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Generalizations of Simpson's Type Inequalities Through Preinvexity and Prequasiinvexity

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Abstract. We establish some new inequalities of simpson's type for differentiable mappings whose third derivatives in absolute values are preinvex and prequasiinvex. These give us new estimates which are better than already presented results.

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1. Introduction

Simpson's inequality states as: If $f : [a, b] \to \mathrm{R}$ be a four times continuously differentiable mapping on (a, b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4 .$$

In [1-2], [4-5], [7], [10], [16] and [18], authors refined and generalized Simpson's type inequality.

For the definition of invex set, reader is refer to [20]. K is invex set with respect to η, if it is invex for all $x \in K$. The invex set K is η-connected set as well.
Every convex set is invex set with respect to $\eta(y, x) = y - x$ [3], but not conversely.
The function f on the invex set K is called preinvex [20] with respect to function η, if

$$f(x + t\eta(y, x)) \le (1-t) f(x) + t f(y), \forall x, y \in K, t \in [0, 1] .$$

The function f is called preconcave if and only if $-f$ is preinvex.
In [20-21], authors showed that every convex function is preinvex but not conversely (see for instance [21]).
The function f on the invex set K is said to be prequasiinvex [11] with respect to η, if

$$f\left(x+t\eta\left(y,x\right)\right)\leq\max\left\{f\left(x\right),f\left(y\right)\right\},\forall x,y\in K,t\in\left[0,1\right].$$

In [22], author showed that prequasiinvex function is the generalization of quasi-convex function.
Recently, many mathematicians generalized the classical convexity. One can refer to [20] and [6], [8], [13-15], [21] and [23]. Hanson [8] significantly generalized the convex function by introducing invex functions. As a special case of invexity, Ben-Israel and Mond [6] presented preinvex function. As a generalization of invex function, Pini[15], presented prequasiinvex functions.

The purpose of this manuscript is to establish some new simpson's type inequalities for the class of functions whose third derivatives in absolute values are preinvex and prequasiinvex.

2. MAIN RESULTS

We begin with the following Lemma.

Lemma 1. *Let $K\subseteq[0,\infty)$ be an open invex subset with respect to $\eta:K\times K\rightarrow\mathrm{R}$ with $a,b\in K$ and $a<a+\eta\left(b,a\right)$. Suppose $f:K\rightarrow\mathrm{R}$ be an absolutely continuous mapping on Ksuch that $f'''\in L\left(\left[a,a+\eta\left(b,a\right)\right]\right)$. If $\left|f'''\right|$ is preinvex on K, then for every $a,b\in K$ with $\eta\left(b,a\right)\neq0$, we have*

$$\begin{aligned}&\left|\int\limits_{a}^{a+\eta(b,a)}f(x)dx-\frac{\eta(b,a)}{6}\left[f\left(a\right)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f\left(a+\eta\left(b,a\right)\right)\right]\right|\\&\leq\left(\eta\left(b,a\right)\right)^{4}\int\limits_{0}^{1}p(\lambda)f'''\left(a+\lambda\eta\left(b,a\right)\right)d\lambda,\end{aligned}$$

where $p(\lambda)=\begin{cases}\frac{1}{6}\lambda^{2}\left(\lambda-\frac{1}{2}\right), & \lambda\in\left[0,\frac{1}{2}\right]\\ \frac{1}{6}\left(\lambda-1\right)^{2}\left(\lambda-\frac{1}{2}\right), & \lambda\in\left[\frac{1}{2},1\right]\end{cases}$.

Proof. Using definition of $p(\lambda)$, we get,

$$\begin{aligned}I&=\int\limits_{0}^{1}p(\lambda)f'''\left(a+\lambda\eta\left(b,a\right)\right)d\lambda\\&=\frac{1}{6}\int_{0}^{1/2}\left|\lambda^{2}\left(\lambda-\frac{1}{2}\right)\right|f'''\left(a+\lambda\eta\left(b,a\right)\right)d\lambda\\&+\frac{1}{6}\int_{1/2}^{1}\left|\left(\lambda-1\right)^{2}\left(\lambda-\frac{1}{2}\right)\right|f'''\left(a+\lambda\eta\left(b,a\right)\right)d\lambda.\end{aligned}$$

Integrating by parts, we get

$$I = \frac{1}{6}\lambda^2\left(\lambda-\frac{1}{2}\right)\frac{f''(a+\lambda\eta(b,a))}{\eta(b,a)}\Big|_0^{1/2} - \frac{1}{6}\lambda\left(3\lambda-1\right)\frac{f'(a+\lambda\eta(b,a))}{(\eta(b,a))^2}\Big|_0^{1/2}$$
$$+\left(\lambda-\frac{1}{6}\right)\frac{f(a+\lambda\eta(b,a))}{(\eta(b,a))^3}\Big|_{1/2}^{1} - \int_0^{1/2}\frac{f(a+\lambda\eta(b,a))}{(\eta(b,a))^3}d\lambda +$$
$$\frac{1}{6}\left(\lambda-1\right)^2\left(\lambda-\frac{1}{2}\right)\frac{f''(a+\lambda\eta(b,a))}{(\eta(b,a))}\Big|_{1/2}^{1} - \frac{1}{6}\left(3\lambda-2\right)\left(\lambda-1\right)\frac{f'(a+\lambda\eta(b,a))}{(\eta(b,a))^2}\Big|_{1/2}^{1}$$
$$+\left(\lambda-\frac{5}{6}\right)\frac{f(a+\lambda\eta(b,a))}{(\eta(b,a))^3}\Big|_{1/2}^{1} - \int_{1/2}^{1}\frac{f(a+\lambda\eta(b,a))}{(\eta(b,a))^3}d\lambda$$
$$= -\frac{1}{24}\frac{f'\left(\frac{2a+\eta(b,a)}{2}\right)}{(\eta(b,a))^2} + \frac{2}{6}\frac{f\left(\frac{2a+\eta(b,a)}{2}\right)}{(\eta(b,a))^3} + \frac{2}{6}\frac{f(a+\eta(b,a))}{(\eta(b,a))^3} - \int_0^{\frac{1}{2}}\frac{f(a+\lambda\eta(b,a))}{(\eta(b,a))^3}d\lambda$$
$$= \frac{1}{24}\frac{f'\left(\frac{2a+\eta(b,a)}{2}\right)}{(\eta(b,a))^2} + \frac{2}{6}\frac{f\left(\frac{2a+\eta(b,a)}{2}\right)}{(\eta(b,a))^3} + \frac{1}{6}\frac{f(a)}{(\eta(b,a))^3} - \int_{1/2}^{1}\frac{f(a+\lambda\eta(b,a))}{(\eta(b,a))^3}d\lambda$$

Change of variable $x = (a+\lambda\eta(b,a))$ and $dx = (a+\eta(b,a))\,d\lambda$, gives

$$I\left(\eta(b,a)\right)^4 =$$
$$\int_a^{a+\eta(b,a)} f(x)\,dx - \frac{\eta(b,a)}{6}\left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a))\right].$$

The proof is completed. □

Theorem 2. *Let the assumption of Lemma 2.1 be satisfied. Then*

$$\left|\int_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right]\right|$$
$$\leq \frac{(\eta(b,a))^4}{1152}\left[|f'''(a)| + |f'''(b)|\right]$$

Proof. Using Lemma 2.1, we get

$$\left|\int_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right]\right| \leq \frac{(\eta(b,a))^4}{6}$$
$$\left\{\int_0^{\frac{1}{2}}\left|\lambda^2\left(\lambda-\frac{1}{2}\right)\right|\left|f'''(a+\lambda\eta(b,a))\right|d\lambda + \int_{\frac{1}{2}}^{1}\left|(\lambda-1)^2\left(\lambda-\frac{1}{2}\right)\right|\left|f'''(a+\lambda\eta(b,a))\right|d\lambda\right\}$$

Using preinvexity of $|f'''|$, we get

$$\left|\int_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right]\right|$$
$$\leq \frac{(\eta(b,a))^4}{6}\left\{\begin{array}{l}\int_0^{\frac{1}{2}}\lambda^2\left(\frac{1}{2}-\lambda\right)\left[(1-\lambda)|f'''(a)|+\lambda|f'''(b)|\right]d\lambda \\ +\int_{\frac{1}{2}}^{1}(\lambda-1)^2\left(\lambda-\frac{1}{2}\right)\left[(1-\lambda)|f'''(a)|+\lambda|f'''(b)|\right]d\lambda\end{array}\right\}.$$

By simple calculations of integrals, we get as required. The proof is completed. □

Let the assumption of Theorem 2.2 be satisfied with $q>1$, such that $p=\frac{q}{q-1}$. If $|f'''|^q$ is preinvex, then we get

$$\left|\int_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right]\right| \leq \frac{(\eta(b,a))^4}{48}\left(\frac{1}{2}\right)^{\frac{1}{p}}$$
$$\left[\frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)}\right]^{\frac{1}{p}}\left\{\left(\frac{3}{8}|f'''(a)|^q+\frac{1}{8}|f'''(b)|^q\right)^q+\left(\frac{1}{8}|f'''(a)|^q+\frac{3}{8}|f'''(b)|^q\right)^q\right\}^{\frac{1}{q}}.$$

Theorem 3. *By using Lemma 2.1 and Holder's inequality, we have*

$$\begin{aligned}&\left|\int\limits_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a))\right]\right| \\ &\leq \frac{(\eta(b,a))^4}{6}\left(\int\limits_0^{1/2}\left|\lambda^2\left(\frac{1}{2}-\lambda\right)\right|^p d\lambda\right)^{1/p}\left(\int_0^{1/2}\left|f'''(a+\lambda\eta(b,a))\right|^q d\lambda\right)^{1/q} \\ &+\frac{(\eta(b,a))^4}{6}\left(\int_{1/2}^1\left(|\lambda-1|^2\left(\lambda-\frac{1}{2}\right)\right)^p d\lambda\right)^{1/p}\left(\int_{1/2}^1\left|f'''(a+\lambda\eta(b,a))\right|^q d\lambda\right)^{1/q}.\end{aligned}$$

Here

$$\int\limits_0^{1/2}\left|\lambda^2\left(\frac{1}{2}-\lambda\right)\right|^p d\lambda = \int_{1/2}^1\left|(\lambda-1)^2\left(\lambda-\frac{1}{2}\right)\right|^p d\lambda = \frac{\Gamma(2p+1)\Gamma(p+1)}{2^{3p+1}2\Gamma(3p+2)}.$$

Using preinvexity of $|f'''|^q$, we get

$$\begin{aligned}&\left|\int\limits_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a))\right]\right| \\ &\leq \frac{(\eta(b,a))^4}{6}\left(\int\limits_0^{1/2}\left|\lambda^2\left(\frac{1}{2}-\lambda\right)\right|^p d\lambda\right)^{1/p}\left(\int_0^{1/2}\left[(1-\lambda)\left|f'''(a)\right|^q+\lambda\left|f'''(b)\right|^q\right]d\lambda\right)^{1/q} \\ &+\frac{(\eta(b,a))^4}{6}\left(\int_{1/2}^1\left|(\lambda-1)^2\left(\lambda-\frac{1}{2}\right)\right|^p d\lambda\right)^{1/p} \\ &\left(\int_{1/2}^1\left[(1-\lambda)\left|f'''(a)\right|^q+\lambda\left|f'''(b)\right|^q\right]d\lambda\right)^{1/q} \\ &= \frac{(\eta(b,a))^4}{48}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left[\frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)}\right]^{\frac{1}{p}} \\ &\left\{\left(\frac{3}{8}\left|f'''(a)\right|^q+\frac{1}{8}\left|f'''(b)\right|^q\right)^q+\left(\frac{1}{8}\left|f'''(a)\right|^q+\frac{3}{8}\left|f'''(b)\right|^q\right)^q\right\}^{\frac{1}{q}}.\end{aligned}$$

This completes the proof.

Theorem 4. *Let the assumption of Theorem 2.1 be satisfied with $q>1$, such that $p=\frac{q}{q-1}$. Then*

$$\begin{aligned}&\left|\int\limits_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a))\right]\right| \\ &\leq \frac{(\eta(b,a))^4}{6}\left[\frac{\Gamma(2p+1)\Gamma(p+1)}{2^{3p+1}2\Gamma(3p+2)}\right]^{\frac{1}{p}}\left[\frac{\left(\left|f'''(a)\right|^q+\left|f'''(b)\right|^q\right)^q}{2}\right]^{\frac{1}{q}}.\end{aligned}$$

Proof. By Lemma 2.1 and Holder's inequality, we have

$$\begin{aligned}&\left|\int\limits_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a))\right]\right| \\ &\leq (\eta(b,a))^4\int\limits_0^1 p(\lambda)f'''(a+\lambda\eta(b,a))\,d\lambda \\ &\leq (\eta(b,a))^4\left(\int\limits_0^1|p(\lambda)|^p d\lambda\right)^{\frac{1}{p}}\left(\int\limits_0^1\left|f'''(a+\lambda\eta(b,a))\right|^q d\lambda\right)^{\frac{1}{q}}\end{aligned}$$

Using preinvexity of $|f'''|$, we have

$$\left|\int_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right]\right|$$
$$\leq \frac{(\eta(b,a))^4}{6}\left(\int_0^{1/2}\left|\lambda^2\left(\frac{1}{2}-\lambda\right)\right|^p d\lambda + \int_{1/2}^1\left|(\lambda-1)^2\left(\lambda-\frac{1}{2}\right)\right|^p d\lambda\right)^{1/p}$$
$$\left(\int_0^1\left[(1-\lambda)|f'''(a)|^q+\lambda|f'''(b)|^q\right]d\lambda\right)^{1/q}$$
$$= \frac{(\eta(b,a))^4}{6}\left[\frac{\Gamma(2p+1)\Gamma(p+1)}{2^{3p+1}2\Gamma(3p+2)}\right]^{\frac{1}{p}}\left[\frac{\left(|f'''(a)|^q+|f'''(b)|^q\right)^q}{2}\right]^{\frac{1}{q}}.$$

By simple calculations, we have

$$\int_0^{1/2}\left|\lambda^2\left(\frac{1}{2}-\lambda\right)\right|^p d\lambda = \int_{1/2}^1\left|(\lambda-1)^2\left(\lambda-\frac{1}{2}\right)\right|^p d\lambda = \frac{\Gamma(2p+1)\Gamma(p+1)}{2^{3p+1}2\Gamma(3p+2)}.$$

This completes the proof.

□

Theorem 5. *Let the assumption of Theorem 2.4 be satisfied with $q \geq 1$, such that $p = \frac{q}{q-1}$. If the function $|f'''|^q$ is preinvex, then we have*

$$\left|\int_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right]\right|$$
$$\leq \frac{(\eta(b,a))^4}{6}\left(\frac{1}{192}\right)^{1-1/q}\left(\frac{|f'''(a)|^q+|f'''(b)|^q}{192}\right)^{\frac{1}{q}}.$$

Proof. Using Lemma 2.1 and power mean inequality, we get

$$\left|\int_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a)+4f\left(\frac{2a+\eta(b,a)}{2}\right)+f(a+\eta(b,a))\right]\right|$$
$$\leq (\eta(b,a))^4\int_0^1 p(\lambda)f'''(a+\lambda\eta(b,a))\,d\lambda$$
$$\leq \frac{(\eta(b,a))^4}{6}\left(\int_0^{1/2}\left|\lambda^2\left(\frac{1}{2}-\lambda\right)\right|d\lambda\right)^{1-1/q}\left(\int_0^{1/2}\left|\lambda^2\left(\frac{1}{2}-\lambda\right)\right||f'''(a+\lambda\eta(b,a))|^q d\lambda\right)^{1/q}$$
$$+\frac{(\eta(b,a))^4}{6}\left(\int_{1/2}^1\left|(\lambda-1)^2\left(\lambda-\frac{1}{2}\right)\right|d\lambda\right)^{1-1/q}$$
$$\left(\int_{1/2}^1\left|(\lambda-1)^2\left(\lambda-\frac{1}{2}\right)\right||f'''(a+\lambda\eta(b,a))|^q d\lambda\right)^{1/q}.$$

Using preinvexity of $|f'''|^q$, we have

$$\left|\int_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a))\right]\right|$$
$$\leq \frac{(\eta(b,a))^4}{6}\left(\int_0^{1/2}\left|\lambda^2\left(\tfrac{1}{2}-\lambda\right)\right|d\lambda\right)^{1-1/q}$$
$$\left(\int_0^{1/2}\left|\lambda^2\left(\tfrac{1}{2}-\lambda\right)\right|\left[(1-\lambda)\left|f'''(a)\right|^q + \lambda\left|f'''(b)\right|^q\right]d\lambda\right)^{\frac{1}{q}} \quad .$$
$$+\frac{(\eta(b,a))^4}{6}\left(\int_{1/2}^1\left|(\lambda-1)^2\left(\lambda-\tfrac{1}{2}\right)\right|d\lambda\right)^{1-1/q}$$
$$\left(\int_{1/2}^1\left|(\lambda-1)^2\left(\lambda-\tfrac{1}{2}\right)\right|\left[(1-\lambda)\left|f'''(a)\right|^q + \lambda\left|f'''(b)\right|^q\right]d\lambda\right)^{\frac{1}{q}}$$
$$\leq \frac{(\eta(b,a))^4}{6}\left(\frac{1}{192}\right)^{1-1/q}\left(\frac{\left|f'''(a)\right|^q + \left|f'''(b)\right|^q}{192}\right)^{\frac{1}{q}}.$$

The proof is completed. □

3. Simpson Type Inequalities For Prequasiinvex Function

Theorem 6. *Let the assumption of Theorem 2.5 be satisfied with $q \geq 1$. If $|f'''|$ is prequasiinvex, for some fixed $q \geq 1$ with $\eta(b,a) \neq 0$, then we have*

$$\left|\int_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a))\right]\right|$$
$$\leq \frac{(\eta(b,a))^4}{576}\left[\left\{\max\left|f'''(a)\right|^q, \left|f'''(b)\right|^q\right\}\right]^{\frac{1}{q}}.$$

Proof. Using Lemma 2.1 and power mean inequality, we get

$$\left|\int_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a))\right]\right|$$
$$\leq \frac{(\eta(b,a))^4}{6}\left(\int_0^{1/2}\left|\lambda^2\left(\tfrac{1}{2}-\lambda\right)\right|d\lambda\right)^{1-1/q}$$
$$\left(\int_0^{1/2}\left|\lambda^2\left(\tfrac{1}{2}-\lambda\right)\right|\left|f'''(a+\lambda\eta(b,a))\right|^q d\lambda\right)^{1/q}$$
$$+\frac{(\eta(b,a))^4}{6}\left(\int_{1/2}^1\left|(\lambda-1)^2\left(\lambda-\tfrac{1}{2}\right)\right|d\lambda\right)^{1-1/q}$$
$$\left(\int_{1/2}^1\left|(\lambda-1)^2\left(\lambda-\tfrac{1}{2}\right)\right|\left|f'''(a+\lambda\eta(b,a))\right|^q d\lambda\right)^{1/q}.$$

Using the prequasiinvexity of $|f'''|^q$, we have

$$\int_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a))\right]$$
$$\leq \frac{(\eta(b,a))^4}{6}\left(\int_0^{1/2}\left|\lambda^2\left(\tfrac{1}{2}-\lambda\right)\right|d\lambda\right)^{1-1/q}$$
$$\left(\int_0^{1/2}\left|\lambda^2\left(\tfrac{1}{2}-\lambda\right)\right|\left[\left\{\max\left|f'''(a)\right|^q, \left|f'''(b)\right|^q\right\}\right]d\lambda\right)^{\frac{1}{q}}$$
$$+\frac{(\eta(b,a))^4}{6}\left(\int_{1/2}^1\left|(\lambda-1)^2\left(\lambda-\tfrac{1}{2}\right)\right|d\lambda\right)^{1-1/q}$$
$$\left(\int_{1/2}^1\left|(\lambda-1)^2\left(\lambda-\tfrac{1}{2}\right)\right|\left[\left\{\max\left|f'''(a)\right|^q, \left|f'''(b)\right|^q\right\}\right]d\lambda\right)^{\frac{1}{q}}$$
$$\leq \frac{(\eta(b,a))^4}{576}\left[\left\{\max\left|f'''(a)\right|^q, \left|f'''(b)\right|^q\right\}\right]^{\frac{1}{q}}.$$

The proof is completed.

□

Corollary 7. *By putting* $q = 1$ *in Theorem 3.1, we get*

$$\left| \int\limits_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6} \left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a)) \right] \right|$$
$$\leq \frac{(\eta(b,a))^4}{576} \left\{ \max |f'''(a)|, |f'''(b)| \right\}.$$

Theorem 8. *Let the assumption of Theorem 3.1 be satisfied with* $q \geq 1$. *If* $|f'''|$ *is prequasiinvex for some fixed* $q \geq 1$ *with* $\eta(b,a) \neq 0$, *then we have*

$$\left| \int\limits_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6} \left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a)) \right] \right|$$
$$\leq \frac{(\eta(b,a))^4}{24} \left(\frac{1}{2}\right)^{\frac{1}{p}} \left[\frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)} \right]^{\frac{1}{p}} \left[\frac{\max\left\{ |f'''(a)|^q, |f'''(b)|^q \right\}}{2} \right]^{\frac{1}{q}}.$$

Proof. By Lemma 2.1 and Holder's inequality, we get

$$\left| \int\limits_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6} \left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a)) \right] \right|$$
$$\leq \frac{(\eta(b,a))^4}{6} \left(\int\limits_0^{1/2} \left| \lambda^2 \left(\frac{1}{2} - \lambda\right) \right|^p d\lambda \right)^{1/p} \left(\int_0^{1/2} |f'''(a+\lambda\eta(b,a))|^q d\lambda \right)^{1/q}$$
$$+ \frac{(\eta(b,a))^4}{6} \left(\int_{1/2}^1 \left| (\lambda-1)^2 \left(\lambda - \frac{1}{2}\right) \right|^p d\lambda \right)^{1/p} \left(\int_{1/2}^1 |f'''(a+\lambda\eta(b,a))|^q d\lambda \right)^{1/q}.$$

By using the fact

$$\int\limits_0^{1/2} \left| \lambda^2 \left(\frac{1}{2} - \lambda\right) \right|^p d\lambda = \int_{1/2}^1 \left| (\lambda-1)^2 \left(\lambda - \frac{1}{2}\right) \right|^p d\lambda = \frac{\Gamma(2p+1)\Gamma(p+1)}{2^{3p+1} 2\Gamma(3p+2)},$$

and prequasiinvexity of $|f'''|^q$, we get

$$\left| \int\limits_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6} \left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a)) \right] \right|$$
$$\leq \frac{(\eta(b,a))^4}{6} \left(\int\limits_0^{1/2} \left| \lambda^2 \left(\frac{1}{2} - \lambda\right) \right|^p d\lambda \right)^{1/p} \left(\int_0^{1/2} \max\left\{ |f'''(a)|^q, |f'''(b)|^q \right\} d\lambda \right)^{1/q}$$
$$+ \frac{(\eta(b,a))^4}{6} \left(\int_{1/2}^1 \left| (\lambda-1)^2 \left(\lambda - \frac{1}{2}\right) \right|^p d\lambda \right)^{1/p} \left(\int_{1/2}^1 \max\left\{ |f'''(a)|^q, |f'''(b)|^q \right\} d\lambda \right)^{1/q}$$
$$= \frac{(\eta(b,a))^4}{24} \left(\frac{1}{2}\right)^{\frac{1}{p}} \left[\frac{\Gamma(2p+1)\Gamma(p+1)}{\Gamma(3p+2)} \right]^{\frac{1}{p}} \left[\frac{\max\left\{ |f'''(a)|^q, |f'''(b)|^q \right\}}{2} \right]^{\frac{1}{q}}.$$

The proof is completed.

□

Theorem 9. *Let the assumption of Theorem 3.3 be satisfied with* $q \geq 1$. *If* $|f'''|$ *is prequasiinvex for some fixed* $q \geq 1$ *with* $\eta(b,a) \neq 0$, *then we have*

$$\left| \int\limits_a^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6} \left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a)) \right] \right|$$
$$\leq \frac{(\eta(b,a))^4}{6} \left[\frac{\Gamma(2p+1)\Gamma(p+1)}{2^{3p+1} 2\Gamma(3p+2)} \right]^{\frac{1}{p}} \left[\frac{\max\left\{ |f'''(a)|^q, |f'''(b)|^q \right\}}{2} \right]^{\frac{1}{q}}.$$

Proof. Using Lemma 2.1, Holder's inequality and prequasiinvexity of $|f'''|$, we get

$$\left|\int\limits_{a}^{a+\eta(b,a)} f(x)dx - \frac{\eta(b,a)}{6}\left[f(a) + 4f\left(\frac{2a+\eta(b,a)}{2}\right) + f(a+\eta(b,a))\right]\right|$$

$$\leq (\eta(b,a))^4 \left(\int\limits_0^1 |p(\lambda)|^p d\lambda\right)^{\frac{1}{p}} \left(\int\limits_0^1 \max\left\{|f'''(a)|^q, |f'''(b)|^q\right\} d\lambda\right)^{\frac{1}{q}}$$

$$\leq \frac{(\eta(b,a))^4}{6}\left(\int\limits_0^{1/2} \left|\lambda^2\left(\frac{1}{2}-\lambda\right)\right|^p d\lambda + \int_{1/2}^1 \left|(\lambda-1)^2\left(\lambda-\frac{1}{2}\right)\right|^p d\lambda\right)^{1/p}$$

$$\left(\int_0^1 \max\left\{|f'''(a)|^q, |f'''(b)|^q\right\} d\lambda\right)^{1/q}$$

$$= \frac{(\eta(b,a))^4}{6}\left[\frac{\Gamma(2p+1)\Gamma(p+1)}{2^{3p+1}2\Gamma(3p+2)}\right]^{\frac{1}{p}} \left[\frac{\max\left\{|f'''(a)|^q, |f'''(b)|^q\right\}}{2}\right]^{\frac{1}{q}}.$$

The proof is completed. □

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