Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 51(1)(2018) pp. 9-15

Some Results Associated with the Diagonals of Derived Plane Permutations

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Received: 02 August, 2017 / Accepted: 07 February, 2018 / Published online: 16 September, 2018

Abstract. A plane permutation is a pair $p = (s, \pi)$ where s is an n-cycle and π is an arbitrary permutation. In this paper, we study the properties of p under two instances; when $\pi = s$ and $\pi = s^{-1}$. We also define the diagonal of the derived plane permutation and establish that it coincides with the diagonal of the underlying plane permutation.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Plane Permutation, Diagonal Plane Permutation, Derived Plane Permutation, Diagonal Derived Plane Permutation

1. Introduction

A plane permutation is represented in a two line notation where the top line lists the elements of the cycle s and the bottom line lists the corresponding images of the elements on the top line. Chen [4] described a plane permutation as a pair of an n-cycle and an arbitrary permutation which gives rise to three other permutations; the upper s, the vertical π and the diagonal D_p. He studied the properties of p under some permutation statistics (exceedance and anti exceedance) and found that if the cardinality of the exceedance sets of two plane permutations p_1 and p_2 are equal, then p_1 and p_2 are said to be equivalent. Recently, Chen [4] applied the theory of plane permutations in the study of genome arrangements and graph embeddings.

The derived plane permutation also gives rise to three distinct permutations: the horizontal permutation (s^h), the vertical permutation (π^h) and the diagonal permutation (D_{p^h}). In this work, we investigate the structure of p when $\pi = s$ and $\pi = s^{-1}$ on a finite set $[n]$. We study the exceedance and anti-exceedance sets in the two instances above and show some inclusion relation on the sets. Furthermore, we prove that the diagonal plane permutation is equal to the diagonal derived plane permutation.

2. Preliminaries

In this section, we give some relevant definitions as they relate to the work.

Definition 2.1. *Let $A = [n]$ and $f : A \longrightarrow A$ be a bijection such that $a_1, a_2, \dots, a_n \in A$. If*

$$f(a_1) = a_2, f(a_2) = a_3, \dots, f(a_n) = a_1,$$

then f is called an n-cycle, written as $(a_1 a_2 \dots a_n)$.

Definition 2.2. (Chen and Reidys [3]) *A plane permutation is a pair $p = (s, \pi)$ of an n-cycle $s = (s_i)_{n=o}^{n-1}$ and an arbitrary permutation π such that p is represented in the form;*

$$p = \begin{pmatrix} s_0 & s_1 & s_2 & \dots & s_{n-1} \\ \pi(s_0) & \pi(s_1) & \pi(s_2) & \dots & \pi(s_{n-1}) \end{pmatrix}$$

Definition 2.3. (Chen and Reidys [3]) *The diagonal of a plane permutation is defined as $D_p = s \circ \pi^{-1}$.*

Definition 2.4. (Chen and Reidys [3])*A derived plane permutation of p is defined as a pair $p^h = (s^h, \pi^h)$ of an n-cycle s^h obtained by transposing the blocks $[s_i, \dots, s_j]$ and $[s_k, \dots, s_l]$ for sequence $h = (i, j, k, l)$ such that $1 \leq i \leq j < k \leq l \leq n-1$ in s of p and $\pi^h = D_p^{-1} s^h$.*

We state here that an n-cycle s(or s^h) induces a partial order $<_s$ (or $<_{s^h}$) on s(or s^h) where $x <_s y$ (or $x <_{s^h} y$) if x appears before y in s(or s^h) from left to right.

Definition 2.5. *An element s_i is an exceedance of p if $s_i <_s \pi(s_i)$ otherwise s_i is an anti-exceedance. Similarly, an element s_i^h is an exceedance of p^h if $s_i^h <_{s^h} \pi^h(s_i^h)$ otherwise s_i^h is an anti-exceedance. The exceedance and anti-exceedance of p are denoted as $Exc(p)$ and $AEx(p)$ respectively.*

3. Diagonal Plane Permutations

We begin this section by stating some results on diagonal plane permutations when $\pi = s$.

Lemma 3.1. (Chen and Reidys [3]). *For a plane permutation $p = (s, \pi)$, we have*

$$|Exc(p)| = |AEx(D_p)| - 1.$$

Proposition 3.2. *Let $p = (s, \pi)$ be a plane permutation such that $\pi = s$. Then D_p is the identity permutation e.*

Proof. The proof follows from Definition 2.3. □

Proposition 3.3. *Let $p = (s, \pi)$ be a plane permutation such that $\pi = s$. The following hold:*

(1) $Exc(D_p) = \emptyset$
(2) $AEx(D_p) = \{s_0, s_1 \ldots s_{n-1}\}$.

Proof. The proof follows from Proposition 3.2. □

Lemma 3.4. *Let $p = (s, \pi)$ be a plane permutation on $[n]$ such that $\pi = s$. Then*

(1) $Exc(p) = \{s_0, s_1, \ldots, s_{n-2}\}$.
(2) $AEx(p) = \{s_{n-1}\}$

Proof. Suppose $s = \pi = (s_0 s_1 \ldots s_{n-1})$ then by construction of p, we have $s_i <_s \pi(s_i)$ for all $0 \leq i \leq n-2$ and $s_{n-1} >_s \pi(s_{n-1})$. This implies:

(1) The elements of the exceedence set of p are the s_i for all $0 \leq i < n-1$, that is, $Exc(p) = \{s_0, s_1, \ldots, s_{n-2}\}$ and thus $|Exc(p)| = n-1$.
(2) Since s_{n-1} is the only element not in $Exc(p)$ then $AEx(p)$ is a singleton of s_{n-1}. This completes the proof.

□

Proposition 3.5. *Let $p = (s, \pi)$ be a plane permutation on such that $\pi = s$. Then $s_0 = \pi(s_{n-1})$ and $s_i = \pi(s_{i-1})$, for all $1 \leq i \leq n-1$.*

Proof. The construction of p when $\pi = s$, is

$$p = \begin{pmatrix} s_0 & s_1 & s_2 & \ldots & s_{n-1} \\ s_1 & s_2 & s_3 & \ldots & s_0 \end{pmatrix}.$$

From p above, it is obvious that $s_0 = \pi(s_{n-1})$ and $s_i = \pi(s_{i-1})$ for all $1 \leq i \leq n-1$. □

The next result shows the inclusion relation of exceedence and anti-exceedence sets of a plane permutation and its diagonal.

Lemma 3.6. *For a plane permutation $p = (s, \pi)$ such that $\pi = s$, then the following hold:*

(1) $Exc(D_p) \subseteq AEx(p) \subseteq AEx(D_p)$
(2) $Exc(D_p) \subseteq Exc(p) \subseteq AEx(D_p)$
(3) $AEx(D_p) \setminus Exc(p) = AEx(p)$
(4) $AEx(D_p) \setminus AEx(p) = Exc(p)$.

In what follows, we state the results when the second instance of our assumption is considered.

Proposition 3.7. *Let $p = (s, \pi)$ such that $\pi = s^{-1}$. Then $s_{n-1} = \pi(s_0)$ and $s_i = \pi(s_{i+1})$, $\forall\ 0 \leq i \leq n-2$.*

Proof. By construction

$$p = \begin{pmatrix} s_0 & s_1 & \dots & s_{n-3} & s_{n-2} & s_{n-1} \\ s_{n-1} & s_0 & \dots & s_{n-4} & s_{n-3} & s_{n-2} \end{pmatrix}.$$

It follows from p above that $s_{n-1} = \pi(s_0)$ for all $0 \leq i \leq n-2$ and $s_i = \pi(s_{i+1})$. □

For a plane permutation $p = (s, \pi)$ on $[n]$ such that $\pi = s^{-1}$, the following hold:
(1) $Exc(p) = \{s_0\}$
(2) $AEx(p) = \{s_1, s_2, \dots, s_{n-1}\}$.

Proof. Suppose$\pi = s^{-1}$, this implies that $\pi = (s_{n-1}s_{n-2}\dots s_1 s_0)$ and

$$p = \begin{pmatrix} s_0 & s_1 & \dots & s_{n-3} & s_{n-2} & s_{n-1} \\ s_{n-1} & s_0 & \dots & s_{n-4} & s_{n-3} & s_{n-2} \end{pmatrix}.$$

Also, we have $s_0 <_s \pi(s_0)$ and $s_i >_s \pi(s_i)$ for all $1 \leq i \leq n-1$. Thus:
(1) s_0 is the only member of the exceedance set of p. Since $s_i <_s \pi(s_0)$ this shows that $Exc(p) = \{s_0\}$.
(2) The anti-exceedance set of p are all s_i such that $1 \leq i \leq n-1$. Therefore $AEx(p) = \{s_1, s_2, \dots, s_{n-1}\}$. Clearly, $|AExc(p)| = n-1$. Hence, the result holds.

□

Proposition 3.8. *Let $p = (s, \pi)$ be a plane permutation such that $\pi = s^{-1}$. Then*

$$D_p = \begin{cases} (s_0 s_2 \dots s_{n-2}), (s_1 s_3 \dots s_{n-1}), & \text{if } n \text{ is even;} \\ (s_0 s_2 \dots s_{n-1} s_1 s_3 \dots s_{n-2}), & \text{if } n \text{ is odd.} \end{cases}$$

Proposition 3.9. *Suppose $p = (s, \pi)$ on $[n]$ such that $\pi = s^{-1}$, then*
(1) $Exc(D_p) = \{s_0, s_1, \dots, s_{n-3}\}$.
(2) $AEx(D_p) = \{s_{n-2}, s_{n-1}\}$.

Proof. By Proposition 3.9, when n is even or odd, $s_i <_s \pi(s_i)$ for $0 \leq i < n-2$. So:
(1) s_i forms the exceedance set of D_p for all $0 \leq i < n-2$ since $s_i <_s \pi(s_i)$. Hence $Exc(D_p) = \{s_0, s_1, \dots, s_{n-3}\}$.
(2) Since s_{n-2}, s_{n-1} are not in $Exc(D_p)$, then it is obvious that they are in $AEx(D_p)$.
□

Lemma 3.10. *Suppose $p = (s, \pi)$ such that $\pi = s^{-1}$. The following hold:*
(1) $Exc(p) \subseteq Exc(D_p)$
(2) $AEx(D_p) \subseteq AEx(p)$
(3) $(Exc(D_p) \setminus Exc(p)) \cup AEx(D_p) = AEx(p)$
(4) $AEx(p) \setminus AEx(D_p) = Exc(D_p) \setminus Exc(p)$.

Proof. The proofs of (1) and (2) are trivial consequences of Corollary 3.8 and Proposition 3.10.

(3). Given

$$(Exc(D_p) \setminus Exc(p)) \cup AEx(D_p) = AEx(p)$$

$$\begin{aligned}(Exc(D_p) \setminus Exc(p)) \cup AEx(D_p) &= (Exc(D_p) \cap (Exc(p))^c) \cup AEx(D_p) \\ &= (Exc(D_p) \cap AEx(p)] \cup AEx(D_p) \\ &= (\{s_0, \ldots, s_{n-3}\} \cap \{s_1, s_2, \ldots, s_{n-1}\}) \cup \{s_{n-2}, s_{n-1}\} \\ &= AEx(p).\end{aligned}$$

(4). Also, given

$$AEx(p) \setminus AEx(D_p) = Exc(D_p) \setminus Exc(p)$$

then

$$\begin{aligned}AEx(p) \setminus AEx(D_p) &= AEx(p) \cap (AEx(D_p))^c \\ &= AEx(p) \cap Exc(D_p) \\ &= Exc(D_p) \cap AEx(p) \\ &= Exc(D_p) \setminus Exc(p).\end{aligned}$$

□

The next lemma gives the inclusion relations of the exceedance and anti-exceedance sets on a plane permutation and its diagonal when the two instances of our assumptions are involved.

Lemma 3.11. *Let $p = (s, \pi)$. The following hold:*

(1) $(Exc(p)_{\pi=s^{-1}}) \subseteq (Exc(p)_{\pi=s})$
(2) $(AEx(p)_{\pi=s}) \subseteq (AEx(p)_{\pi=s^{-1}})$
(3) $(Exc(D_p)_{\pi=s}) \subseteq (Exc(D_p)_{\pi=s^{-1}})$
(4) $(AEx(D_p)_{\pi=s^{-1}}) \subseteq (AEx(D_p)_{\pi=s})$
(5) $(Exc(p)_{\pi=s^{-1}}) \cup (AEx(p)_{\pi=s}) = \{s_0, s_{n-1}\}$
(6) $(Exc(p)_{\pi=s}) \cup (AEx(p)_{\pi=s^{-1}}) = (AEx(D_p)_{\pi=s})$.

Proof. The proofs of (1),(2),(5) and (6) follow from Corollary 3.8 and Lemma 3.5. Also, the proofs of (3) and (4) follow Propositions 3.3 and 3.10. □

4. Diagonal Derived Plane Permutations

Given a derived plane permutation $p^h = (s^h, \pi^h)$, the n-cycle s^h obtained by transposing the blocks $[s_i \ldots s_j]$ and $[s_k \ldots s_l]$ in the n-cycle s of p is defined by

$$s^h = (s_0 s_1 \ldots s_{i-1} \underline{s_k \ldots s_l} s_{j+1} \ldots s_{k-1} \underline{s_i \ldots s_j} s_{l+1} \ldots).$$

For the special case of $j + 1 = k$, s^h is given by

$$s^h = (s_0 s_1 \ldots s_{i-1} \underline{s_k \ldots s_l}\, \underline{s_i \ldots s_j} s_{l+1}).$$

The derived plane permutation $p^h = (s^h, \pi^h)$ is represented as:

$$p^h = \begin{pmatrix} \ldots s_{i-1} & s_k \ldots s_l & s_{j+1} \ldots s_{k-1} & s_i \ldots s_j & s_{l+1} \ldots \\ \ldots \pi(s_{k-1}) & \pi(s_k) \ldots \pi(s_j) & \pi(s_{j+1}) \ldots \pi(s_{i-1}) & \pi(s_i) \ldots \pi(s_l) & \pi(s_{l+1})) \ldots \end{pmatrix}.$$

We build on the settings of the diagonal of plane permutation for its version of the derived plane permutation.

The diagonal of a derived plane permutation is defined as

$$D_{p^h} = s^h \circ (\pi^h)^{-1}.$$

Proposition 4.1. *Let* $p = (s, \pi)$ *and* $p^h = (s^h, \pi^h)$*. Then*

$$|Exc(p)| + |Exc(p^h)| \leq 2n - 2.$$

Proof. In every $n-$cycle there exists at least one anti-exceedance, which implies that the exceedance is bounded by $n - 1$.

Case I: Suppose the $n-$cycle of p and p^h have exactly one anti-exceedance. Then we have exactly $n - 1$ exceedance for p and p^h, so

$$|Exc(p)| + |Exc(p^h)| = 2n - 2.$$

Case II: Suppose the $n-$cycle of p and p^h have more than one anti-exceedance, this implies that exceedance of p and p^h will be strictly less than $2n - 2$.

Combining these two cases we have

$$|Exc(p)| + |Exc(p^h)| \leq 2n - 2.$$

□

Proposition 4.2. *Let* $p = (s, \pi)$ *and* $p^h = (s^h, \pi^h)$*. Then*

$$D_{p^h} = D_p.$$

Proof. Let $s^h = es^h$ where e is an identity permutation. Then

$$\begin{aligned}
s^h &= D_p \circ D_p^{-1} \circ s^h \\
&= D_p \circ (D_p^{-1} \circ s^h) \\
&= D_p \circ \pi^h \\
s^h \circ (\pi^h)^{-1} &= D_p \circ \pi^h \circ (\pi^h)^{-1} \\
D_{p^h} &= D_p \circ (\pi^h \circ (\pi^h)^{-1}) \\
&= D_p \\
\therefore D_{p^h} &= D_p.
\end{aligned}$$

□

Proposition 4.3. *Let* $p = (s, \pi)$ *and* $p^h = (s^h, \pi^h)$*. Then*

$$|Exc(p^h)| = |AEx(D_p)| - 1.$$

Proof. Let p^h be a derived plane permutation. Suppose s_i^h is an exceedance of p^h, then $\pi^h(s_i^h)$ is an anti-exceedance of D_{p^h}, and $\pi^h(s_i^h) \geq_{s^h} D_{p^h}(\pi^h(s_i^h))$ for $0 \leq i < n-1$, so

$$|Exc(p^h)| = |AEx(D_{p^h})|.$$

Considering the last element of the cycle, s^h_{n-1} and $\pi^h(s^h_{n-1})$ are anti-exceedances of p^h and D_{p^h} respectively. So the anti-exceedance of D_{p^h} increases by 1. Therefore

$$|Exc(p^h)| = |AEx(D_{p^h})| - 1.$$

□

Lemma 4.4. *Let* $p = (s^h, \pi^h)$ *and* $p^h = (s^h, \pi^h)$*. Then*

$$|Exc(p^h)| = |AEx(D_p)| - 1.$$

Proof. The proof follows from Proposition 4.2. □

Let $p_1 = (s_1, \pi_1)$ and $p_2 = (s_2, \pi_2)$ be two plane permutations. Then p_1 and p_2 are said to be equivalent if for some permutation ω, $s_1 = \omega \circ s_2 \circ \omega^{-1}$ and $\pi_1 = \omega \circ \pi_2 \circ \omega^{-1}$.

Lemma 4.5. (Chen and Reidys [3]) *For two equivalent plane permutations* $p_1 = (s_1, \pi_1)$ *and* $p_2 = (s_2, \pi_2)$*, we have*

$$|Exc(p_1)| = |Exc(p_2)|.$$

Let $p_1 = (s_1, \pi_1)$ and $p_2 = (s_2, \pi_2)$ be two equivalent plane permutations. Then

$$|AEx(p_1)| = |AEx(p_2)|.$$

5. Acknowledgments

The authors are grateful to Dr. R.X.F. Chen for his meaningful comments and the referees for their invaluable suggestions which have been very helpful. The authors also like to thank F.A. Akinola, S. Buoro and A.H. Ibrahim for their contributions.

References

[1] L. Bulleau, G. Fertin and I. Rusu, *Sorting by Transposition is Difficult*, SIAM J. Discrete Math. **26**, No. 3 (2012) 1148-1180.

[2] R. X. F Chen and C. M Riedys, *A Simple Framework on Sorting Permutations*, arxiv:1502.07971v2 [math.CO].

[3] R. X. F Chen and C. M Riedys, *Plane Permutation and Application to a Result of Zagier-Stanley and Distance of Permutation*, SIAM J. Discrete Math. **30**, 3 (2016) 1660-1684.

[4] R. X. F Chen, *Plane Permutation and their Applications to Graph Embedding and Genome Rearrangements*, Ph.D Thesis, Virginia Polytechnic Institute and State University (2017).

[5] D. A. Christie, *Sorting Permutation by Block-interchanges*, Inf. let. **60**, (1996) 165-169.

[6] R. P. Stanley, *Two Enumeration Results on Cycle of Permutations*, European J. Comb. **12**, (2011) 937-943.

[7] D. Zagier, *On the Distribution of Numbers of Cycle of Elements in Symmetric Group*, Nieuw Arch. Wisk. **13**, No. 3 (1995) 489-495.