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Solving Differential Equations by New Wavelet Transform Method Based on the Quasi-Wavelets and Differential Invariants

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Abstract. In harmonic analysis, wavelets are useful and important tools for analyzing problems and equations. As far as we know, the wavelet applications for solving differential equations are limited to solving either ODE or PDE by numerical means. In this paper, the new mother wavelets with two independent variables are designed in accordance with differential invariants. A new method based on the wavelets is proposed and, new mother wavelets are introduced, while the corresponding wavelet transforms are calculated and applied to differential equations. A lot of methods such as the wavelet-Galerkin method, the wavelet method of moment lead to approximate or numerical solutions. Our method can be used for ODEs and PDEs from every order and accordingly the analytic solutions are obtained.

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1. Introduction

A Norwegian mathematician, Marius Sophus Lie (1842, 1899), largely created the theory of continuous Lie symmetry groups and applied it to the study of geometry and differential equations [4]. These symmetry groups are invertible point transforms of both dependent and independent variables of the differential equations. The symmetry group methods provide an ultimate tool for analyzing the differential equations; it is very important to udnderstand them and construct solutions of differential equations. Several applications and contributions of Lie groups in the theory of differential equations have been discussed in the literature [4], [13] and [14]. The most important ones are the reduction of the order of ODEs, the construction of invariant solutions, the mapping of the solutions to other solutions, and the detection of linearizing transforms (for many other applications of Lie symmetries see [14], [15]).

Differential equations have been appeared in mathematical modelling of scientific fields [2], [8]. Nowadays, a lot of methods have been proposed for solving and analyzing these equations, for example, see [10], [11] and [18]. Meanwhile, the symmetry groups methods and wavelets have many applications [19]. The wavelets are important functions with special properties in functional and harmonic analysis. In 1909, Alphered Haar (a Hungarian mathematician) introduced the first wavelet [7]. In signal-and-image processing, the wavelets can be used as a new tool and are also called the numerical microscopes. They have the desirable advantages of multi-resolution properties and various basis functions, which have great potential for solving and analyzing partial differential equations (PDEs). Multi-dimensional wavelets are very important for applying the wavelet methods to higher-order PDEs with two or more independent variables. Multi-dimensional wavelets, such as those from the tensor product of 1D wavelets, have been widely studied, but the wavelets are proposed here -besides being multi-dimensional- are different.

In 1992, the numerical analysis with wavelets first came to the notice of researchers and, since the topics has gained increasing attention [9]. The wavelets now have numerous applications in some fields of science and technology such as seismology, image-processing, signal-processing, coding theory, biosciences, financial mathematics, fractals and other areas [1]. The application of integral transforms for solving the ODEs and PDEs can be traced back to Leonhard Euler (in 1744), Pierre-Simon Laplace (in 1785) and Joseph Fourier (in 1822). In 1809, Laplace applied his transform to solve the density of a substance diffusing indefinitely in space. The use of Laplace and Fourier transforms for solving differential equations inspired our work (for more information about the Fourier transform method (FTM), we refer the reader to [6] and [17]). As far as we know, the application of wavelets for solving ODEs and PDEs is limited to numerical solutions under special conditions.

In this paper, we build new wavelets with two variables that depend on the differential invariants of Differential equations. Therefore, we can use their transforms for solving differential equations. Indeed, for solving PDEs with two independent variables, we propose a new method based on the wavelets with two independent variables in accordance with differential invariants. This method is called the wavelet transform method (WTM). Because of the need to use differential invariants in the construction of our wavelets, we briefly explain the Lie symmetry method that will be applied for obtaining the differential invariants. We will show the performance of WTM by implementing some examples.

The remainder of this paper is organized as follows. In section 2, we recall some needed results to construct the differential invariants, mother wavelets, quasi-wavelets and wavelet transforms. In section 3, wavelet transform method is proposed. In sections 4, the proposed method will be demonstrated by examples. Finally, the conclusions and future works will be presented.

2. Preliminaries

In this section, we recall some results [needed] to construct differential invariants, the mother wavelets, and their transforms. First, we remember that the Lie symmetry method can be applied for obtaining differential invariants and reducing the order of PDEs. After that, the wavelets and their transforms are discussed. The related definitions and theorems are considered. We refer the reader to [3], [7] and [16] for deeper discussions and the detailed proofs of theorems from the wavelet theory.

2.1. **The Lie symmetry method.** In this section, we remind the general procedure for determining symmetries for any system of PDEs (the best general references for these topics are [15], [14] and [13]). To begin, let us consider the general case of a nonlinear system of partial differential equations of nth-order in p independent and q dependent variables as follows:

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, l, \tag{2. 1}$$

that involving $x = (x^1, \cdots, x^p)$, $u = (u^1, \cdots, u^q)$ and derivatives of u with respect to x up to n, where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n. We also consider a one-parameter Lie group of infinitesimal transforms that acts on the independent and dependent variables of the system (2.1) as below:

$$(\tilde{x}^i, \tilde{u}^j) = (x^i, u^i) + s(\xi^i, \eta^j) + O(s^2), \quad i = 1 \cdots, p, \ j = 1 \cdots, q,$$

where s is the parameter of the transform and ξ^i, η^j are the infinitesimals of the transforms for the independent and dependent variables, respectively. The infinitesimal generator vector field $\mathbf{v}$ associated with the above group of transforms can be written as $\mathbf{v} = \sum_{i=1}^p \xi^i \partial_{x^i} + \sum_{j=1}^q \eta^j \partial_{u^j}$. A symmetry of a differential equation is a transform which maps solutions of the equation to other solutions. The invariance of the system (2.1) under the infinitesimal transforms leads to the invariance conditions (Theorem 2.36 of [3]):

$$\mathrm{Pr}^{(n)}\mathbf{v}\big[\Delta_\nu(x, u^{(n)})\big] = 0, \quad \Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, l,$$

where $\mathrm{Pr}^{(n)}$ is said to be the n^{th} order prolongation of the infinitesimal generator and defined by $\mathrm{Pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^q \sum_J \phi_J^\alpha(x, u^{(n)})\partial_{u_J^\alpha}$, where $J = (j_1, \cdots, j_k)$, $1 \le j_k \le p$, $1 \le k \le n$ and the sum is over all J's of order $0 < \#J \le n$. If $\#J = k$, coefficients ϕ_J^α of $\partial_{u_J^\alpha}$ will only depend on k-th and lower order derivatives of u and $\phi_\alpha^J(x, u^{(n)}) = D_J(\phi_\alpha - \sum_{i=1}^p \xi^i u_i^\alpha) + \sum_{i=1}^p \xi^i u_{J,i}^\alpha$, where $u_i^\alpha := \partial u^\alpha / \partial x^i$ and $u_{J,i}^\alpha := \partial u_J^\alpha / \partial x^i$.

The most advantage of using these infinitesimal symmetries lies in the fact that they form a Lie algebra under the usual Lie bracket. In fact, the symmetry group methods construct new solutions from known solutions. On the other hand, when a nonlinear system of differential equations admits infinite symmetries, so it is possible to transform it to a linear system. The great power of Lie group theory lies in the crucial observation that one can replace the complicated, nonlinear conditions for the invariance of a subset or function under the group transformations themselves by an equivalent linear condition of infinitesimal invariance under the corresponding infinitesimal generators of the group action [14].

Example 2.2. *We applied the Lie symmetry method on the heat equation. Consider the equation for the conduction of heat in a one-dimensional rod $u_t = u_{xx}$, the thermal diffusivity has been normalized to unity. Here there are two independent variables x and t and* *one dependent variable u. Let*

$$\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial x}.$$

be a generated vector field. We wish to determine all possible coefficents ξ, τ and ϕ, so according to the invariant condition, we should have

$$Pr^{(2)}\mathbf{v}\big[\Delta_\nu(x,t,u^{(2)})\big] = 0, \quad \Delta_\nu(x,,t,u^{(2)}) = u_t - u_{xx},$$

but, the second prolongation of **v** *is*

$$\mathbf{v} + \phi^x\frac{\partial}{\partial u_x} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}},$$

After calculation, we get

$$\phi^t = \phi^{xx}$$

which must be satisfied whenever $u_t = u_{xx}$. By equating the coefficients of the various monomials in the first and second order partial derivatives of u, we find the determining equations for the symmetry group of the heat equation to be the following:

$$\begin{gathered} 2\tau_u = 0, \qquad -2\tau_x = 0, \\ \tau_{uu} = 0, \qquad -\xi_u = -2\tau_{xu} - 3\xi_u, \\ \phi_u - \tau_t = -tau_{xx} + \phi_u - 2\xi_x, \qquad -\xi_{uu} = 0, \\ \phi_{uu} - 2\xi_{xu} = 0, \qquad -\xi_t = 2\Phi_{xu} - \xi_{xx}, \qquad \phi_t = \phi_{xx}. \end{gathered}$$

The solution of the determining equations is elementary. Therefore, we conclude that the general infinitesimal symmetry of the heat equation has coefficient functions of the form

$$\begin{gathered} \xi = c_1 + c_4x + 2c_5t + 4c_6xt, \\ \tau = c_2 + 2c_4t + 4c_6t^2, \\ \phi = (c_3 - c_5x - 2c_6t - c_6x^2)u + \alpha(x,t). \end{gathered}$$

where c_is for $(i = 1, ..., 6)$ are arbitrary constants and $\alpha(x,t)$ is an arbitrary solution of the heat equation. Thus, the Lie algebra of infinitesimal symmetries of the heat equation is spanned by the six vector fields

$$\begin{gathered} \mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = \partial_t, \quad \mathbf{v}_3 = u\partial_u, \\ \mathbf{v}_4 = x\partial_x + 2t\partial_t, \quad \mathbf{v}_5 = 2t\partial_x - xu\partial_u, \\ \mathbf{v}_6 = 4xt\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u. \end{gathered}$$

and the infinte-dimensional subalgebra $\mathbf{v}_\alpha = \alpha(x,t)\partial_u$ [14].

2.3. **The wavelets.** Wavelets are important functions in mathematics and other scientific fields. In this section, we introduce them as functions belonging to $L^2(\mathbb{R}^2)$ (The space of squared integrable functions with integral norm).

Definition 2.4. *The function ψ is called wavelet if it satisfies the following admissible condition:*

$$C_\psi = \int_{\mathbb{R}^2} \frac{|F(\psi)(\omega)|^2 d\omega}{|\omega|} < \infty$$

where $F(\psi)(\omega)$ is the Fourier transform of the wavelet ψ and given by:

$$F(\psi)(\omega) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^2} \exp(-ix\cdot\omega)\psi(x)dx$$

Here C_ψ is said to be the wavelet coefficient of ψ and $\omega = (\omega_1, \omega_2)$, while $x = (x_1, x_2)$ belongs to $\mathbb{R}^2$. For further informations and examples, we refer the reader to [3].

Definition 2.5. *The wavelet ψ is called the mother wavelet if it satisfies the following properties:*

$$\int_{\mathbb{R}^2} \psi(x)dx = 0, \tag{2. 2}$$

$$\int_{\mathbb{R}^2} |\psi(x)|^2 dx < \infty, \tag{2. 3}$$

$$\lim_{|\omega|\to\infty} F(\psi(\omega)) = 0 \tag{2. 4}$$

Note that the first property is equivalent to $C_\psi > 0$ (the admissible condition) for the mother wavelet ψ. For more details see [7].

Indeed, the mother wavelets have the admissible condition, n-zero moments and exponential decay properties. The mother wavelet have two parameters: the translation parameter $b = (b_1, b_2)$ and the scaling parameter $a > 0$. The family wavelet related to the mother wavelet ψ with parameters (a, b) is:

$$\psi_{a,b}(x) = \psi(\frac{x-b}{a}) = \psi(\frac{x_1 - b_1}{a}, \frac{x_2 - b_2}{a})$$

If function ψ does not satisfy properties $(2.2), (2.3)$, and (2.4) globally, while approximately satisfies (locally) some properties of mother wavelets, it is called quasi-wavelet (In other words, quasi-wavelets are modified wavelets based on special properties for computational purposes. In fact, quasi-wavelets are modified wavelets based on differential invariants for solving differential equations). The quasi-wavelets have numerous applications in applied mathematics and other scientific fields for solving PDEs (for more details and examples, see [20]). In this paper, we provide quasi-wavelets based on the differential invariants of PDEs. We will analyze PDEs by these quasi-wavelets.

Definition 2.6. *Suppose the mother wavelet ψ, the function $f \in L^2(\mathbb{R}^2)$, and parameters (a, b) are given. Then, the corresponding wavelet transform is defined as follows*

$$W_\psi(f)(a,b) = \frac{1}{\sqrt{|a|.C_\psi}} \int_{\mathbb{R}^2} \psi_{a,b}(x).f(x)dx$$

Thus, the wavelet transform depends on the wavelet ψ, the function f, and parameters (a, b).

Theorem 2.7. *The wavelet transform is an operator from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^3)$ that satisfies in the following properties:*

1 . *Linearity:* $W_\psi[\alpha f(x) + \beta g(x)] = \alpha W_\psi[f(x)] + \beta W_\psi[g(x)]$,
2 . *Translation:* $W_\psi[f(x-k)] = W_\psi[f(x)](a, b-k), \quad \forall k \in \mathbb{R}^2$,
3 . *Scaling:* $W_\psi[\frac{1}{\sqrt{s}} f(\frac{x}{s})] = W_\psi[f(x)](\frac{a}{s}, \frac{b}{s})$,
4 . *Wavelet shifting:* $W_{\psi(x-k)}[f(x)] = W_\psi[f(x)](a, b+ak)$,
5 . *Linear combination:* $W_{\alpha\psi_1+\beta\psi_2}[f(x)] = \alpha W_{\psi_1}[f(x)] + \beta W_{\psi_2}[f(x)]$,
6 . *Wavelet scaling:* $W_{\frac{\psi(x/s)}{\sqrt{(|s|)}}}[f(x)] = W_\psi[f(x)](as, b)$.

Proof. For proof and more details see [7].

Another way of stating Theorem 1 is to say that the wavelet transforms are isometries. Therefore in the smooth manifold *M*, if the collection of wavelet tranforms and the isometry group of *M* are denoted by W(*M*) and I(*M*) respectively, then W(*M*) is a Lie subgroup of I(*M*) [3].

The admissible condition implies that the wavelet transform is invertible (on the other hand, wavelet transforms are isometry). Indeed, the inversion formula for the wavelet transform $W_\psi(f)$ as follows

$$f(x) = f(x_1, x_2) = \frac{1}{C_\psi}\int_{\mathbb{R}^+\times\mathbb{R}^2} W_\psi f(a,b)\,\psi_{a,b}(x)\frac{da\,db_1\,db_2}{a^3}$$

The inversion formula (also called the synthesis formula) can calculate the function $f(x)$ corresponds to the wavelet transform $W_\psi(f)$. Note that, these conceptions are generalizable to $\mathbb{R}^n$ by suitable symbols and assumptions [12].

3. THE WAVELET TRANSFORM METHOD

The wavelet transform method (WTM) has the following steps:

1 . Apply equivalence algorithms (for example, the Lie symmetry method) on differential equations and obtain differential invariants.

2 . Build suitable mother wavelets based on differential invariants.

3 . Multiply the mother wavelet with the both sides of the equation and take the wavelet transform. Solve the reduced differential equation and obtain the wavelet transform.

4 . By the inversion formula, calculate the analytical solution.

In the following, some WTM formulas are proposed.

Theorem 3.1. *Let $\Delta_\nu(x,t,u^{(m)}) = 0$ be the m-th order differential equation with two independent variables (x,t) and ψ be a mother wavelet based on differential invariants (t is taken as a constant, x is taken as a variable). Under these assumptions, we have:*

1) $W_\psi(\partial_t u)(x,t) = \frac{d}{dt}W_\psi(u)(x,t)$,
2) $W_\psi(\partial_t^n u)(x,t) = \frac{d^n}{dt^n}W_\psi(u)(x,t)$,
3) $W_\psi(\partial_x u)(x,t) = -W_{\partial_x\psi}(u)(x,t)$,
4) $W_\psi(\partial_x^n u)(x,t) = (-1)^n W_{\partial_x^n\psi}(u)(x,t)$.

Proof. Without loss of generality, we assume that $(a = 1, b = 0)$.

1) We have:

$$\begin{aligned} W_\psi(\partial_t u)(x,t) &= \tfrac{1}{\sqrt{c_\psi}}\int u_t\psi dx = \tfrac{1}{\sqrt{c_\psi}}\int \lim_{h\to 0}\tfrac{u(x,t+h)-u(x,t)}{h}\psi dx \\ &= \lim_{h\to 0}\tfrac{1}{h}\big\{\tfrac{1}{\sqrt{c_\psi}}\int u(x,t+h)\psi dx - \tfrac{1}{\sqrt{c_\psi}}\int u(x,t)\psi dx\big\} \\ &= \lim_{h\to 0}\tfrac{\tilde{u}(x,t+h)-\tilde{u}(x,t)}{h} = \tfrac{d}{dt}\tilde{u}(x,t). \end{aligned}$$

where $\tilde{u}(x,t) = W_\psi(x,t)$.

2) By induction on the derivation order of t -i.e. n and similar to the above-mentioned procedure- we have

$$\begin{aligned} W_\psi(\partial_t^n u)(x,t) &= \tfrac{1}{\sqrt{c_\psi}}\int u_t^{(n)}\psi dx = \tfrac{1}{\sqrt{c_\psi}}\tfrac{d}{dt}\int \tfrac{\partial^{n-1}u}{\partial t^{n-1}}\psi dx \\ &= \tfrac{d}{dt}\big\{\tfrac{d}{dt}\int \tfrac{\partial^{n-2}u}{\partial t^{n-2}}\psi dx\big\} = \ldots = \tfrac{d^n}{dt^n}\tilde{u}(x,t). \end{aligned}$$

3) We know that

$$W_\psi(\partial_x u)(x,t) = \frac{1}{\sqrt{c_\psi}}\int u_x\psi dx = (\frac{1}{\sqrt{c_\psi}}.\psi.u]_{-\infty}^{+\infty} - \frac{1}{\sqrt{c_\psi}}\int u\frac{\partial\psi}{\partial x}dx$$
$$= -\frac{1}{\sqrt{c_\psi}}\int u\frac{\partial\psi}{\partial x}dx.$$

We assume that

$$\lim_{x,t\to\infty} u(x,t)\frac{\partial\psi(x,t)}{\partial x} = 0,$$

for calculating this integral, we use the integral by part with $U = \psi, dV = \frac{\partial u}{\partial x}$ and we get to $W_{\partial_x\psi}(u)(x,t)$.

4) By following inducely the above procedure according to the derivation order of x -i.e. n, we get:

$$W_\psi(\frac{\partial^n u}{\partial x^n})(x,t) = (\frac{1}{\sqrt{c_\psi}}.\psi\frac{d^{n-1}u}{dx^{n-1}}]_{-\infty}^{+\infty} - (\frac{1}{\sqrt{c_\psi}}.\frac{\partial\psi}{\partial x}.\frac{d^{n-2}u}{dx^{n-2}}]_{-\infty}^{+\infty}$$
$$+ ... + (-1)^n\frac{1}{\sqrt{c_\psi}}\int u\frac{\partial^n\psi}{\partial x^n}dx.$$

where for caculating the integral, we can use the integral by part with $U = \psi, dV = \frac{\partial u}{\partial x}$ and assume that

$$\lim_{x,t\to\infty} u^{(1)}(x,t)\frac{\partial^{(n-1)}\psi(x,t)}{\partial x^{(n-1)}} = ... = \lim_{x,t\to\infty} u^{(n-1)}(x,t)\psi(x,t) = 0,$$

thus, the last integral is $W_{\partial_x^n\psi}(u)(x,t)$.

In practice we take the wavelet transform from both sides of differential equation (t is taken as a constant, x is taken as a variable and $a = 1, b = 0$) and solve the reduced equation according to $\tilde{u}(x,t)$ and its t-derivations and obtain $\tilde{u}(x,t)$. For the given mother wavelet $\psi(x,t)$ and the obtained wavelet transform $\tilde{u}(x,t)$, we calculate $u(x,t)$ from the following formula ($1D$-inversion formula)

$$u(x,t) = \int \tilde{u}(x,t)\psi(x,t)dx \tag{3. 5}$$

where $u(x,t)$ is the desired analytic solution. In this way, the PDE is solved by WTM based on ψ (in accordance with the differential invariants). In the following section, we apply WTM on the heat, wave and KdV equations.

4. Examples

In this section, we demonstrate WTM by examples. We implement WTM on the heat, wave and KdV equations and obtain the solutions. Finally, the WTM results will be proposed.

Example 4.1. *First, the Lie symmetry method is applied to the heat equation $u_t = u_{xx}$, and the symmetry groups, vector fields and differential invariants are obtained (for more detailed calculations and results of the Lie symmetry method implementation on the heat equation, see* [14]*). The Lie symmetry method results for the heat equation proposed in the following table:*

Table 1. The Lie symmetry method results for the heat equation.

Symmetry groups	*V.F.*	$\dim(g)$	*Differential invariants*
Translation	$c\partial_x + \partial_t$	2	$(x - ct), u$
Scaling	$x\partial_x + 2t\partial_t + 2au\partial_u,$	3	$(x/\sqrt{t}), (u/t^a)$
Galilean boost	$2t\partial_x - xu\partial_u$	2	$t, u\exp(x^2/4t)$

In table 1, the symmetry groups are the translation with factor (c), scaling with factor (a) and the Galilean boost (respectively). For each differential invariant and symmetry group, the adequate quasi-wavelets are proposed in table 2.

Table 2. The suitable quasi-wavelets for symmetry groups.

Symmetry groups	*Differential invariants*	*Quasi-wavelets*
Translation	$x - ct, u$	$\exp(-t^2/2)\sin(\pi(x-ct)/2)$ $\exp(-t^2/2)\cos(\pi(x-ct)/2)$
Scaling	$(x/t), (x/\sqrt{t}), (u/t^a)$	$\exp(-t^2/2)\sin(x/t)$ $\exp(-t^2/2)\cos(x/t)$
Galilean boost	$t, u\exp(x^2/4t)$	$\exp(-t^2/2)\sin(x/t)$ $\exp(-t^2/2)\cos(x/t)$

With only a little computation, it can be seen that the offered functions have approximately properties (2.2) *and* (2.3) *of the mother wavelets. For example, we assume that* $\psi_1 := \exp(-t^2/2)\sin(\pi(x-ct)/2)$*, for properties* $(2.2), (2.3)$ *we have:*

$$\int_0^2 \int_0^4 \psi_1(x,t)dx\ dt = 0.0000809453, \tag{4. 6}$$

$$\int_0^1 \int_0^2 |\psi_1(x,t)|^2 dx\ dt = 0.00032350. \tag{4. 7}$$

and for $\psi_2 := \exp(-t^2/2)\cos(\pi(x-ct)/2)$*, the computation is as follows*

$$\int_0^2 \int_0^4 \psi_2(x,t)dx\ dt = 0.0768912, \tag{4. 8}$$

$$\int_0^1 \int_0^2 |\psi_2(x,t)|^2 dx\ dt = 0.01961003. \tag{4. 9}$$

Therefore, these wavelets approximately satisfy the properties of mother wavelets (in their periods) and so are called quasi-wavelets. (Note that, ψ_1 *and* ψ_2 *are periodic functions with* $T_t = 2, T_x = 4$ *and also* ψ_1^2, ψ_2^2 *are periodic functions with* $T_t = 1, T_x = 2$*, where* T_t *and* T_x *are periods of quasi-wavelets in the variables* t *and* x *respectively).*

Figures 1 and 2 show the graphs of quasi-wavelets and make some properties clear.

Now by these quasi-wavelets, we apply WTM on the heat equation. First, consider the quasi-wavelet ψ_1 *as follows:*

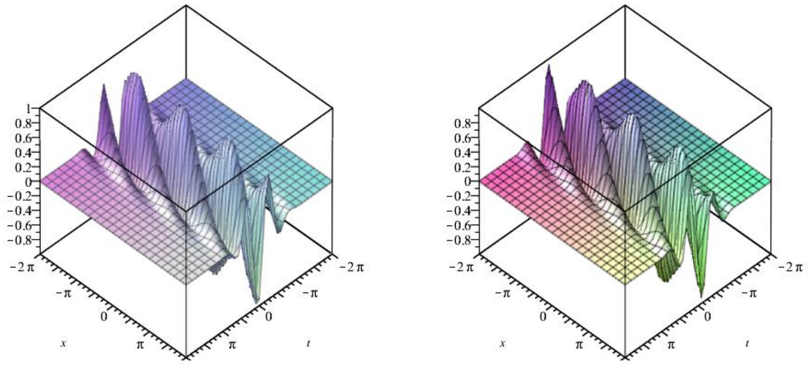
$$\psi_1 := \exp(\frac{-t^2}{2})\sin(\frac{\pi(x-2t)}{2})$$

then by multiplying both sides of heat equation with ψ_1 *and taking the wavelet transform, we have:*

$$\frac{d}{dt}\tilde{u}(x,t) = -\frac{\pi^2}{4}\tilde{u}(x,t)$$

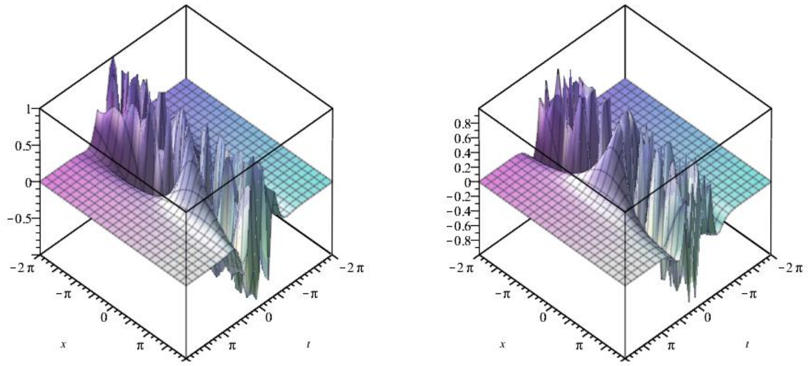
therefore

$$\tilde{u}(x,t) = \tilde{F}(x)\exp(-\frac{\pi^2}{4}t) + \tilde{K}$$



(A) The graph of ψ_1 (B) The graph of ψ_2

FIGURE 1. The graphs of quasi-wavelets



(A) The graph of ψ_3 (B) The graph of ψ_4

FIGURE 2. The graphs of quasi-wavelets

where $\tilde{K}$ and $\tilde{F}(x)$ (respectively) are the wavelet transform related to the constant K at $\mathbb{R}$ and the function F of x. Thus the analytical solution from (3.5) is

$$u(x,t) = F(x)\exp(-\frac{\pi^2}{4}t) + K.$$

Second, consider the quasi-wavelet ψ_2 as follows:

$$\psi_2 := \exp(\frac{-t^2}{2})\cos(\frac{\pi(x-2t)}{2})$$

by multiplying both sides of the heat equation with ψ_2 and taking the wavelet transform, we get:

$$\frac{d}{dt}\tilde{u}(x,t) = -\frac{\pi^2}{4}\tilde{u}(x,t)$$

therefore

$$\tilde{u}(x,t) = \tilde{F}(x)\exp(-\frac{\pi^2}{4}t) + \tilde{K}$$

where $\tilde{K}$ *and* $\tilde{F}(x)$ *are defined as the above. Thus the analytical solution is obtained from* (3.5) *as follows*

$$u(x,t) = F(x)\exp(-\frac{\pi^2}{4}t) + K.$$

By putting the solutions in the heat equation, $F(X)$ *can be obtained as follows:*

$$F(x) = c_1\cos(\tfrac{\pi x}{2}) + c_2\sin(\tfrac{\pi x}{2}),$$

Thus, the final solution is

$$u(x,t) = \exp(-\frac{\pi^2}{4}t)\{c_1\cos(\frac{\pi x}{2}) + c_2\sin(\frac{\pi x}{2})\} + K.$$

where c_1, c_2*, and* K *are real constants that can be found based on initial or boundary conditions.*

Third, let us ψ_3 *as follows*

$$\psi_3 := \exp(\frac{-t^2}{2})\sin(\frac{x}{t})$$

after taking the wavelet transform under ψ_3*, we have:*

$$\frac{d}{dt}\tilde{u}(x,t) = -\frac{1}{t^2}\tilde{u}(x,t)$$

therefore

$$\tilde{u}(x,t) = \tilde{G}(x)\exp(\frac{1}{t}) + \tilde{K}$$

So, the analytical solution from (3.5) *is*

$$u(x,t) = G(x)\exp(\frac{1}{t}) + K.$$

where K *and* $G(x)$ *are same the above and according to initial or boundary conditions will be determined. By putting this solution in the heat equation, we get:*

$$\frac{d^2}{dx^2}G(x) + \frac{1}{t^2}G(x) = 0$$

after solving this second order ODE, $G(x)$ *should be as below:*

$$G(x) = c_1\cos(\frac{x}{t}) + c_2\sin(\frac{x}{t}).$$

Finally, the exact solution obtained as follows

$$u(x,t) = \exp(\frac{1}{t})\{c_1\cos(\frac{x}{t}) + c_2\sin(\frac{x}{t})\} + K.$$

where c_i *for* $(i = 1, ..., 4)$ *are real constants.*

For the quasi-wavelet ψ_4 *the procedure and results are similar.*

Table 3 shows the results of wavelet transform method on the heat equation:

Table 3. WTM on the heat equation.

Quasi-wavelet	The wavelet tranform	The analytical solution
ψ_1, ψ_2	$\tilde{F}(x)\exp(-\frac{\pi^2}{4}t) + \tilde{K}$	$F(x)\exp(-\frac{\pi^2}{4}t) + K$
ψ_3, ψ_4	$\tilde{G}(x)\exp(\frac{1}{t}) + \tilde{K}$	$G(x)\exp(\frac{1}{t}) + K$

Example 4.2. *As another example, we apply WTM on the 1D-wave equation as follows*

$$u_{tt} = u_{xx}$$

We know that this equation has the travelling wave solution (TWS) and so invariants under the translation and dilation (for more details and calculations, we refere the reader to [3]*). So by using ψ_1 and ψ_2, we have*

$$\frac{d^2}{dt^2}\tilde{u}(x,t) = -\frac{\pi^2}{4}\tilde{u}(x,t)$$

We should solve this ODE with characteristics method (for more details about methods for solving ODEs, see [5]*). After calculation we have:*

$$\tilde{u}(x,t) = \tilde{H}(x)\{c_1\cos(\frac{\pi t}{2}) + c_2\sin(\frac{\pi t}{2})\} + \tilde{K}$$

now from (3.5) *the analytical solution is*

$$u(x,t) = H(x)\{c_1\cos(\frac{\pi t}{2}) + c_2\sin(\frac{\pi t}{2})\} + K.$$

By putting this solution in the wave equation, we have

$$H(x) = c_3\cos(\frac{\pi t}{2}) + c_4\sin(\frac{\pi t}{2}).$$

Finally, the exact solution obtained as follows

$$u(x,t) = \{c_1\cos(\frac{\pi t}{2}) + c_2\sin(\frac{\pi t}{2})\}.\{c_3\cos(\frac{\pi t}{2}) + c_4\sin(\frac{\pi t}{2})\}.$$

where c_i for $(i = 1, ..., 4)$ are the real constants.

Table 4 shows the results of wavelet transform method for the wave equation:
Table 4. WTM on the wave equation.

Quasi-wavelet	The wavelet tranform	The analytic solution
ψ_1, ψ_2	$\tilde{F}\{c_1\cos(\frac{\pi t}{2}) + c_2\sin(\frac{\pi t}{2})\} + \tilde{K}$	$\{c_1\cos(\frac{\pi t}{2}) + c_2\sin(\frac{\pi t}{2})\}.H(x)$

where $H(x) = \{c_3\cos(\frac{\pi t}{2}) + c_4\sin(\frac{\pi t}{2})\}$.

Example 4.3. *Finally, we implemented WTM on the generalized version of the Kortewegde Vries (KdV)*

$$u_t + u_{xxx} = u_{xxxxx}$$

We know that the KdV equation has TWS (for more detailed calculations and results, see [14]*). Thus, according to table 2, we can apply WTM by quasi-wavelets ψ_1 and ψ_2. First, we apply WTM with ψ_1, and we have*

$$\frac{d}{dt}\tilde{u}(x,t) - \frac{\pi^3}{8}\tilde{u}(x,t) = \frac{\pi^5}{32}\tilde{u}(x,t),$$

therefore

$$\frac{d}{dt}\tilde{u}(x,t) - 13.44\tilde{u}(x,t) = 0,$$

by solving this ODE [5], *we get*

$$\tilde{u}(x,t) = \tilde{I}(x)\exp(13.44t) + \tilde{K}$$

Now from (3.5), *the analytical solution is*

$$u(x,t) = I(x)\exp(13.44t) + K.$$

For obtaining $I(x)$, *we should put* $u(x,t)$ *in the KdV equation:*

$$\frac{d^5}{dx^5}I(x) - \frac{d^3}{dx^3}I(x) - 13.44I(x) = 0.$$

by solving this 5*-th order ODE,* $I(X)$ *are obtained as follows*

$$c_1\exp(2.05x) - c_2\exp(-2.05x) + c_3\cos(1.78x) + c_4\sin(1.78x) + c_5.$$

where c_i*s for* $(i = 1, ..., 5)$ *are real arbitrary constants.*

Now by employing ψ_2, *we have*

$$\frac{d}{dt}\tilde{u}(x,t) + \frac{\pi^3}{8}\tilde{u}(x,t) = -\frac{\pi^5}{32}\tilde{u}(x,t),$$

therefore

$$\frac{d}{dt}\tilde{u}(x,t) + 13.44\tilde{u}(x,t) = 0,$$

for finding $\tilde{u}(x,t)$, *we should solve this ODE* [5], *after that we get*

$$\tilde{u}(x,t) = \tilde{J}(x)\exp(-13.44t) + \tilde{K}$$

Now from (3.5), *the analytical solution is*

$$u(x,t) = J(x)\exp(-13.44t) + K.$$

By putting this solution in the KdV equation, we have

$$\frac{d^5}{dx^5}J(x) - \frac{d^3}{dx^3}J(x) + 13.44J(x) = 0.$$

after solving this 5*-th order ODE,* $J(X)$ *are obtained as follows*

$$\exp(1.44x)\{c_1\cos(1.26x)+c_2\sin(1.26x)\}-\exp(-1.44x)\{c_3\cos(1.26x)+c_4\sin(1.26x)\}+c_5.$$

where c_i*s for* $(i = 1, ..., 5)$ *are real arbitrary constants.*

Table 5 shows the results of wavelet transform method for KdV equation:
Table 5. WTM on the KdV equation.

Quasi-wavelet	The wavelet tranform	The analytic solution
ψ_1	$\tilde{u}(x,t) = \tilde{I}(x)\exp(13.44t) + \tilde{K}$	$u(x,t) = I(x)\exp(13.44t) + K$
ψ_2	$\tilde{u}(x,t) = \tilde{J}(x)\exp(-13.44t) + \tilde{K}$	$u(x,t) = J(x)\exp(-13.44t) + K$

Note that, in spite of the higher order of KdV (order 5), the final analytical solution is simple. The interesting point about WTM is that the final solution is often separable.

5. Conclusions and future works

In this paper, we proposed a novel method based on wavelets. Indeed, WTM has been inspired from Fourier transform and Laplace transform methods. The idea of using integral transforms for solving differential equations can be employed by wavelets. We produced new quasi-wavelets with two variables in accordance with differential invariants. Next, we calculated the correspondent wavelet transforms to apply them on differential equations and reduced the degree of PDEs. Afterward, the PDEs were solved and the solutions were obtained. Unlike the other applications of wavelets and wavelet transforms, WTM results in analytical and exact solutions and is based on the multi-dimensional wavelets constructed by differential invariants. Indeed, this is what sets our work different from other wavelet applications. As seen before, the crucial step is to provide proper quasi-wavelets based on differential invariants and, therefore this method can be used to solve the PDEs that equivalence methods (like the Lie symmetry method) can be applied on those. In future works, we will propose suitable quasi-wavelets for every differential invariant and symmetry group by implementing WTM on other PDEs. Moreover, we hope to generalize WTM for solving both linear and non-linear PDEs at every order and every number of independent variables.

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