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Estimates of Hermite-Hadamard Inequality for Twice Differentiable Harmonically-Convex Functions with Applications

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Abstract. By using an identity involving a twice differentiable mapping and mathematical analysis techniques, some new estimates are presented for the error bounds of $\left|\frac{1}{k_2-k_1}\int_{k_1}^{k_2} f(s)\,ds - \frac{f(k_1)+f(k_2)}{2}\right|$. We have derived some inequalities of special menas of positive real numbers as applications of the proven results.

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1. Introduction

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called convex function (in the classical sense) if the inequality

$$f(\zeta s_1 + (1-\zeta) s_2) \leq \zeta f(s_1) + (1-\zeta) f(s_2)$$

holds for all $s_1, s_2 \in I$ and $\zeta \in [0,1]$.

The history of the theory convex functions is very long. The commencement of the theory of convex functions can be found to be at the end of the nineteenth century. The roots of the theory of convex functions can be found in the fundamental contributions of O. Hölder [9], J. Hadamard [6] and O. Stolz [24]. In the beginning of the twentieth century, J. L. W. V. Jensen [15] was first mathematician who realized the importance of the convex functions and started the symmetric study of the convex functions. In years thereafter this research has given rise to the theory of convex functions as an independent discipline of mathematical analysis.

Inequalities play an important role in almost all the branches of mathematics as well as in the other areas of sciences. The theory of convex functions plays a pivotal role in the development of the theory of inequalities and hence it has been a subject of extensive research over the past few decades. A number of interesting results have been proved by using the concept of classical convexity.

The most widely studied result for convex functions is stated as follows:

If $f:[k_1,k_2]\subset\mathbb{R}\to\mathbb{R}$ is a convex function, the inequality

$$f\left(\frac{k_1+k_2}{2}\right)\le\frac{1}{k_2-k_1}\int_{k_1}^{k_2}f(s)\,ds\le\frac{f(k_1)+f(k_2)}{2} \tag{1. 1}$$

holds true. The double inequality (1. 1) is well-known in literature as Hermite-Hadamard inequality which was discovered independently by J. Hadamard and Ch. Hermite, see for instance [6] and [8].

The concept of classical convexity has been extended and generalized in several directions. One of the generalizations of classical convexity is the harmonic convexity stated in the definition below.

Definition 1.1. [12] *Let $I\subset\mathbb{R}\backslash\{0\}$ be a real interval. A function $f:I\to\mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{s_1s_2}{\zeta s_1+(1-\zeta)s_2}\right)\le\zeta f(s_2)+(1-\zeta)f(s_1) \tag{1. 2}$$

for all s_1, $s_2\in I$ and $\zeta\in[0,1]$. If the inequality in (1. 2) is reversed, then f is said to be harmonically concave.

The connection between the usual convexity and the harmonic convexity is well explained in Proposition 2.3 from [12].

In recent years, many mathematicians are trying to generalize and extend the notion of harmonic convexity. Most recently, a number of results on Hermite-Hadamard type inequalities and their applications have been produced by using different generalizations and extensions of harmonic convex functions, see for example [1]-[7], [14]-[11], [13], [16]-[23] and [25]-[27].

In Section 2, we present some new estimates by using twice differentiable harmonically-convex mappings.

2. New Results

We begin this section with the definition of some special functions to be used in the sequel of the paper.

These special functions are defined as follows

$$B\left(p,p^{'}\right)=\int_0^1\zeta^{p-1}(1-\zeta)^{p^{'}-1}\,d\zeta,p>0,p^{'}>0 \text{ (The Beta function)},$$

$$\Gamma(p)=\int_0^\infty\zeta^{p-1}e^{-\zeta}d\zeta,p>0 \text{ (The Gamma Functions)}$$

and

$$\begin{aligned}&{}_2F_1\left(p,p^{'};r;z\right)\\&=\frac{1}{B\left(p^{'},r-p^{'}\right)}\int_0^1\zeta^{p^{'}-1}(1-\zeta)^{r-p^{'}-1}(1-z\zeta)^{-p}\,d\zeta \text{ (The hypergeometric function)}\end{aligned}$$

where $|z|<1,r>p^{'}>0$.

The following lemma is important to prove estimates for the Hermite-Hadamard inequality for twice differentiable harmonically-convex mappings. These results provide new error bounds between the middle and the rightmost term in terms of second derivative which satisfies the harmonic convexity assumption. We will use the notation $\Upsilon_{\zeta}(k_1,k_2)=\zeta k_2+(1-\zeta)k_1$ for the convenience of the reader.

Lemma 2.1. *Let $f: I\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a differentiable mapping on I° and k_1, $k_2\in I^{\circ}$ with $k_1<k_2$. If $f^{''}\in L[k_1,k_2]$, then*

$$\frac{(k_2-k_1)^2}{4k_1^2k_2^2}\int_0^1\frac{\zeta(2-2\zeta)}{\Upsilon_{\zeta}^4(k_1,k_2)}f^{''}\left(\frac{k_1k_2}{\Upsilon_{\zeta}(k_1,k_2)}\right)d\zeta = \frac{f(k_1)+f(k_2)}{2}-\frac{1}{k_2-k_1}\int_{k_1}^{k_2}f(s)\,ds. \quad (2.\ 3)$$

Proof. By integration by parts, we have

$$\begin{aligned}
&\int_0^1\frac{\zeta(2-2\zeta)}{\Upsilon_{\zeta}^4(k_1,k_2)}f^{''}\left(\frac{k_1k_2}{\Upsilon_{\zeta}(k_1,k_2)}\right)d\zeta\\
&=-\frac{1}{k_1k_2(k_2-k_1)}\int_0^1\frac{\zeta(2-2\zeta)}{\Upsilon_{\zeta}^2(k_1,k_2)}\cdot\left[-\frac{k_1k_2(k_2-k_1)}{\Upsilon_{\zeta}^2(k_1,k_2)}\right]f^{''}\left(\frac{k_1k_2}{\Upsilon_{\zeta}(k_1,k_2)}\right)d\zeta\\
&=-\frac{1}{k_1k_2(k_2-k_1)}\int_0^1\frac{\zeta(2-2\zeta)}{\Upsilon_{\zeta}^2(k_1,k_2)}\cdot d\left[f^{'}\left(\frac{k_1k_2}{\Upsilon_{\zeta}(k_1,k_2)}\right)\right]d\zeta\\
&=-\frac{1}{k_1k_2(k_2-k_1)}\frac{\zeta(2-2\zeta)}{\Upsilon_{\zeta}^2(k_1,k_2)}f^{'}\left(\frac{k_1k_2}{\Upsilon_{\zeta}(k_1,k_2)}\right)\Bigg|_0^1\\
&\quad+\frac{1}{k_1k_2(k_2-k_1)}\int_0^1\frac{-2k_2\zeta+2(1-\zeta)k_1}{\Upsilon_{\zeta}^3(k_1,k_2)}f^{'}\left(\frac{k_1k_2}{\Upsilon_{\zeta}(k_1,k_2)}\right)d\zeta\\
&=\frac{1}{k_1k_2(k_2-k_1)}\int_0^1\frac{-2k_2\zeta+2(1-\zeta)k_1}{\Upsilon_{\zeta}^3(k_1,k_2)}f^{'}\left(\frac{k_1k_2}{\Upsilon_{\zeta}(k_1,k_2)}\right)d\zeta\\
&=-\frac{1}{k_1^2k_2^2(k_2-k_1)^2}\int_0^1\frac{-2k_2\zeta+2(1-\zeta)k_1}{\Upsilon_{\zeta}(k_1,k_2)}\cdot d\left[f^{'}\left(\frac{k_1k_2}{\Upsilon_{\zeta}(k_1,k_2)}\right)\right]d\zeta\\
&=-\frac{1}{k_1^2k_2^2(k_2-k_1)^2}\frac{-2k_2\zeta+2(1-\zeta)k_1}{\Upsilon_{\zeta}(k_1,k_2)}f\left(\frac{k_1k_2}{\Upsilon_{\zeta}(k_1,k_2)}\right)d\zeta\Bigg|_0^1\\
&\quad+\frac{1}{k_1^2k_2^2(k_2-k_1)^2}\int_0^1\frac{-4k_1k_2}{\Upsilon_{\zeta}^2(k_1,k_2)}f\left(\frac{k_1k_2}{\Upsilon_{\zeta}(k_1,k_2)}\right)d\zeta\\
&=\frac{2f(k_1)+2f(k_2)}{k_1^2k_2^2(k_2-k_1)^2}-\frac{4}{k_1k_2(k_2-k_1)^2}\int_0^1\frac{1}{\Upsilon_{\zeta}^2(k_1,k_2)}f\left(\frac{k_1k_2}{\Upsilon_{\zeta}(k_1,k_2)}\right)d\zeta. \quad (2.\ 4)
\end{aligned}$$

Making use of substitution $\frac{k_1k_2}{\Upsilon_\zeta(k_1,k_2)} = s$ in (2. 4), we get

$$\int_0^1 \frac{\zeta(2-2\zeta)}{\Upsilon_\zeta^4(k_1,k_2)} f''\left(\frac{k_1k_2}{\Upsilon_\zeta(k_1,k_2)}\right) d\zeta$$
$$= \frac{2f(k_1)+2f(k_2)}{k_1^2k_2^2(k_2-k_1)^2} - \frac{4}{k_1k_2(k_2-k_1)^2}\int_{k_2}^{k_1}\left(\frac{s}{k_1k_2}\right)^2\left[\frac{-k_1k_2}{(k_2-k_1)s^2}\right] f(s)\,ds$$
$$= \frac{2f(k_1)+2f(k_2)}{k_1^2k_2^2(k_2-k_1)^2} - \frac{4}{k_1^2k_2^2(k_2-k_1)^3}\int_{k_1}^{k_2} f(s)\,ds. \quad (2.\ 5)$$

Multiplying both sides of (2. 5) by $\frac{(k_2-k_1)^2}{4k_1^2k_2^2}$, we get required equality (2. 3). □

Theorem 2.2. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I° and k_1, $k_2 \in I^\circ$ with $k_1 < k_2$. If $f'' \in L[k_1,k_2]$ and $\left|f''\right|^d$ is harmonically convex on $[k_1,k_2]$ for $d \geq 1$, the following inequality holds*

$$\left|\frac{f(k_1)+f(k_2)}{2} - \frac{1}{k_2-k_1}\int_{k_1}^{k_2} f(s)\,ds\right|$$
$$\leq \frac{(k_2-k_1)^2}{4k_1^2k_2^2}\left(\frac{1}{3k_1^2k_2^2}\right)^{1-\frac{1}{d}}\left[\left\{\frac{2k_1^3+3k_1^2k_2-6k_1k_2^2+k_2^3+6k_1^2k_2\ln\left(\frac{k_2}{k_1}\right)}{3k_1^2k_2(k_2-k_1)^4}\right\}\left|f''(k_1)\right|^d\right.$$
$$\left.+\left\{\frac{k_1^3-6k_1^2k_2+3k_1k_2^2+2k_2^3-6k_1k_2^2\ln\left(\frac{k_2}{k_1}\right)}{3k_1k_2^2(k_2-k_1)^4}\right\}\left|f''(k_2)\right|^d\right]^{\frac{1}{d}}. \quad (2.\ 6)$$

Proof. Talking the absolute value on both sides of (2. 3) and applying the power-mean inequality, we get

$$\left|\frac{f(k_1)+f(k_2)}{2} - \frac{1}{k_2-k_1}\int_{k_1}^{k_2} f(s)\,ds\right|$$
$$\leq \frac{(k_2-k_1)^2}{4k_1^2k_2^2}\int_0^1 \frac{\zeta(2-2\zeta)}{(\zeta k_2+(1-\zeta)k_1)^4}\left|f''\left(\frac{k_1k_2}{\Upsilon_\zeta(k_1,k_2)}\right)\right| d\zeta \leq \frac{(k_2-k_1)^2}{4k_1^2k_2^2}$$
$$\times\left(\int_0^1 \frac{\zeta(2-2\zeta)}{\Upsilon_\zeta^4(k_1,k_2)}d\zeta\right)^{1-\frac{1}{d}}\left(\int_0^1 \frac{\zeta(2-2\zeta)}{\Upsilon_\zeta^4(k_1,k_2)}\left|f''\left(\frac{k_1k_2}{\Upsilon_\zeta(k_1,k_2)}\right)\right|^d d\zeta\right)^{\frac{1}{d}}. \quad (2.\ 7)$$

By integration by parts, we get

$$\begin{aligned}\int_0^1 \frac{\zeta(2-2\zeta)}{\Upsilon_\zeta^4(k_1,k_2)}d\zeta &= \frac{1}{k_2-k_1}\int_0^1 \frac{\zeta(2-2\zeta)(k_2-k_1)}{\Upsilon_\zeta^4(k_1,k_2)}d\zeta \\ &= -\frac{\zeta(2-2\zeta)}{3(k_2-k_1)\Upsilon_\zeta^3(k_1,k_2)}\Bigg|_0^1 + \frac{1}{3(k_2-k_1)}\int_0^1 \frac{(2-4\zeta)}{\Upsilon_\zeta^3(k_1,k_2)}d\zeta \\ &= \frac{1}{3(k_2-k_1)^2}\int_0^1 \frac{(2-4\zeta)(k_2-k_1)}{\Upsilon_\zeta^3(k_1,k_2)}d\zeta \\ &= -\frac{(2-4\zeta)}{6(k_2-k_1)^2\Upsilon_\zeta^2(k_1,k_2)}\Bigg|_0^1 - \frac{2}{3(k_2-k_1)^3}\int_0^1 \frac{(k_2-k_1)}{\Upsilon_\zeta^2(k_1,k_2)}d\zeta \\ &= \frac{1}{3(k_2-k_1)^2k_2^2} + \frac{1}{3(k_2-k_1)^2k_1^2} + \frac{2}{3(k_2-k_1)^3}\left(\frac{1}{k_2}-\frac{1}{k_1}\right) = \frac{1}{3k_1^2k_2^2}. \quad (2.\ 8)\end{aligned}$$

By using the harmonic convexity of $\left|f''\right|^d$ on $[k_1, k_2]$, we have

$$\begin{aligned}&\int_0^1 \frac{\zeta(2-2\zeta)}{\Upsilon_\zeta^4(k_1,k_2)}\left|f''\left(\frac{k_1k_2}{\Upsilon_\zeta(k_1,k_2)}\right)\right|^d d\zeta \\ &\leq \int_0^1 \frac{\zeta(2-2\zeta)}{\Upsilon_\zeta^4(k_1,k_2)}\left[\zeta\left|f''(k_1)\right|^d + (1-\zeta)\left|f''(k_2)\right|^d\right]d\zeta \\ &= \left|f''(k_1)\right|^d\int_0^1 \frac{2\zeta^2(1-\zeta)}{\Upsilon_\zeta^4(k_1,k_2)}d\zeta + \left|f''(k_2)\right|^d\int_0^1 \frac{2\zeta(1-\zeta)^2}{\Upsilon_\zeta^4(k_1,k_2)}d\zeta. \quad (2.\ 9)\end{aligned}$$

By integration by parts, we also get that

$$\begin{aligned}\int_0^1 \frac{2\zeta^2(1-\zeta)}{\Upsilon_\zeta^4(k_1,k_2)}d\zeta &= \frac{1}{k_2-k_1}\int_0^1 2\zeta^2(1-\zeta)\Upsilon_\zeta^{-4}(k_1,k_2)(k_2-k_1)\,d\zeta \\ &= \frac{2k_1^3+3k_1^2k_2-6k_1k_2^2+k_2^3+6k_1^2k_2\ln\left(\frac{k_2}{k_1}\right)}{3k_1^2k_2(k_2-k_1)^4}. \quad (2.\ 10)\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \frac{2\zeta(1-\zeta)^2}{\Upsilon_\zeta^4(k_1,k_2)}d\zeta &= \frac{1}{k_2-k_1}\int_0^1 2\zeta(1-\zeta)^2\Upsilon_\zeta^{-4}(k_1,k_2)(k_2-k_1)\,d\zeta \\ &= \frac{k_1^3-6k_1^2k_2+3k_1k_2^2+2k_2^3-6k_1k_2^2\ln\left(\frac{k_2}{k_1}\right)}{3k_1k_2^2(k_2-k_1)^4} \quad (2.\ 11)\end{aligned}$$

A combination of (2. 8)-(2. 11) and (2. 7) gives the required result. □

The next results can be proved by using the Hölder inequality and the Hypergeometric function.

Theorem 2.3. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° and $k_1, k_2 \in I^\circ$ with $k_1 < k_2$. If $f'' \in L[k_1,k_2]$ and $\left|f''\right|^d$ is harmonically convex on $[k_1,k_2]$ for $d > 1$, the following inequality holds*

$$\left|\frac{f(k_1)+f(k_2)}{2} - \frac{1}{k_2-k_1}\int_{k_1}^{k_2} f(s)\,ds\right| \leq \frac{(k_2-k_1)^2}{2k_1^2k_2^6}\left(\frac{d-1}{2d-1}\right)^{1-\frac{1}{d}}\left(\frac{{}_2F_1\left(4d,d+1,d+3;1-\frac{k_1}{k_2}\right)}{(d+2)(d+1)}\left|f''(k_1)\right|^d + \frac{{}_2F_1\left(4d,d+2,d+3;1-\frac{k_1}{k_2}\right)}{d+2}\left|f''(k_2)\right|^d\right)^{\frac{1}{d}}. \quad (2.\ 12)$$

Proof. Talking the absolute value on both sides of (2. 3) and applying the Hölder inequality, we get

$$\left|\frac{f(k_1)+f(k_2)}{2} - \frac{1}{k_2-k_1}\int_{k_1}^{k_2} f(s)\,ds\right| \leq \frac{(k_2-k_1)^2}{4k_1^2k_2^2}\int_0^1 \frac{\zeta(2-2\zeta)}{\Upsilon_\zeta^4(k_1,k_2)}\left|f''\left(\frac{k_1k_2}{\Upsilon_\zeta(k_1,k_2)}\right)\right|d\zeta \leq \frac{(k_2-k_1)^2}{4k_1^2k_2^2} \times\left(\int_0^1 \zeta^{\frac{d}{d-1}}d\zeta\right)^{1-\frac{1}{d}}\left(\int_0^1 \frac{(2-2\zeta)^d}{\Upsilon_\zeta^{4d}(k_1,k_2)}\left|f''\left(\frac{k_1k_2}{\Upsilon_\zeta(k_1,k_2)}\right)\right|^d d\zeta\right)^{\frac{1}{d}}. \quad (2.\ 13)$$

Since $\left|f''\right|^d$ is harmonically convex on $[k_1,k_2]$ for $d > 1$, we get

$$\int_0^1 \frac{(2-2\zeta)^d}{\Upsilon_\zeta^{4d}(k_1,k_2)}\left|f''\left(\frac{k_1k_2}{\Upsilon_\zeta(k_1,k_2)}\right)\right|^d d\zeta \leq \int_0^1 \frac{(2-2\zeta)^d}{\Upsilon_\zeta^{4d}(k_1,k_2)}\left[\zeta\left|f''(k_1)\right|^d + (1-\zeta)\left|f''(k_2)\right|^d\right]d\zeta = \frac{2^dk_2^{-4d}\,{}_2F_1\left(4d,d+1,d+3;1-\frac{k_1}{k_2}\right)}{(d+2)(d+1)}\left|f''(k_1)\right|^d + \frac{2^dk_2^{-4d}\,{}_2F_1\left(4d,d+2,d+3;1-\frac{k_1}{k_2}\right)}{d+2}\left|f''(k_2)\right|^d. \quad (2.\ 14)$$

The proof follows if we use (2. 14) in (2. 13). □

Theorem 2.4. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° and $k_1, k_2 \in I^\circ$ with $k_1 < k_2$. If $f'' \in L[k_1,k_2]$ and $\left|f''\right|^d$ is harmonically convex on $[k_1,k_2]$ for $d > 1$,* *the following inequality holds*

$$\left|\frac{f(k_1)+f(k_2)}{2}-\frac{1}{k_2-k_1}\int_{k_1}^{k_2} f(s)\,ds\right| \leq \frac{(k_2-k_1)^2}{2k_1^2k_2^6}\left(\frac{d-1}{2d-1}\right)^{1-\frac{1}{d}}$$
$$\times\left(\frac{{}_2F_1\left(4d,1,d+3;1-\frac{k_1}{k_2}\right)}{d+2}\left|f^{''}(k_1)\right|^d\right.$$
$$\left.+\frac{{}_2F_1\left(4d,2,d+3;1-\frac{k_1}{k_2}\right)}{(d+1)(d+2)}\left|f^{''}(k_2)\right|^d\right)^{\frac{1}{d}}. \quad (2.\ 15)$$

Proof. Talking the absolute value on both sides of (2. 3), applying the Hölder inequality and using the harmonic convexity of $\left|f^{''}\right|^d$ on $[k_1,k_2]$, we obtain

$$\left|\frac{f(k_1)+f(k_2)}{2}-\frac{1}{k_2-k_1}\int_{k_1}^{k_2} f(s)\,ds\right|$$
$$\leq \frac{(k_2-k_1)^2}{4k_1^2k_2^2}\int_0^1 \frac{\zeta(2-2\zeta)}{\Upsilon_\zeta^4(k_1,k_2)}\left|f^{''}\left(\frac{k_1k_2}{\Upsilon_\zeta(k_1,k_2)}\right)\right|d\zeta$$
$$\leq \frac{(k_2-k_1)^2}{4k_1^2k_2^2}\left(\int_0^1 (2-2\zeta)^{\frac{d}{d-1}}\,d\zeta\right)^{1-\frac{1}{d}}$$
$$\times\left(\left|f^{''}(k_1)\right|^d\int_0^1\frac{\zeta^{d+1}d\zeta}{\Upsilon_\zeta^{4d}(k_1,k_2)}+\left|f^{''}(k_2)\right|^d\int_0^1\frac{\zeta^d(1-\zeta)\,d\zeta}{\Upsilon_\zeta^{4d}(k_1,k_2)}\right)^{\frac{1}{d}}. \quad (2.\ 16)$$

By the definition of Hypergeometric function, we observe that

$$\int_0^1\frac{\zeta^{d+1}}{\Upsilon_\zeta^{4d}(k_1,k_2)}d\zeta=\frac{k_2^{-4d}\,{}_2F_1\left(4d,1,d+3;1-\frac{k_1}{k_2}\right)}{d+2}$$

and

$$\int_0^1\frac{\zeta^{d}(1-\zeta)}{\Upsilon_\zeta^{4d}(k_1,k_2)}d\zeta=\frac{k_1^{-4d}\,{}_2F_1\left(4d,2,d+3;1-\frac{k_2}{k_1}\right)}{(d+1)(d+2)}.$$

Using the above observations in (2. 16), we get (2. 15). □

Theorem 2.5. *Let* $f: I\subseteq(0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* I° *and* $k_1, k_2\in I^\circ$ *with* $k_1<k_2$. *If* $f^{''}\in L[k_1,k_2]$ *and* $\left|f^{''}\right|^d$ *is harmonically convex on* $[k_1,k_2]$ *for* $d>1$, *the following inequality holds*

$$\left|\frac{f(k_1)+f(k_2)}{2}-\frac{1}{k_2-k_1}\int_{k_1}^{k_2} f(s)\,ds\right| \leq \frac{(k_2-k_1)^2}{2k_1^2k_2^2}$$
$$\times\left(L_{\frac{4d}{1-d}}^{-4}(k_1,k_2)\right)\left[B(d+2,d+1)\right]^{\frac{1}{d}}\left(\left|f''(k_1)\right|^d+\left|f''(k_2)\right|^d\right)^{\frac{1}{d}}. \quad (2.\ 17)$$

Proof. Talking the absolute value on both sides of (2. 3), applying the Hölder inequality and using the harmonic convexity of $\left|f''\right|^d$ on $[k_1, k_2]$, we obtain

$$\left|\frac{f(k_1)+f(k_2)}{2}-\frac{1}{k_2-k_1}\int_{k_1}^{k_2} f(s)\,ds\right|$$
$$\leq \frac{(k_2-k_1)^2}{4k_1^2k_2^2}\int_0^1 \frac{\zeta(2-2\zeta)}{\Upsilon_\zeta^4(k_1,k_2)}\left|f''\left(\frac{k_1k_2}{\Upsilon_\zeta(k_1,k_2)}\right)\right|d\zeta$$
$$\leq \frac{(k_2-k_1)^2}{4k_1^2k_2^2}\left(\int_0^1 \frac{d\zeta}{\Upsilon_\zeta^{\frac{4d}{d-1}}(k_1,k_2)}\right)^{1-\frac{1}{d}}$$
$$\times\left(2^d\left|f''(k_1)\right|^d\int_0^1 \zeta^{d+1}(1-\zeta)^d\,d\zeta+2^d\left|f''(k_2)\right|^d\int_0^1 \zeta^d(1-\zeta)^{d+1}\,d\zeta\right)^{\frac{1}{d}}$$
$$=\frac{(k_2-k_1)^2}{2k_1^2k_2^2}L_{\frac{4d}{1-d}}^{-4}(k_1,k_2)$$
$$\times\left(B(d+2,d+1)\left|f''(k_1)\right|^d+B(d+1,d+2)\left|f''(k_2)\right|^d\right)^{\frac{1}{d}}. \quad (2.\ 18)$$

Since

$$B(d+2,d+1)=B(d+1,d+2),$$

hence the inequality (2. 17) is achieved. □

Theorem 2.6. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° and $k_1, k_2 \in I^\circ$ with $k_1 < k_2$. If $f'' \in L[k_1,k_2]$ and $\left|f''\right|^d$ is harmonically convex on $[k_1,k_2]$ for $d > 1$, the following inequality holds*

$$\left|\frac{f(k_1)+f(k_2)}{2}-\frac{1}{k_2-k_1}\int_{k_1}^{k_2} f(s)\,ds\right| \leq \frac{(k_2-k_1)^2}{2k_1^2k_2^2}\left[B\left(\frac{2d-1}{d-1},\frac{2d-1}{d-1}\right)\right]^{1-\frac{1}{d}}$$
$$\times\left(\chi(k_2,k_1;d)\left|f''(k_1)\right|^d+\chi(k_1,k_2;d)\left|f''(k_2)\right|^d\right)^{\frac{1}{d}}, \quad (2.\ 19)$$

where

$$\chi(k_1,k_2;d)=\frac{k_2^{2-4d}-k_1^{1-4d}\left[2k_2(1-2d)+(4d-1)k_1\right]}{2(k_1-k_2)^2(2d-1)(4d-1)}.$$

Proof. Talking the absolute value on both sides of (2. 3), applying the Hölder inequality and using the harmonic convexity of $\left|f^{''}\right|^{d}$ on $[k_1, k_2]$, we obtain

$$\begin{aligned}
&\left|\frac{f(k_1)+f(k_2)}{2}-\frac{1}{k_2-k_1}\int_{k_1}^{k_2} f(s)\,ds\right| \\
&\qquad\leq \frac{(k_2-k_1)^2}{4k_1^2k_2^2}\int_0^1 \frac{\zeta(2-2\zeta)}{\Upsilon_\zeta^4(k_1,k_2)}\left|f^{''}\left(\frac{k_1k_2}{\Upsilon_\zeta(k_1,k_2)}\right)\right|d\zeta \\
&\qquad\leq \frac{(k_2-k_1)^2}{4k_1^2k_2^2}\left(\int_0^1 \zeta^{\frac{d}{d-1}}(2-2\zeta)^{\frac{d}{d-1}}\,d\zeta\right)^{1-\frac{1}{d}} \\
&\times\left(\left|f^{''}(k_2)\right|^d\int_0^1\frac{(1-\zeta)}{\Upsilon_\zeta^{4d}(k_1,k_2)}d\zeta+\left|f^{''}(k_1)\right|^d\int_0^1\frac{\zeta}{\Upsilon_\zeta^{4d}(k_1,k_2)}d\zeta\right)^{\frac{1}{d}}. \qquad (2.\ 20)
\end{aligned}$$

By evaluating the integrals involved in the above inequality, we have

$$\int_0^1 \zeta^{\frac{d}{d-1}}(2-2\zeta)^{\frac{d}{d-1}}\,d\zeta = 2^{\frac{d}{d-1}}\int_0^1 \zeta^{\frac{d}{d-1}}(1-\zeta)^{\frac{d}{d-1}}\,d\zeta = 2^{\frac{d}{d-1}}B\left(\frac{2d-1}{d-1},\frac{2d-1}{d-1}\right),$$

$$\begin{aligned}
\int_0^1\frac{(1-\zeta)}{\Upsilon_\zeta^{4d}(k_1,k_2)}d\zeta &= \frac{k_2^{2-4d}-k_1^{1-4d}\left[2k_2(1-2d)+(4d-1)k_1\right]}{2(k_1-k_2)^2(2d-1)(4d-1)} \\
&= \chi(k_1,k_2;d)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1\frac{\zeta}{\Upsilon_\zeta^{4d}(k_1,k_2)}d\zeta &= \frac{k_1^{2-4d}-k_2^{1-4d}\left[2k_1(1-2d)+(4d-1)k_2\right]}{2(k_1-k_2)^2(2d-1)(4d-1)} \\
&= \chi(k_2,k_1;d).
\end{aligned}$$

Substituting the values of the above integrals in (2. 20), we get the result (2. 19). Hence the proof of the theorem is accomplished. □

Theorem 2.7. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° and k_1, $k_2 \in I^\circ$ with $k_1 < k_2$. If $f^{''} \in L[k_1,k_2]$ and $\left|f^{''}\right|^d$ is harmonically convex on $[k_1,k_2]$ for $d > 1$, the following inequality holds*

$$\begin{aligned}
\left|\frac{f(k_1)+f(k_2)}{2}-\frac{1}{k_2-k_1}\int_{k_1}^{k_2} f(s)\,ds\right| &\leq \frac{(k_2-k_1)^2}{2k_1^2k_2^{2d}}\left[\psi_1(k_1,k_2;d)\right]^{1-\frac{1}{d}} \\
&\times\left[\chi\left(k_2,k_1;\frac{d}{2}\right)\left|f^{''}(k_1)\right|^d+\chi\left(k_1,k_2;\frac{d}{2}\right)\left|f^{''}(k_2)\right|^d\right]^{\frac{1}{d}}, \qquad (2.\ 21)
\end{aligned}$$

where

$$\chi\left(k_1,k_2;\frac{d}{2}\right) = \frac{k_2^{2-2d}-k_1^{1-2d}\left[2k_2(1-d)+(2d-1)k_1\right]}{2(k_1-k_2)^2(d-1)(2d-1)}$$

and

$$\psi_1(k_1,k_2;d)=B\left(\frac{2d-1}{d-1},\frac{2d-1}{d-1}\right)\ {}_2F_1\left(2d,\frac{2d-1}{d-1},\frac{2(2d-1)}{d-1};1-\frac{k_1}{k_2}\right).$$

Proof. Talking the absolute value on both sides of (2. 3), applying the Hölder inequality and using the harmonic convexity of $\left|f''\right|^d$ on $[k_1,k_2]$, we obtain

$$\begin{aligned}
&\left|\frac{f(k_1)+f(k_2)}{2}-\frac{1}{k_2-k_1}\int_{k_1}^{k_2}f(s)\,ds\right|\\
&\qquad\le\frac{(k_2-k_1)^2}{4k_1^2k_2^2}\int_0^1\frac{\zeta(2-2\zeta)}{\Upsilon_\zeta^4(k_1,k_2)}\left|f''\left(\frac{k_1k_2}{\Upsilon_\zeta(k_1,k_2)}\right)\right|d\zeta\\
&\qquad\le\frac{(k_2-k_1)^2}{4k_1^2k_2^2}\left(\int_0^1\frac{\zeta^{\frac{d}{d-1}}(2-2\zeta)^{\frac{d}{d-1}}}{\Upsilon_\zeta^{2d}(k_1,k_2)}d\zeta\right)^{1-\frac{1}{d}}\\
&\times\left(\left|f''(k_2)\right|^d\int_0^1\frac{(1-\zeta)}{\Upsilon_\zeta^{2d}(k_1,k_2)}d\zeta+\left|f''(k_1)\right|^d\int_0^1\frac{\zeta}{\Upsilon_\zeta^{2d}(k_1,k_2)}d\zeta\right)^{\frac{1}{d}}.
\end{aligned}\tag{2. 22}$$

By evaluating the integrals involved in the above inequality, we have

$$\begin{aligned}
&\int_0^1\frac{\zeta^{\frac{d}{d-1}}(2-2\zeta)^{\frac{d}{d-1}}}{\Upsilon_\zeta^{2d}(k_1,k_2)}d\zeta\\
&=\frac{2^{\frac{d}{d-1}}\cdot B\left(\frac{2d-1}{d-1},\frac{2d-1}{d-1}\right)}{k_2^{2d}}\cdot\frac{1}{B\left(\frac{2d-1}{d-1},\frac{2d-1}{d-1}\right)}\\
&\times\int_0^1\zeta^{\frac{d}{d-1}}(1-\zeta)^{\frac{d}{d-1}}\left(1-\left(1-\frac{k_1}{k_2}\right)\zeta\right)^{-2d}d\zeta\\
&=\frac{2^{\frac{d}{d-1}}\cdot B\left(\frac{2d-1}{d-1},\frac{2d-1}{d-1}\right)}{k_2^{2d}}\ {}_2F_1\left(2d,\frac{2d-1}{d-1},\frac{2(2d-1)}{d-1};1-\frac{k_1}{k_2}\right)\\
&=\frac{2^{\frac{d}{d-1}}}{k_2^{2d}}\psi_1(k_1,k_2;d),
\end{aligned}$$

$$\begin{aligned}
\int_0^1\frac{(1-\zeta)}{\Upsilon_\zeta^{2d}(k_1,k_2)}d\zeta&=\frac{k_2^{2-2d}-k_1^{1-2d}\left[2k_2(1-d)+(2d-1)k_1\right]}{2(k_1-k_2)^2(d-1)(2d-1)}\\
&=\chi\left(k_1,k_2;\frac{d}{2}\right)
\end{aligned}$$

and

$$\int_0^1 \frac{\zeta}{\Upsilon_\zeta^{2d}(k_1,k_2)} d\zeta = \frac{k_1^{2-2d} - k_2^{1-2d}\left[2k_1(1-d) + (2d-1)k_2\right]}{2(k_1-k_2)^2(d-1)(2d-1)}$$
$$= \chi\left(k_2, k_1; \frac{d}{2}\right).$$

By applying the values of the above integrals in (2. 22), we get the result (2. 21). Hence the proof of the theorem is accomplished. □

3. Applications to Means

Let k_1, $k_2 > 0$, then the following means are well-known in mathematical literature

(1) The Arithmetic Mean

$$A = A(k_1,k_2) := \frac{k_1+k_2}{2},$$

(2) The geometric mean

$$G = G(k_1,k_2) := \sqrt{k_1 k_2},$$

(3) The harmonic mean

$$H = H(k_1,k_2) := \frac{2k_1k_2}{k_1+k_2},$$

(4) The Logarithmic mean

$$L = L(k_1,k_2) := \frac{k_2-k_1}{\ln k_2 - \ln k_1},$$

(5) The p-Logarithmic mean

$$L_p = L_p(k_1,k_2) := \left[\frac{k_2^{p+1}-k_1^{p+1}}{(p+1)(k_2-k_1)}\right]^{\frac{1}{p}}, p \in \mathbb{R}\setminus\{-1,0\},$$

and

(6) The Identric mean

$$I = I(k_1,k_2) := \frac{1}{e}\left(\frac{k_2^{k_1}}{k_1^{k_2}}\right)^{\frac{1}{k_2-k_1}}.$$

The inequalities $H \leq G \leq L \leq I \leq A$ hold for these means. It should be noted that L_p is monotonically increasing for $p \in \mathbb{R}$ and L_0 and L_{-1} are denoted by I and L respectively.

We use the result of Theorem 2.2 to get some inequalities of means.

Proposition 3.1. *Let $0 < k_1 < k_2$ and $d \geq 1$, the following inequality holds*

$$\left|A\left(k_1^{\frac{1}{d}+2},k_2^{\frac{1}{d}+2}\right)-L_{\frac{1}{d}+2}^{\frac{1}{d}+2}\left(k_1,k_2\right)\right|\leq\frac{\left(k_2-k_1\right)^2\left(d+1\right)\left(2d+1\right)}{4d^2G^2\left(k_1^2,k_2^2\right)}$$
$$\times\left(\frac{1}{3G^2\left(k_1^2,k_2^2\right)}\right)^{1-\frac{1}{d}}\left\{\frac{2}{\left(k_2-k_1\right)^2L\left(k_1,k_2\right)}\right.$$
$$\left.+\frac{2A\left(k_1,k_2\right)\left[4A^2\left(k_1,k_2\right)-7G^2\left(k_1,k_2\right)\right]}{3\left(k_2-k_1\right)^2G^2\left(k_1^2,k_2^2\right)}\right\}^{\frac{1}{d}}. \quad (3.\ 23)$$

Proof. Let $f(s)=\frac{d^2s^{\frac{1}{d}+2}}{(d+1)(2d+1)}$, $s\in(0,\infty)$, then $\left|f''(s)\right|^d=s$ is harmonically-convex on $[k_1,k_2]$. Applying the result of Theorem 2.2, we get the required result. □

Corollary 3.2. *Let $0<k_1<k_2$, the following inequality holds*

$$\left|A\left(k_1^3,k_2^3\right)-L_3^3\left(k_1,k_2\right)\right|$$
$$\leq\frac{1}{G^2\left(k_1^2,k_2^2\right)}\left\{\frac{A\left(k_1,k_2\right)\left[4A^2\left(k_1,k_2\right)-7G^2\left(k_1,k_2\right)\right]}{G^2\left(k_1^2,k_2^2\right)}+\frac{3}{L\left(k_1,k_2\right)}\right\}. \quad (3.\ 24)$$

Proof. Let $d=1$, then the inequality (3. 24) holds true. □

Proposition 3.3. *Let $0<k_1<k_2$ and $p\in(-1,\infty)\setminus\{0\}$, the following inequality holds*

$$\left|A\left(k_1^{p+2},k_2^{p+2}\right)-L_{p+2}^{p+2}\left(k_1,k_2\right)\right|$$
$$\leq\frac{1}{4G^2\left(k_1^2,k_2^2\right)}\left[\frac{2\left(p+2\right)L_{p+1}^{p+1}\left(k_1,k_2\right)}{L\left(k_1,k_2\right)}-\frac{\left(p+3\right)L_{p+2}^{p+2}\left(k_1,k_2\right)}{3G^2\left(k_1,k_2\right)}\right.$$
$$\left.+\frac{\left(p+5\right)L_{p+5}^{p+5}\left(k_1,k_2\right)}{3G^2\left(k_1^2,k_2^2\right)}-\frac{2}{3}\left(p+1\right)L_p^p\left(k_1,k_2\right)\right]. \quad (3.\ 25)$$

Proof. Consider the function $f(s)=\frac{s^{p+4}}{(p+4)(p+3)}$, $s\in(0,\infty)$. Then $\left|f''(s)\right|=s^{p+2}$ is convex and nondecreasing on $[k_1,k_2]\subset(0,\infty)$. We know that if $I\subset(0,\infty)$ and f is convex and nondecreasing function on I then f is harmonically convex on I. Using the result of Theorem 2.2, we get the required result. □

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