-3465n^8+33726n^6-145158n^4+271209n^2-176418)r +(35n^{10}-910n^8+8540n^6-35070n^4+61425n^2-34020)]f_r, \quad (4.\ 19)$$

$$\eta_4 = \frac{1155}{4\alpha}\sum_{r=-n+1}^{n}[(19305n^2-90090)r^7-(1638n^4+38766n^2-208299)r^6 -(27027n^4-288288n^2+756756)r^5+(2310n^6+26565n^4-393855n^2 +1075305)r^4+(10395n^6-173250n^4+862785n^2-1309770)r^3-(810n^8 -113778n^4+656364n^2-912576)r^2-(945n^8-21420n^6+158193n^4 -449982n^2+383544)r+(42n^{10}-455n^8-1764n^6+32445n^4-100828n^2 +70560)]f_r, \quad (4.\ 20)$$

$$\begin{aligned}\eta_5 = & \frac{-9009}{4\alpha}\sum_{r=-n+1}^{n}[(990n^2-6765)r^7-(3003n^2-18018)r^6-(1638n^4-21294n^2\\ & +61971)r^5+(3465n^4-36960n^2+97020)r^4+(770n^6-14245n^4+75845n^2\\ & -120120)r^3-(945n^6-14490n^4+67767n^2-89082)r^2-(90n^8-2100n^6\\ & +15792n^4+45426n^2+39204)r+(35n^8-700n^6+4655n^4-11550n^2\\ & +7560)]f_r, \qquad (4.\ 21)\end{aligned}$$

$$\begin{aligned}\eta_6 = & \frac{-3003}{4\alpha}\sum_{r=-n+1}^{n}[6435r^7-(462n^2+16863)r^6-(9009n^2-54054)r^5+(630n^4\\ & +14910n^2-80115)r^4+(3465n^4-41580n^2+93555)r^3-(210n^6+2415n^4\\ & -36687n^2+65562)r^2-(315n^6-5670n^4+26271n^2-27396)r+(10n^8\\ & -1470n^4+6500n^2-5040)]f_r, \qquad (4.\ 22)\end{aligned}$$

$$\begin{aligned}\eta_7 = & \frac{6435}{4\alpha}\sum_{r=-n+1}^{n}[858r^7-3003r^6-(1386n^2-9471)r^5+(3465n^2-16170)r^4\\ & +(630n^4-8190n^2+19677)r^3-(945n^4-8820n^2+14847)r^2-(70n^6\\ & -1295n^4+6139n^2-6534)r+(35n^6-490n^4+1715n^2-1260)]f_r, \qquad (4.\ 23)\end{aligned}$$

where

$$\alpha = n(n^2-4)(n^2-1)(n^2-9)(4n^2-9)(4n^2-25)(4n^2-1)(4n^2-49). \qquad (4.\ 24)$$

CONFLICT OF INTRESTS

The authors declare that there is no conflict of interest regarding the publication of this article.

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AUTHORS CONTRIBUTION

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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