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Application of Taylor Expansion for Fredholm Integral Equations of the First Kind

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Abstract. This investigation intends to provide a new application of Taylor expansion approach for solving first kind Fredholm integral equations. The approach is based on employing the νth-degree Taylor polynomial of unknown function at an arbitrary point and integration method such that the first kind Fredholm integral equation is converted into a linear equations system with respect to unknowns and its derivatives up to order ν. Solving this system will result in a desired solution. A considerable advantage of the suggested approach is that for such cases when the true solution is a polynomial function of degree at most ν, the derived νth-degree approximation is equal to true solution. An error analysis is represented and to verify the effectively and the accuracy of the proposed approach six examples are investigated.

AMS (MOS) Subject Classification Codes: 65R20
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1. Introduction

In this paper, we try to find the approximate solution of the linear first kind Fredholm integral equations.

$$\int_a^b K(s,u)\psi(u)du = f(s), \qquad a \le s \le b, \tag{1. 1}$$

where $K(s,u)$ and $f(s)$ are given known functions and $\psi(s)$ is the unknown function to be determined. In general, Eq. (1. 1) is ill-posed; when $K(s,u)$ is considered as a smooth function, then a negligible change in $f(s)$ may cause a large change in $\psi(s)$ and any numerical method must take into account of this [4, 5, 6, 9, 12, 13, 14, 15, 20, 22, 21, 33, 34, 36, 37, 38, 39, 40, 41].

Such equations usually occur in the signal processing theory [46] and appear in many physical models including spectroscopy, cosmic radiation, stereology, radiography, electromagnetic fields, image processing and so on [46].

To solve first kind Fredholm integral equations, several numerical methods have been proposed, in the literature, such as expansion method [7, 27, 35], regularization method [30], Galerkin method [1, 2, 7, 13, 28, 47], wavelets [3, 8, 28, 29, 32, 42], and collocation method [25, 29, 31].

This research aims to propose an approximate approach in order to solve first kind Fredholm integral equation implementing a new application of Taylor expansion which has been proposed by Li [23] (see also [10, 11, 16, 17, 18, 24, 43, 44, 45]) and considering the fact that it has not been used before, for solving first kind Fredholm integral equation. Using νth-degree Taylor polynomial of unknown function and employing integration method, the first kind Fredholm integral equation could be converted into a linear equations system with respect to unknown function and its derivatives. An intended approximate solution is determined by solving the obtained system according to a standard method. The obtained results are compared with those of reported by applying different approaches. In the present paper, the main advantage of the presented approach is that a νth-degree approximation matches the exact solution if the exact solution is a polynomial function of degree at most ν.

The rest of this paper is organized as follows. In Section 2, an approach for solving Fredholm integral equation of the first kind is described. In Section 3, the convergence analysis for approximate solution is discussed. In Section 4, several numerical examples are solved to demonstrate the effectiveness of the approach. In Section 5, some tentative conclusions will be drawn.

2. DESCRIPTION OF THE METHOD

To estimate the solution of first kind Fredholm integral equation (1. 1), Eq. (1. 1) is converted into a linear equations system with respect to unknown and its derivatives. Based on the method used in [10, 11, 16, 17, 18, 23, 24, 43, 44, 45], it is supposed that the unknown function $\psi(u)$ is $\nu+1$ times continuously differentiable. Therefore, $\psi(u)$ is expressed in terms of the νth-degree Taylor series at an arbitrary point s as

$$\psi(u) = \psi(s) + \psi'(s)(u-s) + \cdots + \frac{1}{\nu!}\psi^{(\nu)}(x)(u-s)^{\nu} + E_{\nu}(u,s), \tag{2. 2}$$

where $E_{\nu}(u,s)$ indicates the Lagrange error bound

$$E_{\nu}(u,s) = \frac{\psi^{(\nu+1)}(\xi)}{(\nu+1)!}(u-s)^{\nu+1}, \tag{2. 3}$$

for some point ξ between s and u. Generally, the Lagrange error bound $E_{\nu}(u,s)$ becomes sufficiently small as ν gets great enough provided that $\psi^{(\nu+1)}(s)$ is a uniformly bounded function. Note that the Lagrange error bound becomes zero for a polynomial function of degree at most ν, thus the above νth degree Taylor expansion is equal to exact solution. With due attention to aforementioned assumption, by omitting the last Lagrange error bound, the truncated Taylor expansion $\psi(u)$ will be obtained as

$$\psi(u) \approx \sum_{k=0}^{\nu} \psi^{(k)}(s) \frac{(u-s)^k}{k!}. \qquad (2.\ 4)$$

Inserting the approximate relation (2. 4), for unknown function $\psi(u)$, into Eq. (1. 1) leads to

$$\sum_{k=0}^{\nu} \frac{(-1)^k}{k!} \psi^{(k)}(s) \int_a^b K(s,u)(s-u)^k du = f(s), \qquad (2.\ 5)$$

that can be simplified as

$$v_{00}(s)\psi(s) + v_{01}(s)\psi'(s) + \ldots + v_{0\nu}(s)\psi^{(\nu)}(s) = f(s), \qquad (2.\ 6)$$

where

$$v_{0k}(s) = \frac{(-1)^k}{k!} \int_a^b K(s,u)(s-u)^k du, \qquad k = 0, \ldots, \nu. \qquad (2.\ 7)$$

In fact, Eq. (1. 1) is converted into a νth-order linear ODE with respect to $\psi(s)$ and its derivations up to order ν. In the following, to determine $\psi(s)$,…,$\psi^{(\nu)}(s)$, a linear equations system has to be solved. To accomplish this goal, other ν independent linear equations with respect to $\psi(s)$ and its derivatives up to order ν are required, which is derived by integrating both sides of Eq. (1. 1) ν times with respect to s from a to s. Therefore, we have

$$\int_a^b \int_a^s \frac{(s-t)^{i-1}}{(i-1)!} K(t,u)\psi(u) dt du = f_{(i)}(s), \quad i = 1, \ldots, \nu, \qquad (2.\ 8)$$

where

$$f_{(i)}(s) = \int_a^s \frac{(s-u)^{i-1}}{(i-1)!} f(u) du, \quad i = 1, \ldots, \nu. \qquad (2.\ 9)$$

Now, again we apply approximate relation (2. 4) and after substituting (2. 4) for $\psi(u)$ into Eq. (2. 8), we obtain

$$\sum_{k=0}^{\nu} \frac{(-1)^k}{k!} \psi^{(k)}(s) \int_a^b \int_a^s \frac{(s-t)^{i-1}}{(i-1)!} (s-u)^k K(t,u) dt du = f_{(i)}(s),\ i = 1, \cdots, \nu, \quad (2.\ 10)$$

or equivalently

$$v_{i0}(s)\psi(s) + v_{i1}(s)\psi'(s) + \ldots + v_{i\nu}(s)\psi^{(\nu)}(s) = f_{(i)}(s), \qquad i = 1, \ldots, \nu, \quad (2.\ 11)$$

where

$$v_{ik}(s) = \frac{(-1)^k}{k!(i-1)!} \int_a^b \int_a^s (s-t)^{i-1}(s-u)^k K(t,u) dt du, \quad k = 0, \ldots, \nu. \quad (2.\ 12)$$

Thus, Eqs. (2. 6) and (2. 11) construct a linear equations system with respect to the unknown function $\psi(s)$ and its derivatives up to order n. Now, we rewrite this system as follows

$$V(s)\Psi(s) = F(s), \qquad (2.\ 13)$$

where

$$V(s) = \begin{bmatrix} v_{00}(s) & v_{01}(s) & \cdots & v_{0\nu}(s) \\ v_{10}(s) & v_{11}(s) & \cdots & v_{0\nu}(s) \\ \vdots & \vdots & \ddots & \vdots \\ v_{\nu 0}(s) & v_{\nu 1}(s) & \cdots & v_{\nu\nu}(s) \end{bmatrix}, \tag{2. 14}$$

$$F(s) = \begin{bmatrix} f(s) \\ f_{(1)}(s) \\ \vdots \\ f_{(v)}(s) \end{bmatrix}, \quad \Psi(s) = \begin{bmatrix} \psi(s) \\ \psi'(s) \\ \psi''(s) \\ \vdots \\ \psi^{(\nu)}(s) \end{bmatrix}. \tag{2. 15}$$

Using a standard method to the resulting system of equations yields a νth-degree approximate solution of Eq. (1. 1) which is indicated as $\psi_\nu(s)$.

3. Error analysis

The current section is devoted to giving an error analysis for derived νth-degree approximate solution of the first kind Fredholm integral equation (1. 1) in a similar way that has been discussed in [17]. It is assumed that the exact solution $\psi(u)$ be an infinitely differentiable function on the interval of interest I. As a matter of fact, $\psi(u)$ can be expanded as an uniformly convergent Taylor series in I:

$$\psi(u) = \sum_{k=0}^{\infty} \psi^{(k)}(s) \frac{(u-s)^k}{k!}. \tag{3. 16}$$

Using the above-mentioned method given in Section 2, Eq. (1. 1) can be converted into the following equivalent linear equations system with respect to unknown functions $\psi^{(k)}(s),\ k = 0, 1, \cdots$

$$\mathbf{V\Psi} = \mathbf{F}, \tag{3. 17}$$

where

$$\mathbf{V} = \lim_{\nu \longrightarrow \infty} \mathbf{V}_{\nu\nu}, \qquad \mathbf{\Psi} = \lim_{\nu \longrightarrow \infty} \mathbf{\Psi}_{\nu}, \qquad \mathbf{F} = \lim_{\nu \longrightarrow \infty} \mathbf{F}_{\nu}, \tag{3. 18}$$

in which $\mathbf{V}_{\nu\nu}$, $\mathbf{\Psi}_\nu$, and $\mathbf{F}_\nu$, as shown in the previous section, are defined as

$$\mathbf{V}_{\nu\nu} = \;\; v_{ij}(s) \;\; _{(\nu+1)\times(\nu+1)}, \quad \mathbf{\Psi}_\nu = \left[\psi^{(i)}(s)\right]_{(\nu+1)\times 1}, \quad \mathbf{F}_\nu = \left[f_{(i)}(s)\right]_{(\nu+1)\times 1} \tag{3. 19}$$

Hence, under the solvability conditions of system (3. 17) and letting $\mathbf{B} = \mathbf{V}^{-1}$ the unique solution of system (3. 17) is represented as

$$\mathbf{\Psi} = \mathbf{BF}. \tag{3. 20}$$

We rewrite the relation (3. 20) in an alternative matrix form as

$$\begin{bmatrix} \mathbf{\Psi}_\nu \\ \mathbf{\Psi}_\infty \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{\nu\nu} & \mathbf{B}_{\nu\infty} \\ \mathbf{B}_{\infty\nu} & \mathbf{B}_{\infty\infty} \end{bmatrix} \begin{bmatrix} \mathbf{F}_\nu \\ \mathbf{F}_\infty \end{bmatrix}, \tag{3. 21}$$

where

$$\mathbf{\Psi}_\infty = \begin{bmatrix} \mathbf{\Psi}_{\nu+1} \\ \mathbf{\Psi}_{\nu+2} \\ \vdots \end{bmatrix}. \tag{3. 22}$$

Accordingly, we can find out that the vector $\mathbf{\Psi}_\nu$ consists of the first $\nu+1$ elements of the exact solution vector $\mathbf{\Psi}$ must satisfy the following relation

$$\mathbf{\Psi}_\nu = \mathbf{B}_{\nu\nu}\mathbf{F}_\nu + \mathbf{B}_{\nu\infty}\mathbf{F}_\infty. \tag{3. 23}$$

According to the proposed process in this paper, the unique solution of system $\mathbf{V\Psi} = \mathbf{F}$ can be denoted as

$$\widetilde{\mathbf{\Psi}}_\nu = \mathbf{V}_{\nu\nu}^{-1}\mathbf{F}_\nu. \tag{3. 24}$$

where $\mathbf{\Psi}_\nu$ is replaced by $\widetilde{\mathbf{\Psi}}_\nu$ as its approximate solution, for convenience.

Subtracting (3. 24) from (3. 23) leads to

$$\mathbf{\Psi}_\nu - \widetilde{\mathbf{\Psi}}_\nu = \mathbf{D}_{\nu\nu}\mathbf{F}_\nu + \mathbf{B}_{\nu\infty}\mathbf{F}_\infty, \tag{3. 25}$$

where $\mathbf{D}_{\nu\nu} = \mathbf{B}_{\nu\nu} - \mathbf{V}_{\nu\nu}^{-1}$.

Now, expanding the right-hand side of (3. 25), the first element of the vector at the left-hand side of (3. 25) can be represented as

$$\psi(s) - \widetilde{\psi}(s) = \sum_{j=0}^{\nu} d_{0,j}(s) f_{(j)}(s) + \sum_{j=\nu+1}^{\infty} b_{0,j}(s) f_{(j)}(s), \tag{3. 26}$$

where $d_{i,j}(s)$ and $b_{i,j}(s)$ are the elements of $\mathbf{D}_{\nu\nu}$ and $\mathbf{B}_{\nu\infty}$, respectively. Thus, according to the Cauchy-Schwarz inequality we have

$$\begin{aligned} \left|\psi(s) - \widetilde{\psi}(s)\right| \leq & \left(\sum_{j=0}^{\nu} |d_{0,j}(s)|^2\right)^{\frac{1}{2}} \left(\sum_{j=0}^{\nu} |f_{(j)}(s)|^2\right)^{\frac{1}{2}} + \\ & \left(\sum_{j=\nu+1}^{\infty} |b_{0,j}(s)|^2\right)^{\frac{1}{2}} \left(\sum_{j=\nu+1}^{\infty} |f_{(j)}(s)|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{3. 27}$$

It could be noted that, as $\lim_{\nu\longrightarrow\infty} \mathbf{D}_{\nu\nu} = 0$ and $\lim_{\nu\longrightarrow\infty} \mathbf{B}_{\nu\infty} = 0$, we have $\lim_{\nu\longrightarrow\infty} |\psi(s) - \widetilde{\psi}(s)| = 0$.

4. NUMERICAL EXAMPLES

In this section, we present approximate solutions of several first kind Fredholm integral equations to illustrate the efficiency and the accuracy of the approach described in Section 2. Comparing this method with other selected methods, reveals the validity and applicability of the proposed method. All computations are performed using Mathematica 8.

Example 4.1. Consider the following Fredholm integral equation of the first kind [3, 8, 28, 31, 42]

$$\int_0^1 \sin(su)\psi(u)du = \frac{\sin(s) - s\cos(s)}{s^2}, \tag{4. 28}$$

with the exact solution $\psi(s) = s$. We apply the process discussed in Section 2, to obtain the approximate solution of Eq. (4. 28). In the following, let's consider the first-degree Taylor expansion

$$\psi(u) \approx \psi(s) + \psi'(s)(u-s). \tag{4. 29}$$

Inserting the approximate relation (4. 29), for unknown function $\psi(u)$, into Eq. (4. 28) leads to

$$\frac{(s - s\cos s)}{s^2}\psi(s)+\frac{(-s^2 + (-1 + s)s\cos s + \sin s)}{s^2}\psi'(s) = \frac{\sin(s) - s\cos(s)}{s^2}. \quad (4.\ 30)$$

In fact, Eq. (4. 30) is a first order linear ODE with respect to $\psi(s)$ and $\psi'(s)$. In the following, we want to determine unknowns by solving a linear equations system. So, other one linear equation with respect to $\psi(s)$ and $\psi'(s)$ is needed, which is obtained by integrating both sides of Eq. (4. 28) one time with respect to s from a to s. Therefore, we have

$$\int_0^1 \int_0^s \sin(tu)\psi(u)dtdu = \int_0^s \frac{\sin(u) - u\cos(u)}{u^2}du. \quad (4.\ 31)$$

Now, we apply approximate relation (4. 29) again, and after substituting (4. 29) for $\psi(u)$ into Eq. (4. 31), we obtain

$$\frac{EulerGamma - CosIntegral(s) + Log(s)}{s}\psi(s)+ \frac{s^2 CosIntegral(s) - s(-1 + EulerGamma(s) + sLog(s)) - \sin(s)}{s}\psi'(s) = 1 - \frac{\sin(s)}{s}. \quad (4.\ 32)$$

Thus, Eqs. (4. 30) and (4. 32) construct a linear equations system as

$$\begin{bmatrix} \frac{(s - s\cos s)}{s^2} & \frac{(-s^2 + (-1 + s)s\cos s + \sin s)}{s^2} \\ \frac{EulerGamma - CosIntegral(s) + Log\ s}{s} & \frac{s^2 CosIntegral(s) - s(-1 + EulerGamma(s) + sLog\ s) - \sin s}{s} \end{bmatrix} \begin{bmatrix} \psi(s) \\ \psi'(s) \end{bmatrix} = \begin{bmatrix} \frac{\sin(s) - s\cos(s)}{s^2} \\ 1 - \frac{\sin(s)}{s} \end{bmatrix} \quad (4.\ 33)$$

By solving system (4. 33) the first-degree approximate solution $\psi_1(s)$ yields the exact solution as it is expected. We note that after converting Eq (4. 28) into linear equations system (4. 33), the Mathematica command ‘LinearSolve’ is used for the obtained system. To make a comparison between the results obtained in different references, we list the numerical results in Tables 1, 2 and 3.

TABLE 1. Errors ($\| e_N \|$) for example 4.1 in [8, 28]

N	Coifman wawelets in [28]	Coifman wavelets in [8]	Sinc wavelets in [28]	Sinc wavelets in [8]
2	$.1903e-2$	$8.3e-2$	$.3741e-3$	$3.1e-3$
3	$174.7e-3$	$5.7e-3$	$.5208e-4$	$1.8e-4$
4	$.5178e-5$	$3.8e-5$	$.4153e-5$	$3.3e-5$
5	$.4418e-8$	$8.8e-8$	$.0135e-8$	$6.5e-8$
6	$.7911e-11$	$9.1e-11$	$.8268e-10$	$8.8e-10$
7	$.9426e-12$	$5.6e-12$	$.6782e-11$	$2.2e-11$
8	$.2119e-13$	$4.9e-13$	$.4026e-12$	$7.4e-12$
9	$.3899e-16$	$3.9e-16$	$.9350e-15$	$9.8e-15$

TABLE 2. Absolute errors of Legendre multi-wavelets method [42], Legendre wavelets method [32] and Chebyshev wavelet method [3]

x	Legendre multi-wavelets method [42]	Legendre wavelets method [32]	Chebyshev wavelet method [3]
0	$3.2306194e-5$	$1.1830664e-4$	$8.0915e-6$
0.1	$8.4554541e-6$	$3.5427559e-5$	$8.0916e-6$
0.2	$1.5395286e-5$	$1.1933896e-5$	$8.0917e-6$
0.3	$9.5703736e-6$	$2.3777726e-5$	$8.0917e-6$
0.4	$5.9440235e-7$	$1.0392964e-7$	$8.0918e-6$
0.5	$1.9835266e-6$	$1.1210253e-5$	$1.7887e-5$
0.6	$8.7416855e-7$	$5.3623319e-6$	$9.1574e-6$
0.7	$2.3518955e-7$	$1.4280816e-6$	$8.5192e-6$
0.8	$4.4564133e-8$	$5.9249774e-7$	$7.8810e-6$
0.9	$1.5619527e-8$	$6.9940627e-7$	$7.2428e-6$

This example has recently been solved in [31] using sinc collocation method together with a regularization technique which the obtained results in [31] are tabulated in Table 3 where γ is known as the regularization parameter. The reader can refer to [31] for more details.

TABLE 3. The maximum of the absolute errors in [31] using sinc collocation method for regular parameters $\gamma = 0.01$, and $\gamma = 0.001$.

N	$\gamma = 0.01$	$\gamma = 0.001$
2	$2.5e-2$	$3.4e-3$
3	$5.4e-3$	$6.2e-4$
4	$1.4e-3$	$4.9e-4$
5	$2.8e-4$	$4.7e-5$
6	$3.2e-5$	$8.7e-6$
7	$4.3e-6$	$1.1e-6$
8	$8.5e-7$	$4.2e-7$

Example 4.2. Consider the following Fredholm integral equation of the first kind [3]

$$\int_0^1 e^{su}\psi(u)du = \frac{e^{s+1}-1}{s+1}, \tag{4. 34}$$

with the exact solution $\psi(s) = e^s$. Using the suggested method in this paper, we obtain the approximate results by setting $\nu = 1, \dots, 4$ and our results are given in Table 4. This example has been solved in [3] using Chebyshev wavelet method and Haar wavelet method that was considered in [29] and we list the results in Table 5.

Example 4.3. Consider the following Fredholm integral equation of the first kind [7, 13, 31, 35]

$$\int_0^1 (s^2+u^2)^{\frac{1}{2}}\psi(u)du = \frac{(1+s^2)^{\frac{3}{2}}-s^3}{3}, \tag{4. 35}$$

with the exact solution $\psi(s) = s$. Using the proposed method, we can observe that the first-degree approximate solution $\psi_1(s)$ yields the exact solution, since the exact solution

TABLE 4. Absolute errors of Example 4.2.

x	$\nu = 1$	$\nu = 2$	$\nu = 3$	$\nu = 4$
0.1	$6.45167e-2$	$1.40802e-3$	$1.79020e-4$	$1.23950e-3$
0.2	$1.24959e-2$	$4.60250e-3$	$4.08930e-4$	$2.19367e-3$
0.3	$2.80324e-2$	$6.14073e-3$	$1.43025e-4$	$6.24039e-4$
0.4	$5.57884e-2$	$4.42889e-3$	$1.98471e-4$	$1.40725e-4$
0.5	$6.93572e-2$	$8.23884e-4$	$3.70581e-4$	$2.25595e-5$
0.6	$6.71742e-2$	$3.16848e-3$	$2.67273e-4$	$2.82636e-5$
0.7	$4.75096e-2$	$5.87778e-3$	$5.81542e-5$	$1.58199e-5$
0.8	$8.45132e-3$	$5.45161e-3$	$3.75997e-4$	$7.14451e-6$
0.9	$5.21148e-2$	$1.63631e-4$	$2.60992e-4$	$2.14848e-5$
1.0	$1.36526e-1$	$1.32439e-2$	$9.28844e-4$	$5.21956e-5$

TABLE 5. Errors of Example 4.2 in [29, 3].

x	Chebyshev wavelet [3]		Haar wavelet [29]
	$k=2, M=3$	$k=2, M=4$	
0.1	$0.93756e-3$	$0.15083e-4$	$0.78533e-2$
0.2	$0.26591e-4$	$0.17348e-4$	$0.17394e-2$
0.3	$0.10813e-2$	$0.18632e-4$	$0.56995e-2$
0.4	$0.11006e-2$	$0.15772e-4$	$0.63561e-2$
0.5	$0.12539e-3$	$0.65854e-5$	$0.23140e-2$
0.6	$0.21186e-2$	$0.13025e-5$	$0.12947e-1$
0.7	$0.22688e-2$	$0.23260e-5$	$0.93969e-2$
0.8	$0.72829e-3$	$0.15244e-4$	$0.10479e-1$
0.9	$0.38348e-3$	$0.98384e-5$	$0.38151e-2$
1.0	$0.12761e-2$	$0.10449e-4$	$0.49872e-2$

is a polynomial of degree one. The maximum of the absolute errors between the exact solution and approximate solution which has been given in [35] are tabulated in Table 6.

TABLE 6. The maximum of the absolute errors in [35] for Example 4.3.

N	Error values
2	$4.7e-4$
3	$3.8e-4$
4	$1.5e-4$
5	$8.4e-5$
6	$5.8e-5$
7	$4.7e-5$
8	$4.3e-5$
9	$3.1e-5$
10	$6.4e-6$

This example has recently been solved in [31] applying sinc collocation method together with a regularization technique which the obtained results in [31] are tabulated in Table 7 where γ is known as the regularization parameter. The reader can refer to [31] for more details.

TABLE 7. The maximum of the absolute errors in [31] using sinc collocation method for regular parameters $\gamma = 0.01$, and $\gamma = 0.001$ for Example 4.3.

N	$\gamma = 0.01$	$\gamma = 0.001$
5	$3.9e-2$	$1.4e-2$
10	$7.5e-3$	$3.8e-3$
15	$2.0e-3$	$7.6e-4$
20	$3.2e-4$	$9.3e-5$
25	$4.4e-5$	$2.1e-5$
30	$8.7e-6$	$3.9e-6$
35	$2.3e-6$	$7.8e-7$

Example 4.4. Consider the following Fredholm integral equation of the first kind [25]

$$\int_0^1 e^u \left[\sin(s-u+1)+1\right]\psi(u)du = 1+\cos(s)-\cos(s+1), \tag{4. 36}$$

with the exact solution $\psi(s) = e^{-s}$.
For this problem, a comparison between the exact solution and approximate solutions $\psi_1(s)$ and $\psi_2(s)$ is made, at ten equidistant points in $[0, 1]$, by setting $\nu = 1, 2$, in Table 8. Furthermore, the results are shown in Figs 1 and 2, when $\nu = 1, 2$, respectively. This example was used in [25] and has been solved using Haar wavelet with scaling parameter $J = 4, 6$. Results obtained in [25] are shown in Figs. 3, 4 and it can be observed that the results obtained implementing the proposed approach are much better than those reported in [25].

TABLE 8. Absolute errors of Example 4.4.

x	$\nu = 1$	$\nu = 2$
0.1	$3.08944e-2$	$9.51278e-4$
0.2	$5.62518e-3$	$1.85257e-3$
0.3	$1.15416e-2$	$2.50492e-3$
0.4	$2.13800e-2$	$1.78556e-3$
0.5	$2.45910e-2$	$4.00050e-4$
0.6	$2.18094e-2$	$1.01317e-3$
0.7	$1.36101e-2$	$1.87644e-3$
0.8	$5.13639e-4$	$1.66706e-3$
0.9	$1.70084e-2$	$8.79272e-5$
1.0	$3.85289e-2$	$3.81648e-3$

Example 4.5. Consider the following Fredholm integral equation of the first kind [19]

$$\int_0^1 e^{-su}\psi(u)du = \frac{1-(s+1)e^{-s}}{s^2}, \tag{4. 37}$$

with the exact solution $\psi(s) = s$. For this example, we can find that $\psi_\nu(s)$ yields the exact solution, only by setting $\nu = 1$. The reader can refer to [19] in order to compare the results in depth.

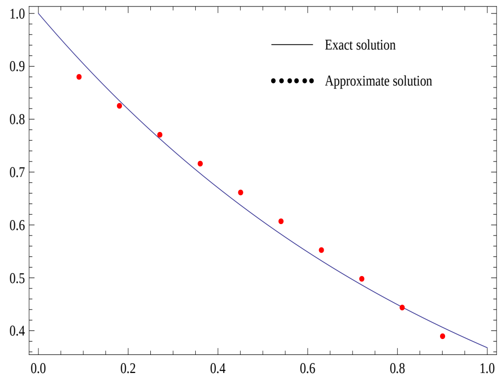

FIGURE 1. Approximate results for Example 4.4 with $\nu = 1$.

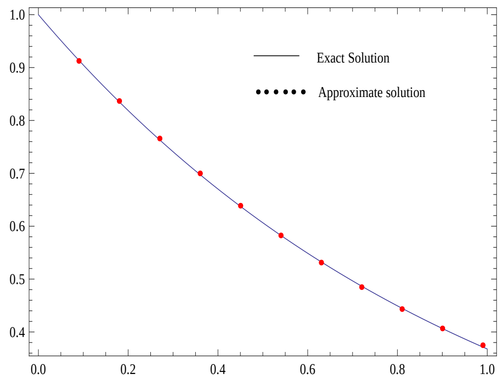

FIGURE 2. Approximate results for Example 4.4 with $\nu = 2$.

Example 4.6. Consider the following Fredholm integral equation of the first kind [29]

$$\int_0^1 \frac{(s+u)^2}{\sqrt{1+u^2}}\psi(u)du =$$
$$\frac{1}{24}\left(-3\sqrt{2} - 16(-2+\sqrt{2})s + 12s^2(\sqrt{2}\,\mathrm{arcsinh}(1)) + 9\,\mathrm{arcsinh}(1)\right), \qquad (4.\ 38)$$

with the exact solution $\psi(s) = s^2$. The first and second degree approximate solutions $\psi_1(s)$ and $\psi_2(s)$ are obtained, by setting $\nu = 1, 2$. The obtained absolute errors between the exact solution and approximate solutions are shown in Table 9. From Table 9, we can find that second-degree approximate solution yields the exact solution as it is expected. This example was used in [29] and has been solved using Haar wavelet with scaling parameter $J = 4, 5$. Results obtained in [29] are shown in Figs. 5, 6.

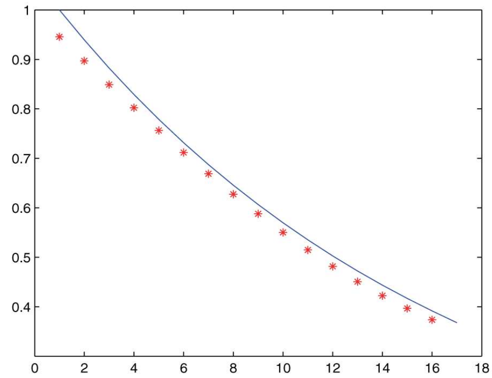

FIGURE 3. Numerical results for Example 4.4 with scaling parameter $J = 4$ in [25] ($-Exact\ solution * Numerical\ solution$).

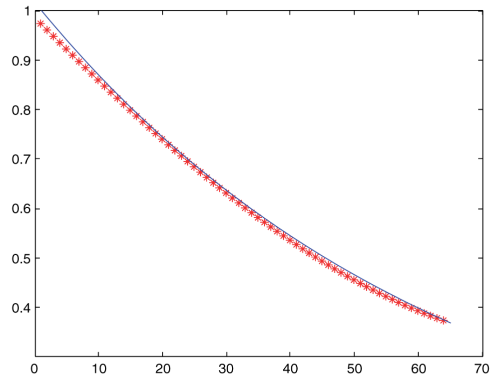

FIGURE 4. Numerical results for Example 4.4 with scaling parameter $J = 6$ in [25] ($-Exact\ solution * Numerical\ solution$).

TABLE 9. Absolute errors of Example 4.4.

x	$\nu = 1$	$\nu = 2$
0.1	$1.52351e-1$ 0	
0.2	$5.58730e-2$	0
0.3	$1.20495e-2$	0
0.4	$5.50158e-2$	0
0.5	$7.48088e-2$	0
0.6	$7.24209e-2$	0
0.7	$4.84532e-2$	0
0.8	$3.29391e-3$	0
0.9	$6.27930e-2$	0
1.0	$1.49621e-1$	0

5. CONCLUSION

In this research, a new application of Taylor expansion has been represented to solve first kind Fredholm integral equations. We have described in detail that the technique is

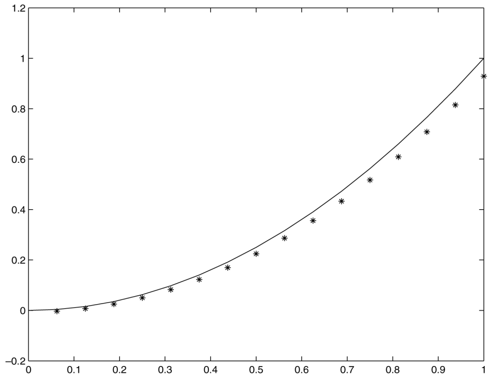

FIGURE 5. Numerical results at scale $J = 4$ for Example 4.6 in [29] $(-Exact\ solution * Numerical\ solution)$.

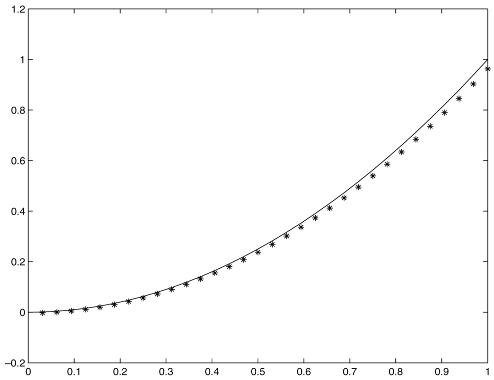

FIGURE 6. Numerical results at scale $J = 5$ for Example 4.6 in [29] $(-Exact\ solution * Numerical\ solution)$.

based on converting first kind Fredholm integral equation into a linear equations system for unknown and its derivatives, by the use of Taylor expansion of unknown function at an arbitrary point and integration method. Six examples were evaluated to demonstrate the ease and the efficiency of the proposed method.

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