Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 50(3)(2018) pp. 13-24

Some Fejer and Hermite-Hadamard Type Inequalities Considering ϵ-Convex and (σ, ϵ)-Convex Functions

Muhammad Amer Latif
Department of Basic Sciences,
Deanship of Preparatory Year Program,
University of Hail,
Hail 2440, Saudi Arabia.
Email: m_amer_latif@hotmail.com

Wajeeha Irshad
Department of Mathematics,
University of Engineering and Technology,
Lahore, Pakistan.
Email: wchattha@hotmail.com

Received: 21 July, 2017 / Accepted: 22 November, 2017 / Published online: 10 April, 2018

Abstract. In current paper, new Hermite-Hadamard and Fejér type inequalities are proved by using the ϵ-convexity and (σ, ϵ)-convexity of differentiable functions and a positive function symmetric with respect to $\frac{\epsilon j+k}{2}$. The results of the paper have been proved to contain previously established results related to differentiable convex functions.

1. Introduction

A function $\eta : U \subseteq \mathbb{R} \to \mathbb{R}$ forenamed as convex function, let

$$\eta\left(t\theta + (1-t)\,y\right) \leq t\eta\left(\theta\right) + (1-t)\,\eta\left(y\right)$$

holds for every θ, $y \in I$ and $t \in [0, 1]$.

The subsequent double integral inequality

$$\eta\left(\frac{j+k}{2}\right) \leq \frac{1}{k-j}\int_j^k \eta(\theta)d\theta \leq \frac{\eta(j)+\eta(k)}{2}. \quad (1.\ 1)$$

holds for convex functions and is notable in literature as the Hermite-Hadamard inequality. The inequalities in (1. 1) holds in reversed order as η is concave function.

The inequality (1. 1) has been a likely of extensive study insomuch as discovery. A number of papers have been written which provide noteworthy extensions, generalizations and refinements for the inequalities (1. 1), see for example [1]-[19].

Dragomir and Agarwal [2], proved subsequent inequalities for differentiable functions which estimate the difference between the middle and rightmost terms in (1. 1).

Theorem 1.1. [2] *Suppose* $\eta : U \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping at* U°*, and* j*,* $k \in U$ *with* $j < k$*, also* $\eta^{'} \in L([j,k])$*. If* $\left|\eta^{'}\right|$ *is convex function on* $[j,k]$*, so subsequent inequality holds:*

$$\left|\frac{\eta(j)+\eta(k)}{2} - \frac{1}{k-j}\int_j^k \eta(\theta)d\theta\right| \le \frac{k-j}{8}\left[\left|\eta^{'}(j)\right| + \left|\eta^{'}(k)\right|\right]. \qquad (1.\ 2)$$

Theorem 1.2. [2] *Let* $\eta : U \subseteq \mathbb{R} \to \mathbb{R}$ *is a differentiable mapping against* I°*, and* j*,* $k \in U$ *with* $j < k$*,including* $\eta^{'} \in L([j,k])$*. Whenever* $\left|\eta^{'}\right|^{\frac{p}{p-1}}$ *is a convex function supported* $[j,k]$*, the coming inequality holds:*

$$\left|\frac{\eta(j)+\eta(k)}{2} - \frac{1}{k-j}\int_j^k \eta(\theta)d\theta\right| \le \frac{k-j}{2(p+1)^{\frac{1}{p}}}\left[\left|\eta^{'}(j)\right|^{\frac{p}{p-1}} + \left|\eta^{'}(k)\right|^{\frac{p}{p-1}}\right], \qquad (1.\ 3)$$

point $p > 1$ *furthermore* $\frac{1}{p} + \frac{1}{q} = 1$*.*

In [17], Pearce attained enhancement and resolution of constant in Theorem 1.2 wherever strengthen this consequence by proving the successive theorem.

Theorem 1.3. [17] *Consider* $\eta : U \subseteq \mathbb{R} \to \mathbb{R}$ *is a differentiable mapping at* I°*, with* j*,* $k \in U$ *and* $j < k$*, together* $\eta^{'} \in L([j,k])$*. If* $\left|\eta^{'}\right|^{q}$ *is a convex function on* $[j,k]$*, also* $q \ge 1$*, then the subsequent inequality exists:*

$$\left|\frac{\eta(j)+\eta(k)}{2} - \frac{1}{k-j}\int_j^k \eta(\theta)d\theta\right| \le \frac{k-j}{4}\left[\frac{\left|\eta^{'}(j)\right|^{q} + \left|\eta^{'}(k)\right|^{q}}{2}\right]^{\frac{1}{q}}. \qquad (1.\ 4)$$

If $\left|\eta^{'}\right|^{q}$ *is concave on* $[j,k]$*, a bit* $q \ge 1$*. Formerly*

$$\left|\frac{\eta(j)+\eta(k)}{2} - \frac{1}{k-j}\int_j^k \eta(\theta)d\theta\right| \le \frac{k-j}{4}\left|\eta^{'}\left(\frac{j+k}{2}\right)\right|. \qquad (1.\ 5)$$

In [6], Dah-Yan Hwang established the following results for convex which affords weighted consolation of results inclined in Theorem 1.1, Theorem 1.2 and the inequality (1. 4) of Theorem1.3.

Theorem 1.4. [6] *Authorize* $\eta : U \subseteq \mathbb{R} \to \mathbb{R}$ *is a differentiable mapping on* I°*, with* j*,* $k \in U^{\circ}$ *along* $j < k$ *and allow* $\rho : [j,k] \to [0,\infty)$ *be continuous positive mapping also symmetric to* $\frac{j+k}{2}$*. Assume* $\left|\eta^{'}\right|$ *is convex function at* $[j,k]$*, succeeding inequality holds:*

$$\left|\left[\frac{\eta(j)+\eta(k)}{2}\right]\int_j^k \rho(\theta)d\theta - \int_j^k \eta(x)\rho(\theta)d\theta\right|$$
$$\le \frac{k-j}{4}\left[\left|\eta^{'}(j)\right| + \left|\eta^{'}(k)\right|\right]\int_0^1\int_{L(j,k,t)}^{U(j,k,t)} \rho(\theta)d\theta dt, \qquad (1.\ 6)$$

where $U(j,k,t) = \frac{1-t}{2}j + \frac{1+t}{2}k$ *and* $L(j,k,t) = \frac{1+t}{2}j + \frac{1-t}{2}k$.

Theorem 1.5. [6] *Confirming considerations of Theorem 1.4 are fulfilled along* $q \geq 1$. *Assuming* $\left|\eta'\right|^q$ *is convex function on* $[j,k]$, *pursuing inequality grips:*

$$\left|\left[\frac{\eta(j)+\eta(k)}{2}\right]\int_j^k \rho(\theta)d\theta - \int_j^k \eta(\theta)\rho(\theta)d\theta\right| \leq \frac{k-j}{2}\left[\frac{\left|\eta'(j)\right|^q + \left|\eta'(k)\right|^q}{2}\right]^{\frac{1}{q}} \int_0^1 \int_{L(j,k,t)}^{U(j,k,t)} \rho(\theta)d\theta dt, \quad (1.7)$$

site $U(j,k,t)$ *with* $L(j,k,t)$ *are decided in Theorem 1.4.*

The classical convexity that is stated above was generalized as ϵ-convexity by G. Toader in [19] as follows:

Definition 1.6. *Function* $\eta : [0,k^*] \to \mathbb{R}$ *named as* ϵ*-convex if*

$$\eta(t\theta + \epsilon(1-t)y) \leq t\eta(\theta) + \epsilon(1-t)\eta(y)$$

grips being $\theta, y \in [0,k^*]$, $\epsilon \in [0,1]$ *and* $t \in (0,1]$, *where* $k^* > 0$. *A function* $\eta : [0,k^*] \to \mathbb{R}$ *forenamed as* ϵ*-concave if* $-\eta$ *is* ϵ*-convex.*

Obviously, for $\epsilon = 1$ the Interpretation 1.6 recaptures perception of standard convex functions which construed on $[0,k^*]$.

Assumption of ϵ-convexity has been further generalized in [12] as declared in successive interpretation.

Definition 1.7. *Function* $\eta : [0,k^*] \to \mathbb{R}$ *is known as* (σ,ϵ)*-convex assuming*

$$\eta(t\theta + \epsilon(1-t)y) \leq t^\sigma\eta(\theta) + \epsilon(1-t^\sigma)\eta(y)$$

exists being $\theta, y \in [0,k^*]$, $(\sigma,\epsilon) \in [0,1]^2$ *with* $t \in (0,1]$, *as* $k^* > 0$. *Function* $\eta : [0,k^*] \to \mathbb{R}$ *forenamed as* (σ,ϵ)*-concave if* $-\eta$ *is* (σ,ϵ)*-convex.*

It can easily be seen that for $\sigma = 1$, the class of ϵ-convex functions are derived from the above interpretation and for $\epsilon = \sigma = 1$ a class of convex functions are derived.

For several declarations concerning Hermite-Hadamard type inequalities for ϵ-convex and (σ,ϵ)-convex functions we specify the attentive reader to [1, 3, 4, 8, 13, 14, 15, 16, 10, 11, 18] and the references cited therein.

In Section 2, we prove some new Fejér and Harmine-Hadamard type inequalities by using the ϵ- and (σ,ϵ)-convexity of the differentiable mappings. The results of this paper contains some previously proved results for convex functions defined over the interval $[0,k^*]$ as special cases.

2. FEJÉR TYPE INEQUALITIES FOR ϵ-CONVEX AND (σ,ϵ)-CONVEX FUNCTIONS

Lemma 2.1. *Consider* $\eta : U \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping at* U° *with* $\rho : [\epsilon j,k] \to [0,\infty)$ *be continuous and symmetric considering* $\frac{\epsilon j+k}{2}$ *for settled* $\epsilon \in (0,1]$, *where* ϵj, $k \in U^{\circ}$ *with* $\epsilon j < k$. *If* $\eta^{'} \in L_1[\epsilon j, k]$, *resulting expression exists*

$$\left[\frac{\eta(\epsilon j)+\eta(k)}{2}\right]\int_{\epsilon j}^{k}\rho(\theta)\,d\theta-\int\limits_{\epsilon j}^{k}\rho(\theta)\,\eta(\theta)\,d\theta$$
$$=\frac{k-\epsilon j}{4}\int\limits_{0}^{1}\left[\int_{L(t,\epsilon)}^{U(t,\epsilon)}\rho(\theta)\,d\theta\right]\left[\eta^{'}(U(t,\epsilon))-\eta^{'}(L(t,\epsilon))\right]dt, \quad (2.\ 8)$$

along

$$U(t,\epsilon)=\epsilon\left(\frac{1-t}{2}\right)j+\left(\frac{1+t}{2}\right)k$$

furthermore

$$L(t,\epsilon)=\epsilon\left(\frac{1+t}{2}\right)j+\left(\frac{1-t}{2}\right)k.$$

Proof. By the integration by parts, we get

$$\begin{aligned}
W_1 &= \int\limits_{0}^{1}\left[\int_{L(t,\epsilon)}^{U(t,\epsilon)}\rho(\theta)\,d\theta\right]\eta^{'}(U(t,\epsilon))\,dt\\
&= \frac{2}{k-\epsilon j}\int\limits_{0}^{1}\left[\int_{L(t,\epsilon)}^{U(t,\epsilon)}\rho(\theta)\,d\theta\right]d\left[\eta(U(t,\epsilon))\right]\\
&= \frac{2}{k-\epsilon j}\left[\int_{L(t,\epsilon)}^{U(t,\epsilon)}\rho(\theta)\,d\theta\right]\eta(U(t,\epsilon))\Bigg|_{0}^{1}\\
&\quad -\int\limits_{0}^{1}\left[\rho(U(t,\epsilon))+\rho(L(t,\epsilon))\right]\eta(U(t,\epsilon))\,dt\\
&= \frac{2}{k-\epsilon j}\eta(k)\int_{\epsilon j}^{k}\rho(\theta)\,d\theta-2\int\limits_{0}^{1}\rho(U(t,\epsilon))\,\eta(U(t,\epsilon))\,dt\\
&= \frac{2}{k-\epsilon j}\eta(k)\int_{\epsilon j}^{k}\rho(\theta)\,d\theta-\frac{4}{k-\epsilon j}\int\limits_{\frac{\epsilon k+j}{2}}^{k}\rho(\theta)\,\eta(\theta)\,d\theta.
\end{aligned}$$

Similarly, we can observe that

$$W_2=-\frac{2}{k-\epsilon j}\eta(\epsilon j)\int_{\epsilon j}^{k}\rho(\theta)\,d\theta+\frac{4}{k-\epsilon j}\int\limits_{\epsilon j}^{\frac{\epsilon j+k}{2}}\rho(\theta)\,\eta(\theta)\,d\theta.$$

Hence

$$W_1 - W_2 = \frac{2}{k-\epsilon j}\left[\eta\left(\epsilon j\right)+\eta\left(k\right)\right]\int_{\epsilon j}^{k}\rho\left(\theta\right)d\theta - \frac{4}{k-\epsilon j}\int\limits_{\epsilon j}^{k}\rho\left(\theta\right)\eta\left(\theta\right)d\theta.$$

Multiplying the above result by $\frac{k-\epsilon j}{4}$, we get what is desired. □

Remark 2.2. *If we choose $\epsilon = 1$ in Lemma 2.1, we obtain the result proved in* [3] *[Lemma 2.1, page 9599].*

Remark 2.3. *If $\rho\left(\theta\right) = \frac{1}{k-\epsilon j}$, $\theta \in [\epsilon j, k]$, then the subsequent equality holds*

$$\frac{\eta\left(\epsilon j\right)+\eta\left(k\right)}{2} - \frac{1}{k-\epsilon j}\int\limits_{\epsilon j}^{k}\eta\left(\theta\right)d\theta$$
$$= \frac{k-\epsilon j}{8}\int\limits_{0}^{1}\left[\eta^{'}\left(\epsilon\left(\frac{1-t}{2}\right)j+\left(\frac{1+t}{2}\right)k\right)-\eta^{'}\left(\epsilon\left(\frac{1+t}{2}\right)j+\left(\frac{1-t}{2}\right)k\right)\right]dt. \tag{2. 9}$$

Now we present some Fejér type inequalities for ϵ-convex functions.

Theorem 2.4. *Let $\eta : W \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $W^{\circ} \supset [0, \infty)$ and $\rho : [\epsilon j, k] \to [0, \infty)$ be continuous and symmetric considering $\frac{\epsilon j+k}{2}$ for settled $\epsilon \in (0, 1]$, where ϵj, $k \in W^{\circ}$ with $\epsilon j < k$. Supposing $\eta^{'} \in L_1[\epsilon j, k]$ and $\left|\rho^{'}\right|$ is ϵ-convex on $[0, k]$, ensuing inequality holds*

$$\left|\left[\frac{\eta\left(\epsilon j\right)+\eta\left(k\right)}{2}\right]\int_{\epsilon j}^{k}\rho\left(\theta\right)d\theta - \int\limits_{\epsilon j}^{k}\rho\left(\theta\right)\eta\left(\theta\right)d\theta\right|$$
$$\leq \frac{k-\epsilon j}{4}\left[\epsilon\left|\eta^{'}\left(j\right)\right|+\left|\eta^{'}\left(k\right)\right|\right]\int\limits_{0}^{1}\int_{L(t,\epsilon)}^{U(t,\epsilon)}\rho\left(\theta\right)d\theta dt. \tag{2. 10}$$

Proof. Taking absolute value on both sides of (2. 8) and employing ϵ-convexity on $[0,k]$, we have

$$\begin{aligned}
&\left|\left[\frac{\eta(\epsilon j)+\eta(k)}{2}\right]\int_{\epsilon j}^{k}\rho(\theta)\,d\theta-\int_{\epsilon j}^{k}\rho(\theta)\,\eta(\theta)\,d\theta\right| \\
\leq\ & \frac{k-\epsilon j}{4}\int_{0}^{1}\left[\int_{L(t,\epsilon)}^{U(t,\epsilon)}\rho(\theta)\,d\theta\right]\left[\left|\eta'(U(t,\epsilon))\right|+\left|\eta'(L(t,\epsilon))\right|\right]dt \\
\leq\ & \frac{k-\epsilon j}{4}\int_{0}^{1}\left[\int_{L(t,\epsilon)}^{U(t,\epsilon)}\rho(\theta)\,d\theta\right]\left[\epsilon\left(\frac{1-t}{2}\right)\left|\eta'(j)\right|+\left(\frac{1+t}{2}\right)\left|\eta'(k)\right|\right. \\
& \left.+\epsilon\left(\frac{1+t}{2}\right)\left|\eta'(j)\right|+\left(\frac{1-t}{2}\right)\left|\eta'(k)\right|\right]dt \\
=\ & \frac{k-\epsilon j}{4}\left[\epsilon\left|\eta'(j)\right|+\left|\eta'(k)\right|\right]\int_{0}^{1}\int_{L(t,\epsilon)}^{U(t,\epsilon)}\rho(\theta)\,d\theta dt.
\end{aligned}$$

Hence argument of theorem is concluded. □

Remark 2.5. *The choice of* $\epsilon=1$, *gives the result of Theorem 2.2 proved in* [3] *for convex functions defined on* $[0,k]$.

Corollary 2.6. *Under the assumptions of Theorem 2.4 and the choice of* $\rho(\theta)=\frac{1}{k-\epsilon j}$, $\theta\in[\epsilon j,k]$, *subsequent inequality holds*

$$\left|\frac{\eta(\epsilon j)+\eta(k)}{2}-\frac{1}{k-\epsilon j}\int_{\epsilon j}^{k}\eta(\theta)\,d\theta\right|\leq\frac{k-\epsilon j}{8}\left[\epsilon\left|\eta'(j)\right|+\left|\eta'(k)\right|\right]. \tag{2. 11}$$

Remark 2.7. *Assuming* $\epsilon=1$ *in Corollary 2.6, we get the result proved in* [2, Theorem 2.2] *for convex functions rationale on* $[0,k]$.

Theorem 2.8. *Let* $\eta: W\subseteq\mathbb{R}\to\mathbb{R}$ *be a differentiable mapping on* $W^{\circ}\supset[0,\infty)$ *and* $\rho:[\epsilon j,k]\to[0,\infty)$ *be continuous and symmetric regarding* $\frac{\epsilon j+k}{2}$ *for settled* $\epsilon\in(0,1]$, *where* $\epsilon j,k\in W^{\circ}$ *with* $\epsilon j<k$. *If* $\eta'\in L_1[\epsilon j,k]$ *and* $\left|\eta'\right|^q$ *is* ϵ*-convex on* $[0,k]$ *for* $q\geq1$, *specified inequality is*

$$\begin{aligned}
&\left|\left[\frac{\eta(\epsilon j)+\eta(k)}{2}\right]\int_{\epsilon j}^{k}\rho(\theta)\,d\theta-\int_{\epsilon j}^{k}\rho(\theta)\,\eta(\theta)\,d\theta\right| \\
&\qquad\leq\frac{k-\epsilon j}{2}\left[\frac{\epsilon\left|\eta'(j)\right|^q+\left|\eta'(k)\right|^q}{2}\right]^{\frac{1}{q}}\int_{0}^{1}\int_{L(t,\epsilon)}^{U(t,\epsilon)}\rho(\theta)\,d\theta dt.
\end{aligned}\tag{2. 12}$$

Proof. Applying Lemma 2.1 and usage of Hölder inequality, gives

$$\left|\left[\frac{\eta(\epsilon j)+\eta(k)}{2}\right]\int_{\epsilon j}^{k}\rho(\theta)\,d\theta-\int_{\epsilon j}^{k}\rho(\theta)\,\eta(\theta)\,d\theta\right|\leq\frac{k-\epsilon j}{4}$$
$$\times\left\{\left(\int_0^1\left[\int_{L(t,\epsilon)}^{U(t,\epsilon)}\rho(\theta)\,d\theta\right]dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left[\int_{L(t,\epsilon)}^{U(t,\epsilon)}\rho(\theta)\,d\theta\right]\left|\eta'(U(t,\epsilon))\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left[\int_{L(t,\epsilon)}^{U(t,\epsilon)}\rho(\theta)\,d\theta\right]dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left[\int_{L(t,\epsilon)}^{U(t,\epsilon)}\rho(\theta)\,d\theta\right]\left|\eta'(U(t,\epsilon))\right|^q dt\right)^{\frac{1}{q}}\right\}. \tag{2. 13}$$

Employing power-mean inequality $\theta^r+y^r\leq 2^{1-r}(\theta+y)^r$ for $j,k>0$ with $r<1$,

$$\left(\int_0^1\left[\int_{L(t,\epsilon)}^{U(t,\epsilon)}\rho(\theta)\,d\theta\right]\left|\eta'(U(t,\epsilon))\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left[\int_{L(t,\epsilon)}^{U(t,\epsilon)}\rho(\theta)\,d\theta\right]\left|\eta'(U(t,\epsilon))\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq 2^{1-\frac{1}{q}}\left(\int_0^1\left[\int_{L(t,\epsilon)}^{U(t,\epsilon)}\eta(\theta)\,d\theta\right]\right)^{\frac{1}{q}}\left(\int_0^1\left|\eta'(U(t,\epsilon))\right|^q dt+\int_0^1\left|\eta'(U(t,\epsilon))\right|^q dt\right)^{\frac{1}{q}}. \tag{2. 14}$$

Since $\left|\eta'\right|^q$ is ϵ-convex on $[0,b]$ for settled $\epsilon\in(0,1]$ and $q\geq 1$, we attained

$$\int_0^1\left|\eta'(U(t,\epsilon))\right|^q dt+\int_0^1\left|\eta'(U(t,\epsilon))\right|^q dt$$
$$\leq\epsilon\left(\frac{1-t}{2}\right)\left|\eta'(a)\right|^q+\left(\frac{1+t}{2}\right)\left|\eta'(b)\right|^q$$
$$+\epsilon\left(\frac{1+t}{2}\right)\left|\eta'(a)\right|^q+\left(\frac{1-t}{2}\right)\left|\eta'(b)\right|^q=\epsilon\left|\eta'(a)\right|^q+\left|\eta'(b)\right|^q \tag{2. 15}$$

Using (2. 15) in (2. 14) and then resulting inequality in (2. 13), we grab which was desired. □

Remark 2.9. *Assuming* $\epsilon=1$, *we accomplished result of Theorem 2.4 proved in* [3].

Corollary 2.10. *Under the assumptions of Theorem 2.8 and the choice of* $g(\theta) = \frac{1}{k-\epsilon j}$, $x \in [\epsilon j, k]$, *subsequent result exists*

$$\left| \frac{\eta(\epsilon j) + \eta(k)}{2} - \frac{1}{k-\epsilon j} \int_{\epsilon j}^{k} \eta(\theta)\, d\theta \right| \leq \frac{k-\epsilon j}{4} \left[\frac{\epsilon \left|\eta^{'}(j)\right|^{q} + \left|\eta^{'}(k)\right|^{q}}{2} \right]^{\frac{1}{q}}. \quad (2.\ 16)$$

Remark 2.11. *Consider* $\epsilon = 1$ *in Corollary 2.10, we draw the result proved in* [17, Theorem 1].

Now we present some Fejér type inequalities for (σ, ϵ)-convex functions.

Theorem 2.12. *Endorse* $\eta : W \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* $W^{\circ} \supset [0, \infty)$ *and* $\rho : [\epsilon j, k] \to [0, \infty)$ *be continuous and symmetric by* $\frac{\epsilon j + k}{2}$ *for established* $\epsilon \in (0, 1]$, *where* ϵj, $k \in W^{\circ}$ *with* $\epsilon j < k$. *Wherever* $\eta^{'} \in L_1[\epsilon j, k]$ *and* $\left|\eta^{'}\right|$ *is* (σ, ϵ)*-convex on* $[0, k]$ *for* $(\sigma, \epsilon) \in (0, 1] \times (0, 1]$, *resulting inequality is*

$$\left| \left[\frac{\eta(\epsilon j) + \eta(k)}{2} \right] \int_{\epsilon j}^{k} \rho(\theta)\, d\theta - \int_{\epsilon j}^{k} \rho(\theta)\, \eta(\theta)\, d\theta \right| \leq \frac{(k-\epsilon j)^2}{4} \|\rho\|_{\infty} \left[\epsilon \chi(\sigma) \left|\eta^{'}(j)\right| + (1 - \chi(\sigma)) \left|\eta^{'}(k)\right| \right], \quad (2.\ 17)$$

spot

$$\chi(\sigma) = \frac{2(2^{-\sigma} + \sigma)}{(\sigma+2)(\sigma+1)} \text{ and } \|\rho\|_{\infty} = \sup_{\theta \in [\epsilon j, k]} |\rho(\theta)|.$$

Proof. We observed the consequences of Lemma 2.1 can be drafted as

$$\begin{aligned} &\left[\frac{\eta(\epsilon j) + \eta(k)}{2} \right] \int_{\epsilon j}^{k} \rho(\theta)\, d\theta - \int_{\epsilon j}^{k} \rho(\theta)\, \eta(\theta)\, d\theta \\ &= \frac{k-\epsilon j}{4} \int_{0}^{1} \left[\int_{L(t,\epsilon)}^{U(t,\epsilon)} \rho(\theta)\, d\theta \right] \left[\eta^{'}(U(t,\epsilon)) - \eta^{'}(L(t,\epsilon)) \right] dt \\ &\leq \frac{(k-\epsilon j)^2}{4} \|\rho\|_{\infty} \int_{0}^{1} t \left[\eta^{'}(U(t,\epsilon)) - \eta^{'}(L(t,\epsilon)) \right] dt, \quad (2.\ 18) \end{aligned}$$

where $\|\rho\|_{\infty} = \sup_{\theta \in [\epsilon j, k]} |\rho(\theta)|$.
Taking the absolute value on both sides of (2. 18), we gained

$$\left| \left[\frac{\eta(\epsilon j) + \eta(k)}{2} \right] \int_{\epsilon j}^{k} \rho(\theta)\, d\theta - \int_{\epsilon j}^{k} \rho(\theta)\, \eta(\theta)\, d\theta \right| \leq \frac{(k - \epsilon j)^2}{4} \|\rho\|_{\infty} \int_0^1 t \left[\left| \eta^{'}(U(t, \epsilon)) \right| + \left| \eta^{'}(L(t, \epsilon)) \right| \right] dt. \quad (2.\ 19)$$

Adopting (σ, ϵ)-convexity of $\left|\eta^{'}\right|$ on $[0, k]$, we have

$$\begin{aligned} &\int_0^1 t \left[\left| \eta^{'}(U(t, \epsilon)) \right| + \left| \eta^{'}(L(t, \epsilon)) \right| \right] dt \\ &\leq \int_0^1 t \left\{ \left(\frac{1+t}{2} \right)^{\sigma} \left| \eta^{'}(b) \right| + \epsilon \left[1 - \left(\frac{1+t}{2} \right)^{\sigma} \right] \left| \eta^{'}(j) \right| \right. \\ &\quad \left. + \left(\frac{1-t}{2} \right)^{\sigma} \left| \eta^{'}(k) \right| + \epsilon \left[1 - \left(\frac{1-t}{2} \right)^{\sigma} \right] \left| \eta^{'}(a) \right| \right\} dt \\ &= \left| \eta^{'}(k) \right| \int_0^1 t \left[\left(\frac{1+t}{2} \right)^{\sigma} + \left(\frac{1-t}{2} \right)^{\alpha} \right] dt \\ &\quad + \epsilon \left| \eta^{'}(j) \right| \int_0^1 t \left[2 - \left(\frac{1-t}{2} \right)^{\sigma} - \left(\frac{1+t}{2} \right)^{\sigma} \right] dt \\ &= \left\{ \frac{2(2^{-\sigma} + \sigma)}{(\sigma+2)(\sigma+1)} \right\} \left| \eta^{'}(k) \right| + \epsilon \left\{ 1 - \frac{2(2^{-\sigma} + \sigma)}{(\sigma+2)(\sigma+1)} \right\} \left| \eta^{'}(j) \right|. \quad (2.\ 20) \end{aligned}$$

Applying the inequality (2. 20) in (2. 19), we scored the result given by (2. 17). □

Corollary 2.13. *Presume conditions of Theorem 2.12 are fulfilled and* $\rho(\theta) = \frac{1}{k - \epsilon j}$, $\theta \in [\epsilon j, k]$, *subsequent inequality holds*

$$\left| \frac{\eta(\epsilon j) + \eta(k)}{2} - \frac{1}{k - \epsilon j} \int_{\epsilon j}^{k} \eta(x)\, dx \right| \leq \frac{k - \epsilon j}{4} \left[\epsilon \chi(\sigma) \left| \eta^{'}(j) \right| + (1 - \chi(\sigma)) \left| \eta^{'}(k) \right| \right], \quad (2.\ 21)$$

position $\chi(\sigma)$ *is specified in Theorem 2.12.*

Remark 2.14. *If* $\sigma = \epsilon = 1$ *in (2. 21), we get the result proved in* [2, Theorem 2.2] *for convex functions defined on* $[0, k]$.

Theorem 2.15. *Let $\eta : W \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $W^{\circ} \supset [0,\infty)$ and $\rho : [\epsilon j, k] \to [0,\infty)$ be continuous and symmetric by $\frac{\epsilon j+k}{2}$, settle $\epsilon \in (0,1]$, where ϵj, $k \in W^{\circ}$ with $\epsilon j < k$. Granted $\eta^{'} \in L_1[\epsilon j, k]$ and $\left|\eta^{'}\right|^{q}$ is (σ,ϵ)-convex on $[0,k]$ for $q \geq 1$, $(\sigma,\epsilon) \in (0,1] \times (0,1]$, coming inequality grips*

$$\left|\left[\frac{\eta(\epsilon j)+\eta(k)}{2}\right]\int_{\epsilon j}^{k}\rho(\theta)\,d\theta-\int\limits_{\epsilon j}^{k}\rho(\theta)\,\eta(\theta)\,d\theta\right| \leq \frac{(k-\epsilon j)^{2}}{4}\left\|\rho\right\|_{\infty}\left[\epsilon\chi(\sigma)\left|\eta^{'}(j)\right|^{q}+(1-\chi(\sigma))\left|\eta^{'}(k)\right|^{q}\right]^{\frac{1}{q}}, \quad (2.\ 22)$$

where $\chi(\sigma)$ and $\|\rho\|_{\infty}$ are construe in Theorem 2.12.

Proof. Continuing from (2. 19) and employing Hölder inequality, we achieved

$$\left|\left[\frac{\eta(\epsilon j)+\eta(k)}{2}\right]\int_{\epsilon j}^{k}\rho(\theta)\,d\theta-\int\limits_{\epsilon j}^{k}\rho(\theta)\,\eta(\theta)\,d\theta\right| \leq \frac{(k-\epsilon j)^{2}}{4}\left\|\rho\right\|_{\infty}\left(\int\limits_{0}^{1}t\,dt\right)^{1-\frac{1}{q}} \times\left\{\left(\int\limits_{0}^{1}t\left|\eta^{'}(U(t,\epsilon))\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int\limits_{0}^{1}t\left|\eta^{'}(U(t,\epsilon))\right|^{q}dt\right)^{\frac{1}{q}}\right\}. \quad (2.\ 23)$$

Accepting power-mean inequality $\theta^{r}+y^{r} \leq 2^{1-r}(\theta+y)^{r}$ for $j,k>0$ and $r<1$, we attain

$$\left(\int\limits_{0}^{1}t\left|\eta^{'}(U(t,\epsilon))\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int\limits_{0}^{1}t\left|\eta^{'}(U(t,\epsilon))\right|^{q}dt\right)^{\frac{1}{q}} \leq 2^{1-\frac{1}{q}}\left(\int\limits_{0}^{1}t\left|\eta^{'}(U(t,\epsilon))\right|^{q}dt+\int\limits_{0}^{1}t\left|\eta^{'}(U(t,\epsilon))\right|^{q}dt\right)^{\frac{1}{q}} \quad (2.\ 24)$$

Since $\left|\eta^{'}\right|^{q}$ is (σ,ϵ)-convex on $[0,k]$ for $q \geq 1$, $(\sigma,\epsilon) \in (0,1] \times (0,1]$, we have

$$\int_{0}^{1} t\left|\eta^{'}\left(U\left(t,\epsilon\right)\right)\right|^{q} dt + \int_{0}^{1} t\left|\eta^{'}\left(U\left(t,\epsilon\right)\right)\right|^{q} dt$$

$$\leq \int_{0}^{1} t\left\{\left(\frac{1+t}{2}\right)^{\sigma}\left|\eta^{'}\left(k\right)\right|^{q} + \epsilon\left[1-\left(\frac{1+t}{2}\right)^{\sigma}\right]\left|\eta^{'}\left(j\right)\right|^{q}\right.$$

$$\left. + \left(\frac{1-t}{2}\right)^{\sigma}\left|\eta^{'}\left(k\right)\right|^{q} + \epsilon\left[1-\left(\frac{1-t}{2}\right)^{\sigma}\right]\left|\eta^{'}\left(j\right)\right|^{q}\right\} dt$$

$$= \left\{\frac{2\left(2^{-\sigma}+\sigma\right)}{\left(\sigma+2\right)\left(\sigma+1\right)}\right\}\left|\eta^{'}\left(k\right)\right|^{q} + \epsilon\left\{1-\frac{2\left(2^{-\sigma}+\sigma\right)}{\left(\sigma+2\right)\left(\sigma+1\right)}\right\}\left|\eta^{'}\left(j\right)\right|^{q}. \quad (2.\ 25)$$

Using (2. 25) in (2. 24) and then the resulting inequality in (2. 23), we get the appropriate inequality. □

Corollary 2.16. *Expect the conditions of Theorem 2.15 are convinced and* $\rho\left(\theta\right) = \frac{1}{k-\epsilon j}$*,* $\theta \in [\epsilon j, k]$*, ensuing inequality grips*

$$\left|\frac{\eta\left(\epsilon j\right)+\eta\left(k\right)}{2} - \int_{\epsilon j}^{k} \eta\left(x\right) dx\right|$$

$$\leq \frac{k-\epsilon j}{4}\left[\epsilon\chi\left(\sigma\right)\left|\eta^{'}\left(j\right)\right|^{q} + \left(1-\chi\left(\sigma\right)\right)\left|\eta^{'}\left(k\right)\right|^{q}\right]^{\frac{1}{q}}, \quad (2.\ 26)$$

spot $\chi\left(\alpha\right)$ *is defined in Theorem 2.12.*

Remark 2.17. *Assuming* $\sigma = \epsilon = 1$ *in (2. 26), we get the result craved in* [17, Theorem 1] *for convex functions decided on* $[0,k]$.

REFERENCES

[1] M. K. Bakula and M. E. Özdemir,*Hadamard type inequalities for m-convex and (α,m)-convex functions*, J. Inequal. Pure Appl. Math. **9**, No. 4 (2008) Art. 96, 12 pages.

[2] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11**, No. 5 (1998) 91-95.

[3] S. S. Dragomir, *On some new inequalities of Hermite-Hadamard type for m-convex functions*, Tamkang J. Math. **33**, (2002) 45-55.

[4] S. S. Dragomir and G. Toader, *Some inequalities for m-convex functions*, Studia Univ. Babes Bolyai Math. **38**, (1993) 21-28.

[5] S. S. Dragomir, *Generalization, Refinement and Reverses of the Right Fejer Inequality for Convex Functions*, Punjab Univ. j. math. **49**, No. 2 (2017) 1-13.

[6] D-Y. Hwang, *Some inequalities for differentiable convex mapping with application to weighted trapezoidal formula and higher moments of random variables*, Applied Mathematics and Computation **217**, (2011) 9598-9605.

[7] D-Y. Hwang, *Some inequalities for differentiable convex mapping with application to weighted midpoint formula and higher moments of random variables*, Applied Mathematics and Computation **232**, (2014) 68-75.

[8] H. Kavurmacı, M. Emin Özdemir and M. Avcı, *New Ostrowski type inequalities for m-convex functions and applications*, Hacettepe Journal of Mathematics and Statistics **40**, N. 2 (2011) 135 – 145.

[9] M. A. Latif, *Estimates of Hermite-Hadamard Inequality for Twice Differentiable Harmonically-Convex Functions with Applications*, Punjab Univ. j. math. **50**, No. 1 (2018) 1-13.

[10] M. A. Latif, S. S. Dragomir and E. Momoniat, *Some -analogues of Hermite-Hadamard Inequality for s-convex Functions in the Second Sense and Related Estimates*, Punjab Univ. j. Mmath. **48**, No. 2 (2016) 147-166.

[11] M. A. Latif, *Sever Dragomir and Ebrahim Momoniat, Some Weighted Hermite-Hadamard-Noor Type Inequalities for Differentiable Preinvex and Quasi Preinvex Functions*, Punjab Univ. j. math. **47**, No. 1 (2015) 57-72.

[12] V. G. Mihesan, *A generalization of the convexity, Seminar on Functional Equations*, Approx. Convex, Cluj-Napoca, 1993 (Romania).

[13] M. E. Özdemir, M. Avci and E. Set, *On some inequalities of Hermite–Hadamard type via m-convexity*, Appl. Math. Lett. **23**, No. 9 (2010) 1065–1070.

[14] M. E. Özdemir, H. Kavurmaci and E. Set, *Ostrowski's type inequalities for (α, m)-convex functions*, Kyungpook Math. J. **50**, (2010) 371–378.

[15] M. E. Özdemir, M. Avcı and H. Kavurmacı, *Hermite–Hadamard-type inequalities via (α, m)-convexity*, Comput. Math. Appl. **61**, (2011) 2614–2620.

[16] M. E. Özdemir, E. Set and M. Z. Sarıkaya, *Some new Hadamard's type inequalities for coordinated m-convex and (α, m)-convex functions*, Hacet. J. Math. Stat. **40**, (2011) 219-229.

[17] C. E. M. Pearce, *Inequalities for differentiable mappings with application to special means and quadrature formulae*, Applied Mathematics Letters **13**, (2000) 51-55.

[18] E. Set, M. Sardari, M. E. Özdemir and J. Rooin, *On generalizations of the Hadamard inequality for (α, m)-convex functions*, Kyungpook Math. J. **52**, (2012) 307-317.

[19] G. Toader, *Some generalizations of the convexity, Proceedings of the Colloquium on Approximation and Optimization*, Univ. Cluj-Napoca, Cluj-Napoca (1985) 329–338.