Punjab University Journal of Mathematics
ISSN: 1016-2526
Vol.51(11)(2019) pp. 17-29

Modified Optimal Homotopy Perturbation Method to Investigate Jeffery-Hamel Flow

Liaqat Ali
Department of Basic Sciences and Humanities, CECOS University Peshawer, Khyber Pakhtunkhwa, Pakistan
Email: liaqat@cecos.edu.pk
Saeed Islam
Department of Mathematics, Abdul Wali Khan University Mardan, KPK, Pakistan.
Email: saeedislam @awkum.edu.pk
Taza Gul
Department of Mathematics, City University of Science and Information Technology, Peshawar 25000, Khyber Pakhtunkhwa, Pakistan,
Email: tazagulsafi@yahoo.com

Abstract. A simple analytical approach is used to solve nonlinear problem forming in the phenomenon of Jeffery-Hamel Flow. The suggested technique consists of a homotopy with an embedding parameter, Daftardar-Gejji and Jafari polynomials (DJPs) and auxiliary functions. It has also some constants, used for controlling the convergence of the solution. The suggested technique is named Modified Optimal Homotopy Perturbation Method (MOHPM). The method is simple but effective and the results gained by this are in good agreement with numerical outcomes. The achieved results are compared with the results gained by Homotopy asymptotic method (HAM), Optimal homotopy asymptotic method (OHAM), Homotopy perturbation method (HPM), Modified Homotopy Perturbation Method (MHPM), Adomain decomposition method (ADM) and Differential transform method (DTM) to authenticate the code.

Key Words: Jeffery-Hamel Flow, Modified Optimal Homotopy Perturbation Method, Daftardar-Gejji and Jafari Polynomials, Least Square Method.

1. Introduction

The flow in converging and diverging channel has a significant role in engineering and industries. This flow is applicable in fluid mechanics, civil, environmental, mechanical and bio mechanical engineering. The applications of the flow are; pressure driven transport of particles through a symmetric converging and diverging channel, heat transfer of heat exchangers for milk flowing and many others as in [1,2]. In various fields of science and engineering, nonlinear equations, as well as their analytic and numerical solutions are fundamentally important. The eminent Jeffery-Hamel problem between the non-parallel walls has investigated by Jeffery [3] and Hamel [4] for the first time. This type of work can also be seen in [5]. The Jeffery-Hamel flow problem has been solved by numerical methods as well as analytical methods like; Homotopy Perturbation Method (HPM), Differential Transform Method (DTM), Homotopy Analysis Method (HAM), Adomain Decomposition Method (ADM) and Optimal Homotopy Asymptotic Method (OHAM). Joneidi et al. [6] and Smaeilpour et al [7] also investigate the phenomenon of Jeffery-Hamel flow by using analytical methods such as HAM, HPM, DTM and OHAM. Such type of research is found in [8-10]. Here in our research we have used an analytical approach MOHPM to investigate the Jeffery-Hamel flow problem. It consists of auxiliary convergence control parameters, initial guess, least square method and auxiliary function. The effectiveness of the applied technique depends upon construction and determination of the auxiliary functions combined with a suitable technique to optimally control the convergence of the solution. In this method, a nonlinear term is expanded in terms of Daftardar-Gejji and Jafari polynomials [11,12] instead of normal expansion due to which produces more accurate and reliable results than the other analytical techniques. The relevant work can be seen in [13-19]. The same work is also found in [20-22]. It can be observed from the solved nonlinear equation of governing Jeffery hamel flow in section 5 that MOHPM is better for different problems in the fluids. It consists of few steps and solves difficult problem in a simple way with more accuracy. The suggested method may be used for the solution of ODEs, IDEs and their system forming in different physical phenomena. The manuscript has seven sections given below. Section 1 consists of introduction while section 2 is devoted to the statement of the model and its mathematical formulation. The explanation about daftardar-Gejji and Jafari polynomials as well as convergence are studied in section 3. Section 4 contains the introduction of MOHPM. Section 5 is devoted to the application of MOHPM to investigate Jeffery-Hamel flow. In section 6, the results are shown in the form of figures and tables. In section 7 some conclusions are mentioned to facilitate the readers.

2. Statement of the Model and its Mathematical Formulation:

Assume a two-dimensional flow of an incompressible conducting viscous fluid between two rigid plane walls with an angle $\pm\alpha$, which is steady and fully developed, as illustrated in Figure-1. The channel walls are supposed to be convergent if α is less than 0 and divergent if α is greater than 0. The velocity depends upon r and θ and is taken along the radial direction only explained in [6,7,10].

Now the equation of continuity and Navier- Stokes equations in polar coordinates given as,

$$\frac{\rho}{r}\frac{\partial}{\partial r}(r\,U(r,\theta)) = 0, \tag{2. 1}$$

$$\begin{aligned} U(r,\theta)\,\frac{\partial\,U(r,\theta)}{\partial r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\Big[\frac{\partial^2 U(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial U(r,\theta)}{\partial r} \\ &+ \frac{1}{r^2}\frac{\partial^2 U(r,\theta)}{\partial \theta^2} - \frac{U(r,\theta)}{r^2}\Big], \end{aligned} \tag{2. 2}$$

$$-\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \frac{2\nu}{r^2}\,\frac{\partial U(r,\theta)}{\partial \theta} = 0, \tag{2. 3}$$

where ν the kinematic viscosity, p the pressure, ρ is the fluid density. Use Eq.(2. 1), we obtained:

$$g(\theta) = r\,U(r,\theta), \tag{2. 4}$$

$$\mu(x) = \frac{g(\theta)}{g_{\max}},\ \ x = \frac{\theta}{\alpha}. \tag{2. 5}$$

Now using Eq.(2. 5) in Eq.(2. 2) and Eq.(2. 3) to get:

$$\mu''' + 2\alpha\,\mathrm{Re}\,\mu\,\mu' + 4\,\alpha^2\mu' = 0, \tag{2. 6}$$

With boundary conditions

$$\mu(0) = 1,\quad \mu'(0) = 0,\ \ \mu(1) = 0. \tag{2. 7}$$

. Where the velocity $U_{\max}$ is maximum at centre of the channel. $\alpha < 0$, $U_{\max} < 0$ is taken for convergent channel while for divergent channel $\alpha > 0$, $U_{\max} > 0$. Re $= \frac{\alpha\, g_{\max}}{v} = U_{\max} r\,\alpha$ shows Reynolds number.

TABLE 1. **Nomenclature**

$Symbols$	Defined as	Symbols	Defined as
$U(r,\theta)$	Radial Velocity	v	Coefficient of kinematic viscosity
p	The pressure	ρ	Fluid density (kgm^{-3})
$\alpha < 0$, $U_{\max} < 0$	is taken for convergent channel	$\alpha > 0$, $U_{\max} > 0$	for divergent channel
Re $= \frac{\alpha\, g_{\max}}{v} = U_{\max} r\,\alpha$	shows Reynolds number	α	Channel angle
ℓ	embedding parameter	r, θ	Radial and Angular coordinates
μ, g	Similarity functions	$c_0, c_1, \ldots$	Similarity constants
L	Linear	N	Nonlinear
$U_{\max}$	Maximum velocity at centre of the channel		

3. EXPLANATION OF THE POLYNOMIALS USED IN OHPM AND MOHPM (DAFTARDAR-GEJJI AND JAFARI POLNOMIALS (DJPS))

3.1. Explanation about the polynomials of OHPM and MOHPM:. Suppose we have a nonlinear term $N(\mu) = \mu^2$ in a functional equation such that

$$\mu = \mu_0 + \ell\mu_1 + \ell^2\mu_2 + \ell^3\mu_3$$

. If this function is expanded in taylor series about μ_0 and considered $\ell = 1$ then we have:

$$\begin{aligned} N(\mu) &= \mu^2 = (\mu_0 + \mu_1 + \mu_2 + \mu_3)^2 \\ &= \underbrace{\mu_0^2} + \underbrace{(\mu_1^2 + 2\mu_0\mu_1)} + \underbrace{(\mu_2^2 + 2\mu_0\mu_2 + 2\mu_1\mu_2)} \\ &+ \underbrace{(2\mu_0\mu_3 + 2\mu_1\mu_3 + 2\mu_2\mu_3 + \mu_3^2)} \end{aligned} \tag{3. 8}$$

Relating to the power of ℓ, the OHPM built-in polynomials for this nonlinear term are:

$$\begin{aligned} N_0(\mu_0) &= \underbrace{\mu_0^2}, \\ N_1(\mu_0,\mu_1) &= \underbrace{2\mu_0\mu_1}, \\ N_2(\mu_0,\mu_1,\mu_2) &= \underbrace{\mu_1^2+2\mu_0\mu_2}, \\ N_3(\mu_0,\mu_1,\mu_2,\mu_3) &= \underbrace{2\mu_1\mu_2+2\mu_0\mu_3}. \end{aligned} \tag{3. 9}$$

We notice that these polynomials took only six terms of the series while the series $Eq.(3.8)$ is consisting of ten terms. Relating to the power of ℓ, the polynomials in MOHPM for this nonlinear term are:

$$\begin{aligned} N_0(\mu_0) &= \underbrace{\mu_0^2}, \\ N_1(\mu_0,\mu_1) &= \underbrace{\mu_1^2+2\mu_0\mu_1}, \\ N_2(\mu_0,\mu_1,\mu_2) &= \underbrace{\mu_2^2+2\mu_0\mu_2+2\mu_1\mu_2}, \\ N_3(\mu_0,\mu_1,\mu_2,\mu_3) &= \underbrace{2\mu_0\mu_3+2\mu_1\mu_3+2\mu_2\mu_3+\mu_3^2}, \end{aligned} \tag{3. 10}$$

From the above we notice that these polynomials took all the terms of the expansion $Eq.(3.8)$. It clearly indicates the superiority of MOHPM-polynomials (Daftardar-Gejji and Jafari Polynomials) over the OHPM-polynomials.

3.2. **Convergence.** Here our mean is to find the condition for the convergence of DJPs used in the method: The nonlinear function $N(\mu)$ is written in the following form,

$$\begin{aligned} N(\mu) = N(\mu_0) + [N(\mu_0+\mu_1) - N(\mu_0)] + \\ [N(\mu_0+\mu_1+\mu_2) - N(\mu_0+\mu_1)] + \dots \end{aligned} \tag{3. 11}$$

Let $G_0 = N(\mu_0)$ and

$$G_n = N(\sum_{i=0}^{n}\mu_i) - N(\sum_{i=0}^{n-1}\mu_i), \tag{3. 12}$$

for $n = 1, 2, \dots$ therefore $N(\mu) = \sum_{i=0}^{\infty} G_i$. Also by the use of taylors theorem given below,

Taylor's Theorem. Suppose that $\mu \in C^n(u)$, where u is an open subset of X (Banach space) containing the line segment from x_0 to h, then

$$\begin{aligned} N(x_0+h) = N(x_0) + N'(x_0)h + N''(x_0)\tfrac{(h^2)}{2!} + \dots + N^{(n-1)}(x_0)\tfrac{(h^{(n-1)})}{(n-1)!} \\ = \textstyle\sum_{k=0}^{n} N^{(k)}(x_0)\frac{h^k}{k!} + q(x), \end{aligned} \tag{3. 13}$$

where $q(x)$ is such that $\|q(x)\| = O\|x\|^n$
and $N^{(k)}(x)$ is symmetric.

We get as:

$$G_1 = N(\mu_0+\mu_1) - N(\mu_0) = N(\mu_0) + N'(\mu_0)\mu_1 + N''(\mu_0)\frac{\mu_1^2}{2!} + \ldots - N(\mu_0)$$
$$= \sum_{k=1}^{\infty} N^{(k)}(\mu_0)\frac{\mu_1^k}{k!}, \qquad (3.\ 14)$$

$$G_2 = N(\mu_0+\mu_1+\mu_2) - N(\mu_0+\mu_1) =$$
$$N(\mu_0) + N'(\mu_0)\mu_1 + N''(\mu_0)\frac{\mu_1^2}{2!} + \ldots - N(\mu_0)$$
$$= \sum_{j=1}^{\infty} N^{(j)}(\mu_0+\mu_1)\frac{\mu_2^j}{j!}$$
$$= \sum_{j=1}^{\infty}[\sum_{i=0}^{\infty} N^{(i+j)}(\mu_0)\frac{\mu_1^i}{i!}]\frac{\mu_2^j}{j!}, \qquad (3.\ 15)$$

$$G_3 = \sum_{i_{3=1}}^{\infty}\sum_{i_{2=0}}^{\infty}\sum_{i_{1=0}}^{\infty} N^{(i_{1+}i_{2+}i_3)}(\mu_0)\,\frac{\mu_3^{i_3}}{i_3!}\frac{\mu_2^{i_2}}{i_2!}\ \frac{\mu_1^{i_1}}{i_1!}, \qquad (3.\ 16)$$

In general,

$$G_n = \sum_{i_{n=1}}^{\infty}\sum_{i_{n-1=0}}^{\infty}\ldots\sum_{i_{1=0}}^{\infty}[N^{(\sum_{k=1}^{n} i_k)}(\mu_0)\,\prod_{j=1}^{n}\frac{\mu_j^{i_j}}{i_j!}] \qquad (3.\ 17)$$

3.3. **Theorem:** If N is C^{∞} in a neighbourhood of μ_0 and

$$\|N^n(\mu_0)\| = SupN^n(\mu_0)(\hbar_1,\ldots\hbar_n) : \|\hbar\|_i \leq 1, 1 \leq i \leq n \leq \Re, \qquad (3.\ 18)$$

for any n and some real $\Re > 0$ and $\|\mu_i\| \leq M < \frac{1}{e}$, the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent, $i = 1, 2, 3, \ldots,$. Moreover

$$\|G_n\| \leq LM^n e^{n-1}(e-1), forn = 1, 2, \ldots \qquad (3.\ 19)$$

Proof: Consider Eq.(3. 17)

$$\|G_n\| \leq \Re M^n \sum_{i_{n=1}}^{\infty}\sum_{i_{n-1=0}}^{\infty}\ldots\sum_{i_{1=0}}^{\infty}[N^{(\sum_{k=1}^{n} i_k)}(\mu_0)(\prod_{i_1=1}^{n}\frac{1}{i_j!})] =$$
$$\Re M^n e^{n-1}(e-1),\ n = 1, 2, 3, \ldots. \qquad (3.\ 20)$$

Thus by the convergent series $\|\sum_{n=1}^{\infty} G_n\|$ is dominated, $\Re M(e-1)\sum_{n=1}^{\infty}(Me)^{n-1}$ where $M \prec \frac{1}{e}$. Hence by the comparison test $\sum_{n=1}^{\infty}\|G_n\|$ is absolutely convergent. Now to present the boundedness of all $\|\mu_i\|$, for all i , we have to prove the following theorem.

3.4. **Theorem:** When N is C^{∞} and $\|N^n(\mu_0)\| \leq M \leq e^{-1}$, for all n , then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent.

Proof: Let the relation

$$\varsigma_n = \varsigma_0 \exp(\varsigma_{n-1}) \qquad (3.\ 21)$$

for $n = 1, 2, \ldots.$

where $\varsigma_0 = M$. Let $\kappa_n = \varsigma_n - \varsigma_{n-1}$. Using $\mu_n = G_{n-1}$ and $G_n = \sum_{i_{n=1}}^{\infty}\sum_{i_{n-1=0}}^{\infty}\ldots$ $\sum_{i_{1=0}}^{\infty}[N^{(\sum_{k=1}^{n} i_k)}(\mu_0)\ \prod_{j=1}^{n}\frac{\mu_j^{ij}}{i_j!}]$ we get as :

$$\|G_n\| \leq \kappa_n,\, n = 1, 2, \ldots. \tag{3. 22}$$

Let

$$\omega_n = \sum_{i=1}^{n} \kappa_i = \varsigma_n - \varsigma_0 \tag{3. 23}$$

. Note that $\varsigma_0 = e^{-1} > 0$, $\varsigma_1 = \varsigma_0 \exp(\varsigma_0) > \varsigma_0$ and $\varsigma_2 = \varsigma_0 \exp(\varsigma_1) = \varsigma_1$. Generally, $\varsigma_n > \varsigma_{n-1} > 0$,

Therefore, $\sum \kappa_n$ is a series of $+$ ive real numbers. Note that:

$$\begin{aligned} 0 < \varsigma_0 = M = e^{-1} < 1, \\ 0 < \varsigma_1 = \varsigma_0 \exp(\varsigma_0) < \varsigma_0\, e^1 = e^{-1}e^1 = 1, \\ 0 < \varsigma_2 = \varsigma_0 \exp(\varsigma_1) < \varsigma_0\, e^1 = e^{-1}e^1 = 1. \end{aligned} \tag{3. 24}$$

Generally, $0 < \varsigma_n < 1$. So, $\omega_n = \varsigma_n - \varsigma_0 < 1$ which proves that $\{\omega_n\}_{n=1}^{\infty}$ is bounded above by 1 , and hence convergent. Therefore by comparison test $\sum G_n$ is absolutely convergent.

4. Introduction of MOHPM

Consider the problem:

$$L(\,\mu(x)) + N(\mu(x)) + G(x) = 0, x \in \Omega \tag{4. 25}$$

,

$$\beta\left(\mu(x), \frac{\partial \mu(x)}{\partial x}\right) = 0, x \in T, \tag{4. 26}$$

where L denotes linear operator while N nonlinear. $G(x)$ is a given function . Construct a homotopy, $\omega(x, \ell) : \Omega \times [0, 1] \to R$, by

$$\begin{aligned} H(\omega, \ell) = (1 - \ell)[L(\omega(x, \ell)) - L(u_{ini}(x))] + \\ \ell[L(\omega(x, \ell)) + N(\omega(x, \ell) + G(x))] = 0. \end{aligned} \tag{4. 27}$$

Where, $\ell \in [0, 1]$ and presents an embedding parameter, $\omega(x, \ell)$ is an unknown function and $u_{ini}(x)$ is an initial approximation for the solution of Eq.(4. 25 , which satisfies the BCs. Clearly, when $\ell = 0$ and $\ell = 1$ Eq.(4. 27) holds and takes the forms as:

$$H(\omega, 0) = L(\omega(x, 0)) - L(u_{ini}(x)) = 0, \tag{4. 28}$$

$$H(\omega, 1) = L(\omega(x, 1)) + N(\omega(x, 1)) + G(x) = 0. \tag{4. 29}$$

The paths $L(\omega(x, 0)) - L(\mu_{ini}(x))$ and $L(\omega(x, 1)) + N(\omega(x, 1)) + G(x)$ are homotopic to each other. For the solution of Eq.(4. 28) and Eq.(4. 29) assume the perturbation series:

$$\omega(x, \ell) = \mu_0(x) + \ell\mu_1(x) + \ell^2\mu_2(x) + ... \tag{4. 30}$$

When $\ell = 0$, then $\omega(x,0) = \mu_0(x)$ and when $\ell = 1$, then $\omega(x,1) = \bar{\mu}(x) = \mu_0(x) + \mu_1(x) + \mu_2(x) + ...$, which is the essence of He‚s HPM.
Now for MOHPM, the nonlinear function $N(\omega(x,\ell))$ is decomposed as:

$$N(\omega(x,\ell)) = N(\mu_0) + \ell\,[N(\mu_0+\mu_1) - N(\mu_0)] + \ell^2\,[N(\mu_0+\mu_1+\mu_2) - N(\mu_0+\mu_1)] + ..., \quad (4.\ 31)$$

The expressions $N(\mu_0), [N(\mu_0+\mu_1) - N(\mu_0)], [N(\mu_0+\mu_1+\mu_2) - N(\mu_0+\mu_1)], ...$on the right hand side in equation Eq.(4. 31)

are Daftardar-Jafari polynomials [11,12]. For simplicity and convenience, these polynomials are expressed as:

$N_0 = N(\mu_0), N_m = N\left(\sum_{i=0}^{m}\mu_i\right) - N\left(\sum_{i=0}^{m-1}\mu_i\right).$

We can now express

$$N(\omega(x,\ell)) = N_0 + \sum_{k=1}^{\infty}\ell^k\,N_k \quad (4.\ 32)$$

Substituting back, equation Eq.(4. 32) for equation Eq.(4. 27), also by introducing a number of unknown auxiliary functions, $\gamma_0(x,c_i)$, $\gamma_1(x,c_j)$, $\gamma_2(x,c_l)$, for $i,\ j,\ l = 0,1,2,3,\ldots$ that depend on the variable x and some constants $c_0, c_1, c_2, \ldots$, we get a new homotopy as:

$$\begin{aligned} H(\omega,\ell) &= (1-\ell)[L(\omega(x,\ell)) - L(u_{ini}(x))] + \\ &\ell[L(\omega(x,\ell)) + \gamma_0(x,c_i)N_0 + \ \gamma_1(x,c_j)\,\ell N_1 + \ell^2\gamma_2(x,c_l)N_2 + \cdots + G(x))] \\ &= 0. \end{aligned} \quad (4.\ 33)$$

Now, comparing the coefficients of similar powers of ℓ in Eq.(4. 33), we get linear differential equations of zeroth order, first order, second order and so on, which can be solved very easily.

Zeroth order problem:

$$L(\mu_0(x)) = L(\mu_{ini}(x)),\ \ \beta\left(\mu_0(x), \frac{d\mu_0(x)}{dx}\right) = 0. \quad (4.\ 34)$$

First order problem:

$$L(\mu_1(x)) + \gamma_0(x,c_i)(N_0) + G(x) = 0,\ \ \beta\left(\mu_1(x), \frac{d\mu_1(x)}{dx}\right) = 0\,. \quad (4.\ 35)$$

Second order problem:

$$L(\mu_2(x)) + \gamma_1(x,c_j)(N_1) = 0,\ \ \beta\left(\mu_2(x), \frac{d\mu_2(x)}{dx}\right) = 0. \quad (4.\ 36)$$

Third order problem:

$$L(\mu_3(x)) + \gamma_2(x,c_l)(N_2) = 0,\ \ \beta\left(\mu_3(x), \frac{d\mu_3(x)}{dx}\right) = 0. \quad (4.\ 37)$$

$\vdots$ where $\gamma_0(x,c_i)$, $\gamma_1(x,c_j)$, $\gamma_2(x,c_l), \ldots$ for $i,\ j,\ l = 0,1,2,3,\ldots$, are not unique and can be chosen as the same form of nonlinear operator N [15]. We can make higher order problems but solutions up to the third order problems are enough to get more accurate results.

Let $\bar{\mu}(x)$ be the m^{th} order approximate solution then

$$\bar{\mu}(x) = \mu_0(x) + \mu_1(x) + \mu_2(x) + \mu_3(x) + \ldots + \mu_m(x). \quad (4.\ 38)$$

This depends upon the auxiliary functions $\gamma_0(x, c_i)$, $\gamma_1(x, c_j)$, $\gamma_2(x, c_l), \ldots$ for $i,\ j,\ l = 0, 1, 2, 3, \ldots$. The constants $c_0,\ c_1, c_2, \ldots$ which are present in the expressions of auxiliary functions can be determined by different methods such as the Galerkins method, Collocation method and the least square method. The residual is achieved by putting $\bar{\mu}(x)$ in Eq.(4. 25):

$$R(x, c_0, c_1, c_2, \ldots) = L(\bar{\mu}(x, c_0, c_1, c_2, \ldots)) + N(\bar{\mu}(x, c_0, c_1, c_2, \ldots)) + G(x). \quad (4.\ 39)$$

If $R = 0$, then the solution will be exact. Other wise, it is minimized by different methods mentioned in [14,15]. The values of $c_0, c_1, c_2, \ldots$ can be calculated by various techniques mentioned above. Here least square method is used for this purpose as given bellow: Defined the function as:

$$\zeta(c_0, c_1, c_2, \ldots) = \int_a^b (R(x, c_0, c_1, c_2, \ldots)^2) dx \quad (4.\ 40)$$

and then minimizing it, we have:

$$\frac{\partial \zeta}{\partial c_0} = \frac{\partial \zeta}{\partial c_1} = \frac{\partial \zeta}{\partial c_2} = 0 \cdots \quad (4.\ 41)$$

In Galerekins method, the following system is used for finding the auxiliary constants as:

$$\int_a^b R\frac{\partial \bar{\mu}(x)}{\partial c_0} dx = 0, \int_a^b R\frac{\partial \bar{\mu}(x)}{\partial c_1} dx = 0, \int_a^b R\frac{\partial \bar{\mu}(x)}{\partial c_2} dx = 0, \ldots \quad (4.\ 42)$$

5. Application of MOHPM to investigate Jeffery Hamel Flow

Let,

$$L(\omega(x, \ell)) = \frac{\partial^3(\omega(x, \ell))}{\partial x^3}, L(\mu_{ini}(x)) = 0 \quad (5.\ 43)$$

$$N(\omega(x, \ell)) = 2\,\alpha \mathrm{Re}\ \omega(x, \ell)\,\frac{\partial(\omega(x, \ell))}{\partial x} + 4\,\alpha^2 \frac{\partial(\omega(x, \ell))}{\partial x}, \quad (5.\ 44)$$

and boundary conditions are

$$\omega(0, \ell) = 1,\ \frac{\partial(\omega(0, \ell))}{\partial x} = 0,\ \omega(1, \ell) = 0 \quad (5.\ 45)$$

Using Eq.(4. 33)we get from Eq.(4. 34)
Zeroth order problem:

$$L(\mu_0(x)) + L(\mu_{ini}(x)) = 0, \quad \beta(\mu_0(x)) = 0, \quad (5.\ 46)$$

$$\frac{\partial^3(\mu_0(x))}{\partial x^3}=0,\ \mu_0(0)=1\ ,\mu_0'(0)=0\ ,\ \mu_0(1)=0, \tag{5. 47}$$

We obtained

$$\mu_0(x)=1-x^2. \tag{5. 48}$$

Putting, $\gamma_0(x,c_i)=c_1+c_2x+c_3x^2+c_4x^3$ in Eq.(4. 35), we obtained
First order problem:

$$\begin{aligned}&4\alpha^2c_1\mu_0'(x)+4\alpha^2x\,c_2\mu_0'(x)+4\alpha^2x^2c_3\mu_0'(x)+4\alpha^2x^3c_4\mu_0'(x)+\\&2\alpha\mathrm{Re}c_1\mu_0x\mu_0'(x)+2\alpha\mathrm{Re}x\,c_2\mu_0x\mu_0'(x)+2\alpha\mathrm{Re}x^2c_3\mu_0(x)\mu_0'(x)\\&+2\alpha\mathrm{Re}x^3c_4\mu_0(x)\mu_0'(x)+\mu_1^3(x)=0\,,\mu_1(0)=0,\ \mu_1'(0)=0,\ \mu_1(1)=0.\end{aligned} \tag{5. 49}$$

Its solution is:

$$\begin{aligned}\mu_1(x)=\ &\tfrac{1}{1260}(-420\alpha^2x^2c_1-168\alpha\mathrm{Re}x^2c_1+\\&420\alpha^2x^4c_1+210\alpha\mathrm{Re}x^4c_1-42\alpha\mathrm{Re}x^6c_1\\&-168\alpha^2x^2c_2-60\alpha\mathrm{Re}x^2c_2+168\alpha^2x^5c_2+\\&84\alpha\mathrm{Re}x^5c_2-24\alpha\mathrm{Re}x^7c_2-84\alpha^2x^2c_3-\mathrm{Re}x^6c_1-\\&27\alpha\mathrm{Re}x^2c_3+84\alpha^2x^6c_3+42\alpha\mathrm{Re}x^6c_3-15\alpha\mathrm{Re}x^8c_3\\&-48\alpha^2x^2c_4-14\alpha\mathrm{Re}x^2c_4+48\alpha^2x^7c_4+\\&24\alpha\mathrm{Re}x^7c_4-10\alpha\mathrm{Re}x^9c_4)\end{aligned} \tag{5. 50}$$

Now, putting the values of $\mu_0(x)$, $\mu_1(x)$ in Eq.(4. 38), We obtained the 1st order approximate solution:

$$\begin{aligned}\bar{\mu}(x)=\ &1+\tfrac{1}{6}x^4(2\alpha^2c_1+\alpha\mathrm{Re}c_1)+\tfrac{1}{15}x^5(2\alpha^2c_2+\alpha\mathrm{Re}c_2)-\\&\tfrac{1}{84}\alpha\mathrm{Re}x^8c_3+\tfrac{1}{30}x^6(-\alpha\mathrm{Re}c_1+2\alpha^2c_3+\alpha\mathrm{Re}c_3)-\tfrac{1}{126}\alpha\mathrm{Re}x^9c_4+\\&\tfrac{x^2(-1260-420\alpha^2c_1-168\alpha\mathrm{Re}c_1-168\alpha^2c_2-60\alpha\mathrm{Re}c_2-84\alpha^2c_3-27\alpha\mathrm{Re}c_3-48\alpha^2c_4-14\alpha\mathrm{Re}c_4)}{1260}\\&+\tfrac{2}{105}x^7(-\alpha\mathrm{Re}c_2+2\alpha^2c_4+\alpha\mathrm{Re}c_4).\end{aligned} \tag{5. 51}$$

Now, applying the Least square method: Eq.(4. 40), Eq.(4. 41), we obtained
$c_1=0.37221496329045584$, $c_2=0.36317546662571465$, $c_3=-0.35469830635505323$, $c_4=2.4238340045161983$, for, $\alpha=-5^\circ$, $\mathrm{Re}=80$.
So, the 1st order approximate solution is:

$$\begin{aligned}\bar{\mu}(x)=\ &1-0.39967340538381296x^2-0.4321469159595225x^4-\\&0.16866077222370218x^5+0.16898032028521215x^6-\\&0.2733179694321173x^7-0.029479301414967643x^8+\\&0.13429804412891036x^9.\end{aligned} \tag{5. 52}$$

Now, applying the Least square method: Eq.(4. 40), Eq.(4. 41), we obtained
$c_1=1.8253284184260292$, $c_2=-0.4706368493812372$, $c_3=-3.3075346177317275$, $c_4=2.3204912912600206$, for, $\alpha=5^\circ$, $\mathrm{Re}=50$.

So, the 1st order approximate solution is:

$$\begin{aligned}\bar{\mu}(x) = 1 - 1.7705541863794897x^2 + 1.332049841598825x^4 - \\ 0.13738059065763727x^5 - 0.748223888059061x^6 + \\ 0.23264642053911208x^7 + 0.17180764643782936x^8 - \\ 0.08035756606417079x^9. \end{aligned} \quad (5.\ 53)$$

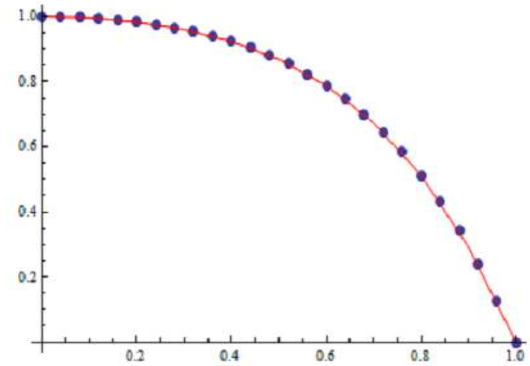

FIGURE 1. Dotted curve-sol: (MOHPM Eq.(5. 52)) for $\alpha = -5°$, Re $= 80,$ and solid curve-sol: (Numerical Method)

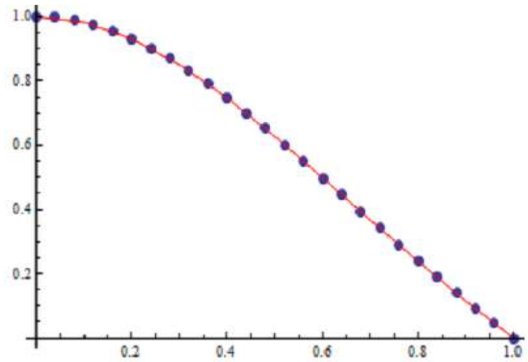

FIGURE 2. Dotted curve-sol: (MOHPM Eq.(5. 53)) for $\alpha = 5°$, Re $=$ $50,$ and solid curve-sol: (Numerical Method)

TABLE 2. Shows the comparisons of the results obtained by MOHPM Eq.(5. 52) with the obtained results of HAM [6], DTM [6], HPM [6], MHPM [10] for $\alpha = -5°$, Re $= 80,$ and E^*= (Numerical values-MOHPM).

x	HAM [6]	DTM [6]	HPM [6]	MHPM [10]	MOHPM Eq.(5. 52)	Num: results	E^* MOHPM
0.	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000	0.0
0.1	0.9995960242	0.9959603887	0.9960671874	0.9962165196	0.9959580613	0.9959606278	2.1×10^{-6}
0.2	0.9832755258	0.9832745481	0.9836959424	0.9843230775	0.9832749668	0.9832755383	5.7×10^{-7}
0.3	0.9601798911	0.9601775551	0.9610758773	0.9625668179	0.9601832790	0.96017991139	-3.4×10^{-6}
0.4	0.9235215737	0.9235170706	0.9249245156	0.9276517677	0.9235224328	0.9235215894	-8.4×10^{-7}
0.5	0.8684588997	0.8684511349	0.8701997697	0.8743082951	0.8684539854	0.86845887772	4.9×10^{-6}
0.6	0.7880910186	0.7880785402	0.7898325937	0.7949430464	0.7880873398	0.78809092032	3.6×10^{-6}
0.7	0.6731437690	0.673248448	0.6745334968	0.6795990006	0.6731468393	0.6731436346	-2.5×10^{-6}
0.8	0.51199099937	0.5119644061	0.51283730959	0.5164879998	0.5119924986	0.5119910891	-1.4×10^{-6}
0.9	0.2915580178	0.2915280122	0.2918936991	0.2933661078	0.2915563248	0.29155874261	2.4×10^{-6}
1.	-0.000001149	0	0	0.0000000005	0	0	0

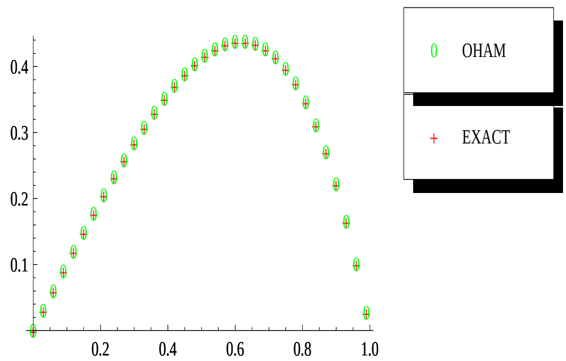

FIGURE 3. Geometry of the model.

TABLE 3. Shows the comparisons of the results obtained by MOHPM Eq.(5. 52) with the obtained results of OHAM [7] for $\alpha = 5°$, Re $= 50,$

x	OHAM [7]	MOHPM Eq.(5. 52)	Num. values	E^* MOHPM
0.	1.00000000	1.00000000	1.000000000	0.0
0.1	0.98251809	0.982425689	0.98243124	5.6×10^{-6}
0.2	0.93156589	0.931221133	0.93122596	4.8×10^{-6}
0.3	0.85138155	0.850622116	0.85061062	-1.1×10^{-5}
0.4	0.74826040	0.746814973	0.74823445	-2.4×10^{-5}
0.5	0.62953865	0.626965232	0.62694817	-1.7×10^{-5}
0.6	0.50242894	0.498235240	0.49823445	-7.9×10^{-7}
0.7	0.37293383	0.366963390	0.36696634	2.9×10^{-6}
0.8	0.24508198	0.238130168	0.23812375	-6.4×10^{-6}
0.9	0.12071562	0.115159727	0.11515193	-7.8×10^{-6}
1.	0.	0	0	0

5.1. **Figures and Tables.**

6. Conclusions

In this endeavor a modified analytical method has been used to solve nonlinear equation forming in the phenomenon of Jeffery-Hamel flow and compared the results with other methods to authenticate the code. The proposed procedure is valid even if the nonlinear equation does not contain any small or large parameters. MOHPM provides a simple and rigorous way to control and adjust the convergence of the solution through the auxiliary functions involving several constants which are optimally calculated. Actually the capital strength of the proposed procedure is its fast convergence, since after only two iterations it converges to the exact solution, which proves that this method is very effective in practice. Here the code is authenticated by comparing the achieved analytic solutions with the solutions obtained via numerical simulations or other known procedures.
From the results presented above, we can conclude the following.
1)The velocity $U_{\max}$ is maximum at centre of the channel.
2) $\alpha < 0\,, U_{\max} < 0$ is taken for convergent channel while for divergent channel $\alpha > 0\,, U_{\max} > 0$.
3) The use of DJMs in the expansion of nonlinear term increases the accuracy.
4) The optimal convergence control parameters in the auxiliary function have great influence on the solution and so the accuracy increases with the increase of parameters.
5) The achieved results are compared with various techniques given in the tables.
6) The results obtained by MOHPM have also good concurrence with the results obtained numerically.
7) The suggested method can solve all linear and nonlinear problems of any order and their systems.
8) The proposed technique can be used to solve PDEs, IDEs and Their systems.
9) Increase in the value of Reynolds number is directly proportional to velocity
10) Figure 1 shows the geometry of the flow.
11) Nomenclature is given in Table 1.
12) The obtained results and numerical results are compared in Figure-2, figure-3, Table-2 and Table-3.

Authors Contribution Statement

Dr. Liaqat Ali: Conceived and designed the analysis; Solved the problem and Wrote the paper.
Dr. Saeed Islam, Dr. Taza Gul: Analyzed and interpreted the data; Contributed analysis data.

References

[1] L. Ali, I. Islam, T. Gul, I. Khan, L. C. C. Dennis, *New Version of Optimal Homotopy Asymptotic Method for the solution of nonlinear boundary value problems in finite and infinite intervals*, Alexandria Engineering Journal. **55**, (2016) 2811-2819.

[2] L. Ali, S. Islam, T. Gul, A. Alshomrani, I. Khan, A. Khan, *Magnetohydrodynamics thin film fluid flow under the effect of thermophoresis and variable fluid properties*, American Institute of Chemical Engineering Journal (AIChE Journal) (2017), DOI: 10. 1002/aic.15794.

[3] L. Ali , S. Islam, T. Gul, *A New Analytical Approach for Solving Nonlinear Boundary Value Problems Arising in Nonlinear Phenomena*, FILOMAT. **32**, (2018) 2489-2497.
[4] W. I. Axford, *The magneto-hydrodynamic Jeffery-Hamel problem for a weekly conducting fluid*, The Quarterly Journal of Mechanics and Applied Mathematics. **14** (1961) 335-351.
[5] A. Basiri Parsa, M.M. Rashidi, O. Anwar Bg, S.M. Sadri, *Semi-Computational Simulation of Magneto-Hemodynamic Flow in a Semi-Porous Channel Using Optimal Homotopy and Differential Transform Methods*, Computers in Biology and Medicine. **43** (9),(2013) 1142-1153.
[6] S. Bhalekar and V. Daftardar-Gejji, *Convergence of the New Iterative Method*, International Journal of Differential Equations. **11** (2011) Article ID 989065, 10 pages.
[7] V. Daftardar-Gejji and H. Jafari, *An iterative method for solving nonlinear functional equations*, Journal of Mathematical Analysis and Applications. **316**, (2006) 753763.
[8] Q. Esmaili, A. Ramiar, E. Alizadeh, D.D. Ganji, *An approximation of the analytical solution of the Jeffery-Hamel flow by decomposition method*, Physics Letters A. **372**, (2008) 3434-3439.
[9] M. Esmaeilpour and D. D. Ganji, *Solution of the Jeffery-Hamel flow problem by optimal homotopy asymptotic method*, Computer and Mathematics with Applications. **59** (2010) 3405-3411.
[10] L.E. Fraenkel, *Laminar flow in symmetrical channels with slightly curved walls. I: On the Jeffery-Hamel solutions for flow between plane walls*, Proceedings of the Royal Society A. **267**, (1962) 119-38.
[11] N. Freidoonimehr, M.M. Rashidi, *Dual Solutions for MHD JefferyHamel Nano-fluid Flow in Non-parallel Walls using Predictor Homotopy Analysis Method*, Journal of Applied Fluid Mechanics. **9** (1) (2016).
[12] F. Ghani, S. Islam, C. Ozel, L. Ali, M. M. Rashidi. *Application of Modified Optimal Homotopy Perturbation Method Higher Order Boundary Value problems in a Finite Domain*, Hacettepe Journal of Mathematics and Statistics. **45**, (2016) 1049-1060.
[13] M. Hamadiche, J. Scott,D, Jeandel, *Temporal stability of Jeffery-Hamel flow*, Journal of Fluid Mechanics. **268**, (1994) 71-88.
[14] G. Hamel, *Spiralformige.Bewgungen.Zaher Flussigkeiten, Jahresbericht der. Deutchen*, Mathematiker Vereinigung. **25**, (1916) 34-60.
[15] S.Islam, R.A. Shah, and I. Ali, *Couette and Poiseuille flows for fourth grade fluids using Optimal homotopy asymptotic method*, World Applied Sciences Journal. **9**, (2010) 1228-1236.
[16] G.B. Jeffery, *The two-dimensional steady motion of a viscous fluid*, Philosophical Magzine. **6**, (1915) 455-465.
[17] A. A. Joneidi, Domairy, and M. Babaelahi, *Three analytical method applied to Jeffery-Hamel flow*, Communications in Nonlinear Science and Numerical Simulation. **15**, (2010) 3423-3434.
[18] V.Marinca, N.Herisanu and I. Nemes, *Optimal homotopy asymptotic method with application to thin film flow*, European Journal of Physics. **6**, (2008) 648-653.
[19] V. Marinca, N. Herisanu, *Optimal homotopy perturbation method for strongly nonlinear differential equations*, Nonlinear Science Letters A. **1**,(2010) 273-280.
[20] S. M. Moghimi, D. D. Ganji, H. Bararnia, M. Hosseini, and M. Jalaal, *Homotopy perturbation method for nonlinear MHD Jeffery-Hamel problem*, Computers and Mathematics with Applications. **61**, (2011) 2213-2216.
[21] J. Singh, M.M. Rashidi, S. Kumar, D. Kumar, *A Hybrid Computational Approach for Jeffery-Hamel Flow in Non-parallel Walls, Neural Computing and Applications* in press.
[22] Sushila, S. Jagdev, Y.S. Shishodia, *A modified analytical technique for Jeffery-Hamel flow.using sumudu.trasform*, Journal of the Association of Arab Universities for basic and Applied Sciences. **16**, (2014) 11-15.