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Fractional Hermite-Hadamard Type Integral Inequalities for Functions whose Modulus of Derivatives are Co-ordinated $\log$-Preinvex

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Abstract. In this paper we introduce the concept of co-ordinated $\log$-preinvex functions, we establish a new fractional identity involving a function of two independent variables, and then we derive some fractional Hermite-Hadamard's type inequalities which are co-ordinated $\log$-preinvex.

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1. Introduction

Let f be a convex function on $[u, v]$, then

$$f\left(\tfrac{u+v}{2}\right) \leq \tfrac{1}{v-u}\int_{u}^{v} f(x)dx \leq \tfrac{f(u)+f(v)}{2}, \tag{1. 1}$$

if the function f is concave then (1. 1) holds in the reverse direction (see [26]).

The above inequality is known as Hermite-Hadamard integral inequality

In [6] Dragomir established the bidimentionnal anologue of (1. 1) given by

$$\begin{aligned} f\left(\tfrac{u+v}{2}, \tfrac{c+d}{2}\right) &\leq \tfrac{1}{2}\left(\tfrac{1}{v-u}\int_{u}^{v} f(\tau, \tfrac{c+d}{2})d\tau + \tfrac{1}{d-c}\int_{c}^{d} f(\tfrac{u+v}{2}, y)dy\right) \\ &\leq \tfrac{1}{(v-u)(d-c)}\int_{u}^{v}\int_{c}^{d} f(\tau, y)dyd\tau \end{aligned}$$

$$\leq \frac{1}{4}\left(\frac{1}{v-u}\int_{u}^{v} f(\tau,c)d\tau + \frac{1}{v-u}\int_{u}^{v} f(\tau,d)d\tau + \frac{1}{d-c}\int_{c}^{d} f(u,y)dy + \frac{1}{d-c}\int_{c}^{d} f(v,y)dy\right)$$
$$\leq \frac{f(u,c)+f(u,d)+f(v,c)+f(v,d)}{4}. \tag{1. 2}$$

Inequalities (1. 1) and (1. 2) have attracted many researchers, we note that the literature in this context is rich and various. About some papers related to the integral inequalities we mention [1, 3, 5, 6, 7, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 25] and references therein.

Hanson [8] gave a new generalization of the classical convexity, called invexity or generalized convexity. Many authors studied the properties, and applications of this new concept, see for instance [4, 23, 24, 27, 29, 30].

Alomari et al. [2] gave the following Hermite-Hadamard for co-ordinated log-convex functions

Theorem 1.1. *Assume that $f:\Delta\to\mathbb{R}_+$ is co-ordinated $\log$-convex functions on $\Delta = [u,v]\times[c,d]\subset\mathbb{R}^2$. Then*

$$4\log f\left(\tfrac{u+v}{2},\tfrac{c+d}{2}\right) \leq \frac{4}{(v-u)(d-c)}\int_{u}^{v}\int_{c}^{d} \log f(x,y)dydx \leq \log\left[f(u,c)\,f(u,d)\,f(v,c)\,f(v,d)\right].$$

In this paper we introduce the concept of co-ordinated log-preinvex functions, we establish a new fractional identity involving a function of two independent variables, and then we derive some fractional Hermite-Hadamard type inequalities for functions whose modulus of the mixed derivatives are co-ordinated log-preinvex.

2. Preliminaries

In this section we recall some definitions and lemmas that's well known in the literature, and assume that $\Omega := [u,v]\times[\theta,\omega]$ is a bidimensional interval in $\mathbb{R}^2$ with $u<v$ and $\theta<\omega$.

Definition 2.1. [2] *A positive function $f:\Omega\to\mathbb{R}$ is said to be co-ordinated $\log$-convex on Ω, if the following inequality:*

$$f\left(m\theta+(1-m)u, s\omega+(1-s)v\right) \leq f^{ms}(\theta,\omega)f^{m(1-s)}(\theta,v)f^{(1-m)s}(u,\omega) \times f^{(1-m)(1-s)}(u,v)$$

holds for all $m,s\in[0,1]$ and $(\theta,\omega),(u,v)\in\Omega$.

Definition 2.2. [19] *Let H_1,H_2 be two nonempty subsets of $\mathbb{R}^n$, $(\theta,\omega)\in H_1\times H_2$. We say that the set $K_1\times K_2$ is invex at point (θ,ω) with respect to ξ_1 and ξ_2, if for each*

$(u, v) \in H_1 \times H_2$ *and* $m, s \in [0, 1]$, *we have*

$$(\theta + m\xi_1 (u, \theta), \omega + s\xi_2 (v, \omega)) \in H_1 \times H_2.$$

$H_1 \times H_2$ is said to be an invex set with respect to ξ_1 and ξ_2 if $H_1 \times H_2$ is invex at each points $(\theta, \omega) \in H_1 \times H_2$.

Definition 2.3. [14] *Assume that* $f \in L([u, v])$. *The Riemann-Liouville fractional integrals* $J^{\alpha}_{u^+} f$ *and* $J^{\alpha}_{v^-} f$ *of order* $\alpha > 0$ *with* $u \geq 0$ *are defined by*

$$\begin{aligned} J^{\alpha}_{u^+} f(x) &= \frac{1}{\Gamma(\alpha)} \int_u^x (x-t)^{\alpha-1} f(t) dt, \quad x > u \\ J^{\alpha}_{v^-} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^v (t-x)^{\alpha-1} f(t) dt, \quad v > x, \end{aligned}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$, *is the Gamma function and* $J^0_{u^+} f(x) = J^0_{v^-} f(x) = f(x)$.

Definition 2.4. [14] *Assume that* $f \in L(\Omega)$. *The Riemann-Liouville fractional integrals* $J^{\alpha,\beta}_{u^+,\theta^+}$, $J^{\alpha,\beta}_{u^+,\omega^-}$, $J^{\alpha,\beta}_{v^-,\theta^+}$ *and* $J^{\alpha,\beta}_{v^-,\omega^-}$ *of order* $\alpha, \beta > 0$ *with* $u, \theta \geq 0$ *are defined by*

$$J^{\alpha,\beta}_{u^+,\theta^+} f(v, \omega) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_u^v \int_\theta^\omega (v-x)^{\alpha-1} (\omega-y)^{\beta-1} f(x, y) \, dy dx \tag{2. 3}$$

$$J^{\alpha,\beta}_{u^+,\omega^-} f(v, \theta) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_u^v \int_\theta^\omega (v-x)^{\alpha-1} (y-\theta)^{\beta-1} f(x, y) \, dy dx \tag{2. 4}$$

$$J^{\alpha,\beta}_{v^-,\theta^+} f(u, \omega) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_u^v \int_\theta^\omega (x-u)^{\alpha-1} (\omega-y)^{\beta-1} f(x, y) \, dy dx \tag{2. 5}$$

$$J^{\alpha,\beta}_{v^-,\omega^-} f(u, \theta) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_u^v \int_\theta^\omega (x-u)^{\alpha-1} (y-\theta)^{\beta-1} f(x, y) \, dy dx. \tag{2. 6}$$

Definition 2.5. [28] *Assume that* $f \in L(\Omega)$. *The Riemann-Liouville fractional integrals* $J^{\alpha}_{v^-} f(u, \theta)$, $J^{\alpha}_{u^+} f(v, \theta)$, $J^{\beta}_{\omega^-} f(u, \theta)$, *and* $J^{\alpha}_{\theta^+} f(u, \omega)$ *of order* $\alpha, \beta > 0$ *with* $u, \theta \geq 0$, $u < v$, *and* $\theta < \omega$ *are defined by*

$$J^{\alpha}_{v^-} f(u, \theta) = \frac{1}{\Gamma(\alpha)} \int_u^v (x-u)^{\alpha-1} f(x, \theta) \, dx \tag{2. 7}$$

$$J^{\alpha}_{u^+} f(v, \theta) = \frac{1}{\Gamma(\alpha)} \int_u^v (v-x)^{\alpha-1} f(x, \theta) \, dx \tag{2. 8}$$

$$J_{\omega^-}^{\beta} f(u,\theta) = \frac{1}{\Gamma(\beta)} \int_{\theta}^{\omega} (y-\theta)^{\beta-1} f(u,y)\, dy \tag{2. 9}$$

$$J_{\theta^+}^{\alpha} f(u,\omega) = \frac{1}{\Gamma(\beta)} \int_{\theta}^{\omega} (\omega-y)^{\beta-1} f(u,y)\, dy, \tag{2. 10}$$

where Γ is the Gamma function.

We also recall that the weighted arithmetic-geometric mean inequality can be says that for $a, b \geq 0$ and $0 \leq \nu \leq 1$

$$a^{\nu} b^{1-\nu} \leq \nu a + (1-\nu) b.$$

3. Main results

In what follows we assume that $K = [a, a+\xi_1(b,a)] \times [c, c+\xi_2(d,c)]$ be an invex subset of $\mathbb{R}^2$ with respect to ξ_1, ξ_2 where $\xi_1, \xi_2 : \mathbb{R}^2 \to \mathbb{R}$ are two bifunctions such that $\xi_1(b,a) > 0$ and $\xi_2(d,c) > 0$.

We first introduce the class of log-preinvex functions on the co-ordinates

Definition 3.1. *A positive function $f : K \to \mathbb{R}$ is said to be co-ordinated* log*-preinvex on K with respect to ξ_1 and ξ_2, if*

$$f(x+t\xi_1(y,x), u+s\xi_2(v,u)) \leq f^{(1-t)(1-s)}(x,u)\, f^{(1-t)s}(x,v) \times f^{t(1-s)}(y,u)\, f^{ts}(y,u)$$

holds for all $(x,u), (y,v) \in [a, a+\xi_1(b,a)] \times [c, c+\xi_2(d,c)]$ and $t, s \in [0,1]$.

Remark 3.2. *Definition 3.1 recapture Definition 2.1, if we choose $\xi_1(y,x) = \xi_2(y,x) = y-x$.*

We will start with the following lemma which is an auxiliary result.

Lemma 3.3. *Let $f : K \to \mathbb{R}$ be a partially differentiable function on K, if $\frac{\partial^2 f}{\partial t \partial s} \in L(K)$, then the following equality holds*

$$\begin{aligned}
&\tfrac{f(a,c)+f(a,c+\xi_2(d,c))+f(a+\xi_1(b,a),c)+f(a+\xi_1(b,a),c+\xi_2(d,c))}{4} - A \\
&+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\xi_1(b,a))^{\alpha}(\xi_2(d,c))^{\beta}} \Big(J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,(c+\xi_2(d,c))^-} f(a,c) \\
&+J^{\alpha,\beta}_{a^+,(c+\xi_2(d,c))^-} f(a+\xi_1(b,a),c) + J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,c^+} f(a, c+\xi_2(d,c)) \\
&+ J^{\alpha,\beta}_{a^+,c^+} f(a+\xi_1(b,a), c+\xi_2(d,c)) \Big) \\
= \;&\tfrac{\xi_1(b,a)\xi_2(d,c)}{4} \int_0^1\!\!\int_0^1 (t^{\alpha}-(1-t)^{\alpha}) \left(s^{\beta}-(1-s)^{\beta}\right) \\
&\times \tfrac{\partial^2 f}{\partial t \partial s}(a+t\xi_1(b,a), c+s\xi_2(d,c))\, ds dt,
\end{aligned} \tag{3. 11}$$

where

$$\begin{aligned}
A &= \tfrac{\Gamma(\alpha+1)}{4(\xi_1(b,a))^\alpha} \Big(J^\alpha_{(a+\xi_1(b,a))^-} f\left(a, c+\xi_2\left(d,c\right)\right) + J^\alpha_{(a+\xi_1(b,a))^-} f\left(a,c\right) \\
&+ J^\alpha_{a^+} f\left(a+\xi_1\left(b,a\right), c+\xi_2\left(d,c\right)\right) + J^\alpha_{a^+} f\left(a+\xi_1\left(b,a\right), c\right)\Big) \\
&+ \tfrac{\Gamma(\beta+1)}{4(\xi_2(d,c))^\beta} \Big(J^\beta_{(c+\xi_2(d,c))^-} f\left(a+\xi_1\left(b,a\right), c\right) + J^\beta_{(c+\xi_2(d,c))^-} f\left(a,c\right) \\
&+ J^\alpha_{c^+} f\left(a+\xi_1\left(b,a\right), c+\xi_2\left(d,c\right)\right) + J^\alpha_{c^+} f\left(a, c+\xi_2\left(d,c\right)\right)\Big). \qquad (3.\ 12)
\end{aligned}$$

Proof. By integration by parts, we get

$$\begin{aligned}
&\int_0^1\int_0^1 \left(t^\alpha-(1-t)^\alpha\right)\left(s^\beta-(1-s)^\beta\right) \tfrac{\partial^2 f}{\partial t\partial s}\left(a+t\xi_1\left(b,a\right), c+s\xi_2\left(d,c\right)\right) dsdt \\
=& \int_0^1 \left(t^\alpha-(1-t)^\alpha\right) \\
&\times\left(\int_0^1 \left(s^\beta-(1-s)^\beta\right) \tfrac{\partial^2 f}{\partial t\partial s}\left(a+t\xi_1\left(b,a\right), c+s\xi_2\left(d,c\right)\right) ds\right) dt \\
=& \int_0^1 \left(t^\alpha-(1-t)^\alpha\right) \Big(\tfrac{1}{\xi_2(d,c)} \tfrac{\partial f}{\partial t}\left(a+t\xi_1\left(b,a\right), c+\xi_2\left(d,c\right)\right) \\
&+\tfrac{1}{\xi_2(d,c)} \tfrac{\partial f}{\partial t}\left(a+t\xi_1\left(b,a\right), c\right) \\
&-\tfrac{\beta}{\xi_2(d,c)} \int_0^1 \left(s^{\beta-1}+(1-s)^{\beta-1}\right) \tfrac{\partial f}{\partial t}\left(a+t\xi_1\left(b,a\right), c+s\xi_2\left(d,c\right)\right) ds\Big) dt \\
=& \tfrac{1}{\xi_2(d,c)} \int_0^1 \left(t^\alpha-(1-t)^\alpha\right) \tfrac{\partial f}{\partial t}\left(a+t\xi_1\left(b,a\right), c+\xi_2\left(d,c\right)\right) dt \\
&+\tfrac{1}{\xi_2(d,c)} \int_0^1 \left(t^\alpha-(1-t)^\alpha\right) \tfrac{\partial f}{\partial t}\left(a+t\xi_1\left(b,a\right), c\right) dt \\
&-\tfrac{\beta}{\xi_2(d,c)} \int_0^1\int_0^1 \left(s^{\beta-1}+(1-s)^{\beta-1}\right)\left(t^\alpha-(1-t)^\alpha\right) \\
&\times\tfrac{\partial f}{\partial t}\left(a+t\xi_1\left(b,a\right), c+s\xi_2\left(d,c\right)\right) dtds \\
=& \tfrac{1}{\xi_1(b,a)\xi_2(d,c)} f\left(a+\xi_1\left(b,a\right), c+\xi_2\left(d,c\right)\right) + \tfrac{1}{\xi_1(b,a)\xi_2(d,c)} f\left(a, c+\xi_2\left(d,c\right)\right) \\
&-\tfrac{\alpha}{\xi_1(b,a)\xi_2(d,c)} \int_0^1 \left(t^{\alpha-1}+(1-t)^{\alpha-1}\right) f\left(a+t\xi_1\left(b,a\right), c+\xi_2\left(d,c\right)\right) dt \\
&+\tfrac{1}{\xi_1(b,a)\xi_2(d,c)} f\left(a+\xi_1\left(b,a\right), c\right) + \tfrac{1}{\xi_1(b,a)\xi_2(d,c)} f\left(a,c\right)
\end{aligned}$$

$$
\begin{aligned}
& -\tfrac{\alpha}{\xi_1(b,a)\xi_2(d,c)}\int\limits_0^1\left(t^{\alpha-1}+(1-t)^{\alpha-1}\right)f\left(a+t\xi_1\left(b,a\right),c\right)dt \\
& -\tfrac{\beta}{\xi_1(b,a)\xi_2(d,c)}\int\limits_0^1\left(s^{\beta-1}+(1-s)^{\beta-1}\right)f\left(a+\xi_1\left(b,a\right),c+s\xi_2\left(d,c\right)\right)ds \\
& -\tfrac{\beta}{\xi_1(b,a)\xi_2(d,c)}\int\limits_0^1\left(s^{\beta-1}+(1-s)^{\beta-1}\right)f\left(a,c+s\xi_2\left(d,c\right)\right)ds \\
& +\tfrac{\alpha\beta}{\xi_1(b,a)\xi_2(d,c)}\int\limits_0^1\int\limits_0^1\left(s^{\beta-1}+(1-s)^{\beta-1}\right)\left(t^{\alpha-1}+(1-t)^{\alpha-1}\right) \\
& \times f\left(a+t\xi_1\left(b,a\right),c+s\xi_2\left(d,c\right)\right)dsdt \\
= \; & \tfrac{1}{\xi_1(b,a)\xi_2(d,c)}f\left(a+\xi_1\left(b,a\right),c+\xi_2\left(d,c\right)\right)+\tfrac{1}{\xi_1(b,a)\xi_2(d,c)}f\left(a,c+\xi_2\left(d,c\right)\right) \\
& -\tfrac{\alpha}{\xi_1(b,a)\xi_2(d,c)}\int\limits_0^1\left(t^{\alpha-1}+(1-t)^{\alpha-1}\right)f\left(a+t\xi_1\left(b,a\right),c+\xi_2\left(d,c\right)\right)dt \\
& +\tfrac{1}{\xi_1(b,a)\xi_2(d,c)}f\left(a+\xi_1\left(b,a\right),c\right)+\tfrac{1}{\xi_1(b,a)\xi_2(d,c)}f\left(a,c\right) \\
& -\tfrac{\alpha}{\xi_1(b,a)\xi_2(d,c)}\int\limits_0^1\left(t^{\alpha-1}+(1-t)^{\alpha-1}\right)f\left(a+t\xi_1\left(b,a\right),c\right)dt \\
& -\tfrac{\beta}{\xi_1(b,a)\xi_2(d,c)}\int\limits_0^1\left(s^{\beta-1}+(1-s)^{\beta-1}\right)f\left(a+\xi_1\left(b,a\right),c+s\xi_2\left(d,c\right)\right)ds \\
& -\tfrac{\beta}{\xi_1(b,a)\xi_2(d,c)}\int\limits_0^1\left(s^{\beta-1}+(1-s)^{\beta-1}\right)f\left(a,c+s\xi_2\left(d,c\right)\right)ds \\
& +\tfrac{\alpha\beta}{\xi_1(b,a)\xi_2(d,c)}\int\limits_0^1\int\limits_0^1\left(s^{\beta-1}+(1-s)^{\beta-1}\right)\left(t^{\alpha-1}+(1-t)^{\alpha-1}\right) \\
& \times f\left(a+t\xi_1\left(b,a\right),c+s\xi_2\left(d,c\right)\right)dsdt \\
= \; & \tfrac{f(a,c)+f(a,c+\xi_2(d,c))+f(a+\xi_1(b,a),c)+f(a+\xi_1(b,a),c+\xi_2(d,c))}{\xi_1(b,a)\xi_2(d,c)} \\
& -\tfrac{\alpha}{\xi_1(b,a)\xi_2(d,c)}\int\limits_0^1 t^{\alpha-1}f\left(a+t\xi_1\left(b,a\right),c+\xi_2\left(d,c\right)\right)dt \\
& -\tfrac{\alpha}{\xi_1(b,a)\xi_2(d,c)}\int\limits_0^1 (1-t)^{\alpha-1}f\left(a+t\xi_1\left(b,a\right),c+\xi_2\left(d,c\right)\right)dt
\end{aligned}
$$

$$
\begin{aligned}
&-\tfrac{\alpha}{\xi_1(b,a)\xi_2(d,c)}\int_0^1 t^{\alpha-1} f\left(a+t\xi_1\left(b,a\right),c\right)dt \\
&-\tfrac{\alpha}{\xi_1(b,a)\xi_2(d,c)}\int_0^1 \left(1-t\right)^{\alpha-1} f\left(a+t\xi_1\left(b,a\right),c\right)dt \\
&-\tfrac{\beta}{\xi_1(b,a)\xi_2(d,c)}\int_0^1 s^{\beta-1} f\left(a+\xi_1\left(b,a\right),c+s\xi_2\left(d,c\right)\right)ds \\
&-\tfrac{\beta}{\xi_1(b,a)\xi_2(d,c)}\int_0^1 \left(1-s\right)^{\beta-1} f\left(a+\xi_1\left(b,a\right),c+\xi_2\left(d,c\right)\right)ds \\
&-\tfrac{\beta}{\xi_1(b,a)\xi_2(d,c)}\int_0^1 s^{\beta-1} f\left(a,c+s\xi_2\left(d,c\right)\right)ds \\
&-\tfrac{\beta}{\xi_1(b,a)\xi_2(d,c)}\int_0^1 s^{\beta-1} f\left(a,c+s\xi_2\left(d,c\right)\right)ds \\
&-\tfrac{\beta}{\xi_1(b,a)\xi_2(d,c)}\int_0^1 \left(1-s\right)^{\beta-1} f\left(a,c+s\xi_2\left(d,c\right)\right)ds \\
&+\tfrac{\alpha\beta}{\xi_1(b,a)\xi_2(d,c)}\int_0^1\int_0^1 s^{\beta-1}t^{\alpha-1} f\left(a+t\xi_1\left(b,a\right),c+s\xi_2\left(d,c\right)\right)dsdt \\
&+\tfrac{\alpha\beta}{\xi_1(b,a)\xi_2(d,c)}\int_0^1\int_0^1 s^{\beta-1}\left(1-t\right)^{\alpha-1} f\left(a+t\xi_1\left(b,a\right),c+s\xi_2\left(d,c\right)\right)dsdt \\
&+\tfrac{\alpha\beta}{\xi_1(b,a)\xi_2(d,c)}\int_0^1\int_0^1 \left(1-s\right)^{\beta-1} t^{\alpha-1} f\left(a+t\xi_1\left(b,a\right),c+s\xi_2\left(d,c\right)\right)dsdt \\
&+\tfrac{\alpha\beta}{\xi_1(b,a)\xi_2(d,c)}\int_0^1\int_0^1 \left(1-s\right)^{\beta-1}\left(1-t\right)^{\alpha-1} f\left(a+t\xi_1\left(b,a\right),c+s\xi_2\left(d,c\right)\right)dsdt.
\end{aligned}
\tag{3. 13}
$$

Putting $x=a+t\xi_1\left(b,a\right)$ and $y=c+s\xi_2\left(d,c\right)$ in (3. 13) we obtain

$$
\begin{aligned}
&\int_0^1\int_0^1 \left(t^{\alpha}-\left(1-t\right)^{\alpha}\right)\left(s^{\beta}-\left(1-s\right)^{\beta}\right)\tfrac{\partial^2 f}{\partial t\partial s}\left(a+t\xi_1\left(b,a\right),c+s\xi_2\left(d,c\right)\right)dsdt \\
=\quad &\tfrac{f(a,c)+f(a,c+\xi_2(d,c))+f(a+\xi_1(b,a),c)+f(a+\xi_1(b,a),c+\xi_2(d,c))}{\xi_1(b,a)\xi_2(d,c)}
\end{aligned}
$$

$$-\frac{\alpha}{(\xi_1(b,a))^{\alpha+1}\xi_2(d,c)}\int\limits_a^{a+\xi_1(b,a)}(x-a)^{\alpha-1}f(x,c+\xi_2(d,c))\,dx$$

$$-\frac{\alpha}{(\xi_1(b,a))^{\alpha+1}\xi_2(d,c)}\int\limits_a^{a+\xi_1(b,a)}(a+\xi_1(b,a)-x)^{\alpha-1}f(x,c+\xi_2(d,c))\,dx$$

$$-\frac{\alpha}{(\xi_1(b,a))^{\alpha+1}\xi_2(d,c)}\int\limits_a^{a+\xi_1(b,a)}(x-a)^{\alpha-1}f(x,c)\,dx$$

$$-\frac{\alpha}{(\xi_1(b,a))^{\alpha+1}\xi_2(d,c)}\int\limits_a^{a+\xi_1(b,a)}(a+\xi_1(b,a)-x)^{\alpha-1}f(x,c)\,dx$$

$$-\frac{\beta}{\xi_1(b,a)(\xi_2(d,c))^{\beta+1}}\int\limits_c^{c+\xi_2(d,c)}(y-c)^{\beta-1}f(a+\xi_1(b,a),y)\,dy$$

$$-\frac{\beta}{\xi_1(b,a)(\xi_2(d,c))^{\beta+1}}\int\limits_c^{c+\xi_2(d,c)}(c+\xi_2(d,c)-y)^{\beta-1}f(a+\xi_1(b,a),y)\,dy$$

$$-\frac{\beta}{\xi_1(b,a)(\xi_2(d,c))^{\beta+1}}\int\limits_c^{c+\xi_2(d,c)}(y-c)^{\beta-1}f(a,y)\,dy$$

$$-\frac{\beta}{\xi_1(b,a)(\xi_2(d,c))^{\beta+1}}\int\limits_c^{c+\xi_2(d,c)}(c+\xi_2(d,c)-y)^{\beta-1}f(a,y)\,dy$$

$$+\frac{\alpha\beta}{(\xi_1(b,a))^{\alpha+1}(\xi_2(d,c))^{\beta+1}}$$

$$\times\int\limits_a^{a+\xi_1(b,a)}\int\limits_c^{c+\xi_2(d,c)}(x-a)^{\alpha-1}(y-c)^{\beta-1}f(x,y)\,dydx$$

$$+\frac{\alpha\beta}{(\xi_1(b,a))^{\alpha+1}(\xi_2(d,c))^{\beta+1}}$$

$$\times\int\limits_a^{a+\xi_1(b,a)}\int\limits_c^{c+\xi_2(d,c)}(a+\xi_1(b,a)-x)^{\alpha-1}(y-c)^{\beta-1}f(x,y)\,dydx$$

$$+\frac{\alpha\beta}{(\xi_1(b,a))^{\alpha+1}(\xi_2(d,c))^{\beta+1}}$$

$$\times\int\limits_a^{a+\xi_1(b,a)}\int\limits_c^{c+\xi_2(d,c)}(x-a)^{\alpha-1}(c+\xi_2(d,c)-y)^{\beta-1}f(x,y)\,dydx$$

$$+\frac{\alpha\beta}{(\xi_1(b,a))^{\alpha+1}(\xi_2(d,c))^{\beta+1}}$$
$$\times\int_a^{a+\xi_1(b,a)}\int_c^{c+\xi_2(d,c)}(a+\xi_1(b,a)-x)^{\alpha-1}(c+\xi_2(d,c)-y)^{\beta-1}f(x,y)\,dydx. \tag{3. 14}$$

Multplying both sides of (3. 14) by $\frac{\xi_1(b,a)\xi_2(d,c)}{4}$, and using (2. 3)-(2. 10), we get the desired result. □

Theorem 3.4. *Let $f:K\to\mathbb{R}$ be a partially differentiable function on K. If $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$ is co-ordinated* log*-preinvex function on K with respect to ξ_1 and ξ_2 such that $\xi_1(b,a)>0$ and $\xi_2(d,c)>0$, then the following fractional inequality holds*

$$\begin{aligned}&\Bigg|\frac{f(a,c)+f(a,c+\xi_2(d,c))+f(a+\xi_1(b,a),c)+f(a+\xi_1(b,a),c+\xi_2(d,c))}{4}-A\\&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\xi_1(b,a))^{\alpha}(\xi_2(d,c))^{\beta}}\Big(J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,(c+\xi_2(d,c))^-}f(a,c)\\&+J^{\alpha,\beta}_{a^+,(c+\xi_2(d,c))^-}f(a+\xi_1(b,a),c)+J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,c^+}f(a,c+\xi_2(d,c))\\&+J^{\alpha,\beta}_{a^+,c^+}f(a+\xi_1(b,a),c+\xi_2(d,c))\Big)\Bigg|\\\leq\ &\frac{\xi_1(b,a)\xi_2(d,c)}{4}\frac{\sigma+3\tau}{(\alpha+1)(\beta+1)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,\end{aligned}$$

where

$$\sigma=\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|, \tag{3. 15}$$

$$\tau=\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|, \tag{3. 16}$$

and A is defined as in (3. 12).

Proof. From Lemma 3.3, and properties of modulus we have

$$\begin{aligned}&\Bigg|\frac{f(a,c)+f(a,c+\xi_2(d,c))+f(a+\xi_1(b,a),c)+f(a+\xi_1(b,a),c+\xi_2(d,c))}{4}-A\\&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\xi_1(b,a))^{\alpha}(\xi_2(d,c))^{\beta}}\Big(J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,(c+\xi_2(d,c))^-}f(a,c)\end{aligned}$$

$$+ J^{\alpha,\beta}_{a^+,(c+\xi_2(d,c))^-} f\left(a+\xi_1(b,a),c\right) + J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,c^+} f\left(a,c+\xi_2(d,c)\right)$$
$$+ \ J^{\alpha,\beta}_{a^+,c^+} f\left(a+\xi_1(b,a),c+\xi_2(d,c)\right)\Big)\Big|$$
$$\leq \tfrac{\xi_1(b,a)\xi_2(d,c)}{4}$$
$$\times \int_0^1\int_0^1 |t^\alpha-(1-t)^\alpha| \left|\lambda^\beta-(1-\lambda)^\beta\right| \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\xi_1(b,a),c+\lambda\xi_2(d,c)\right)\right| d\lambda dt$$
$$\leq \tfrac{\xi_1(b,a)\xi_2(d,c)}{4}$$
$$\times \int_0^1\int_0^1 \left(t^\alpha+(1-t)^\alpha\right)\left(\lambda^\beta+(1-\lambda)^\beta\right) \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\xi_1(b,a),c+\lambda\xi_2(d,c)\right)\right| d\lambda dt. \tag{3. 17}$$

Using $\log$-preinvexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$, we get

$$\left|\tfrac{f(a,c)+f(a,c+\xi_2(d,c))+f(a+\xi_1(b,a),c)+f(a+\xi_1(b,a),c+\xi_2(d,c))}{4} - A\right.$$
$$+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\xi_1(b,a))^\alpha(\xi_2(d,c))^\beta}\Big(J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,(c+\xi_2(d,c))^-} f(a,c)$$
$$+J^{\alpha,\beta}_{a^+,(c+\xi_2(d,c))^-} f\left(a+\xi_1(b,a),c\right) + J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,c^+} f\left(a,c+\xi_2(d,c)\right)$$
$$\left. + \ J^{\alpha,\beta}_{a^+,c^+} f\left(a+\xi_1(b,a),c+\xi_2(d,c)\right)\Big)\right|$$
$$\leq \quad \tfrac{\xi_1(b,a)\xi_2(d,c)}{4}$$
$$\times\int_0^1\int_0^1 \Big(\left(t^\alpha+(1-t)^\alpha\right)\left(\lambda^\beta+(1-\lambda)^\beta\right)\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^{(1-t)(1-\lambda)}$$
$$\times\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^{(1-t)\lambda}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^{t(1-\lambda)}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^{t\lambda}\Big) d\lambda dt$$
$$\leq \quad \tfrac{\xi_1(b,a)\xi_2(d,c)}{4}$$
$$\times\int_0^1\int_0^1 \Big(\left(t^\alpha+(1-t)^\alpha\right)\left(\lambda^\beta+(1-\lambda)^\beta\right)\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^{1+t\lambda}$$
$$\times\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^{1-t\lambda}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^{1-t\lambda}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^{t\lambda}\Big) d\lambda dt$$
$$= \quad \tfrac{\xi_1(b,a)\xi_2(d,c)}{4}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|$$
$$\times\int_0^1\int_0^1 \left(t^\alpha+(1-t)^\alpha\right)\left(\lambda^\beta+(1-\lambda)^\beta\right)\sigma^{t\lambda}\tau^{1-t\lambda} d\lambda dt, \tag{3. 18}$$

where σ and τ are defined as in (3. 15) and (3. 16) respectively.

Now, applying the weighted arithmetic-geometric mean inequality, (3. 18) becomes

$$\begin{aligned}
&\left|\tfrac{f(a,c)+f(a,c+\xi_2(d,c))+f(a+\xi_1(b,a),c)+f(a+\xi_1(b,a),c+\xi_2(d,c))}{4}-A\right.\\
&+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\xi_1(b,a))^\alpha(\xi_2(d,c))^\beta}\Big(J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,(c+\xi_2(d,c))^-}f(a,c)\\
&+J^{\alpha,\beta}_{a^+,(c+\xi_2(d,c))^-}f(a+\xi_1(b,a),c)+J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,c^+}f(a,c+\xi_2(d,c))\\
&\left.+\ J^{\alpha,\beta}_{a^+,c^+}f(a+\xi_1(b,a),c+\xi_2(d,c))\Big)\right|\\
\leq\ &\tfrac{\xi_1(b,a)\xi_2(d,c)}{4}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|
\end{aligned}$$

$$\begin{aligned}
&\times\int_0^1\int_0^1\left(t^\alpha+(1-t)^\alpha\right)\left(\lambda^\beta+(1-\lambda)^\beta\right)\left(t\lambda\sigma+(1-t\lambda)\tau\right)d\lambda dt\\
=\ &\tfrac{\xi_1(b,a)\xi_2(d,c)}{4}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|\\
&\times\left((\sigma-\tau)\left(\int_0^1\left(t^{\alpha+1}+t(1-t)^\alpha\right)dt\right)\left(\int_0^1\left(\lambda^{\beta+1}+\lambda(1-\lambda)^\beta\right)d\lambda\right)\right.\\
&\left.+\tau\left(\int_0^1\left(t^\alpha+(1-t)^\alpha\right)dt\right)\left(\int_0^1\left(\lambda^\beta+(1-\lambda)^\beta\right)d\lambda\right)\right)\\
=\ &\tfrac{\xi_1(b,a)\xi_2(d,c)}{4}\tfrac{\sigma+3\tau}{(\alpha+1)(\beta+1)}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,
\end{aligned}$$

The proof is achieved. □

Corollary 3.5. *In Theorem 3.4 if we choose $\xi_1(b,a)=\xi_2(b,a)=b-a$, we obtain the following fractional inequality*

$$\begin{aligned}
&\left|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\right.\\
&\left.\times\left(J^{\alpha,\beta}_{b^-,d^-}f(a,c)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{a^+,c^+}f(b,d)\right)\right|\\
\leq\ &\tfrac{(b-a)(d-c)}{4}\tfrac{\sigma+3\tau}{(\alpha+1)(\beta+1)}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|.
\end{aligned}$$

Theorem 3.6. *Let $f:K\to\mathbb{R}$ be a partially differentiable function on K. If $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is co-ordinated* log*-preinvex function on K with respect to ξ_1 and ξ_2 such that $\xi_1(b,a)>0$ and $\xi_2(d,c)>0$, where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following fractional inequality* *holds*

$$\begin{aligned} & \left| \tfrac{f(a,c)+f(a,c+\xi_2(d,c))+f(a+\xi_1(b,a),c)+f(a+\xi_1(b,a),c+\xi_2(d,c))}{4} - A \right. \\ & + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\xi_1(b,a))^{\alpha}(\xi_2(d,c))^{\beta}} \Big(J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,(c+\xi_2(d,c))^-} f(a,c) \\ & + J^{\alpha,\beta}_{a^+,(c+\xi_2(d,c))^-} f(a+\xi_1(b,a),c) + J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,c^+} f(a,c+\xi_2(d,c)) \\ & \left. + \ J^{\alpha,\beta}_{a^+,c^+} f(a+\xi_1(b,a),c+\xi_2(d,c)) \Big) \right| \\ \leq \ & \tfrac{\xi_1(b,a)\xi_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}} \left| \tfrac{\partial^2 f}{\partial t \partial \lambda}(a,c) \right| \left(\tfrac{l+3m}{4} \right)^{\frac{1}{q}}, \end{aligned} \tag{3. 19}$$

where

$$l = \left| \tfrac{\partial^2 f}{\partial t \partial \lambda}(a,c) \right|^q \left| \tfrac{\partial^2 f}{\partial t \partial \lambda}(b,d) \right|^q, \tag{3. 20}$$

$$m = \left| \tfrac{\partial^2 f}{\partial t \partial \lambda}(a,d) \right|^q \left| \tfrac{\partial^2 f}{\partial t \partial \lambda}(b,c) \right|^q, \tag{3. 21}$$

and A is defined as in (3. 12).

Proof. From Lemma 3.3, properties of modulus, and Hölder inequality, we have

$$\begin{aligned} & \left| \tfrac{f(a,c)+f(a,c+\xi_2(d,c))+f(a+\xi_1(b,a),c)+f(a+\xi_1(b,a),c+\xi_2(d,c))}{4} - A \right. \\ & + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\xi_1(b,a))^{\alpha}(\xi_2(d,c))^{\beta}} \Big(J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,(c+\xi_2(d,c))^-} f(a,c) \\ & + J^{\alpha,\beta}_{a^+,(c+\xi_2(d,c))^-} f(a+\xi_1(b,a),c) + J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,c^+} f(a,c+\xi_2(d,c)) \\ & \left. + \ J^{\alpha,\beta}_{a^+,c^+} f(a+\xi_1(b,a),c+\xi_2(d,c)) \Big) \right| \\ \leq \ & \tfrac{\xi_1(b,a)\xi_2(d,c)}{4} \left(\left(\int_0^1\int_0^1 t^{\alpha p}\lambda^{\beta p} d\lambda dt \right)^{\frac{1}{p}} + \left(\int_0^1\int_0^1 t^{\alpha p}(1-\lambda)^{\beta p} d\lambda dt \right)^{\frac{1}{p}} \right. \\ & \left. + \left(\int_0^1\int_0^1 (1-t)^{p\alpha}\lambda^{p\beta} d\lambda dt \right)^{\frac{1}{p}} + \left(\int_0^1\int_0^1 (1-t)^{p\alpha}(1-\lambda)^{p\beta} d\lambda dt \right)^{\frac{1}{p}} \right) \\ & \times \left(\int_0^1\int_0^1 \left| \tfrac{\partial^2 f}{\partial t \partial \lambda}(a+t\xi_1(b,a),c+\lambda\xi_2(d,c)) \right|^q d\lambda dt \right)^{\frac{1}{q}} \\ = \ & \tfrac{\xi_1(b,a)\xi_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}} \left(\int_0^1\int_0^1 \left| \tfrac{\partial^2 f}{\partial t \partial \lambda}(a+t\xi_1(b,a),c+\lambda\xi_2(d,c)) \right|^q d\lambda dt \right)^{\frac{1}{q}} \end{aligned} \tag{3. 22}$$

Since $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q$ is co-ordinated $\log$-preinvex, we deduce

$$\begin{aligned}
&\Bigg|\tfrac{f(a,c)+f(a,c+\xi_2(d,c))+f(a+\xi_1(b,a),c)+f(a+\xi_1(b,a),c+\xi_2(d,c))}{4} - A \\
&+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\xi_1(b,a))^{\alpha}(\xi_2(d,c))^{\beta}}\Big(J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,(c+\xi_2(d,c))^-}f(a,c) \\
&+J^{\alpha,\beta}_{a^+,(c+\xi_2(d,c))^-}f(a+\xi_1(b,a),c)+J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,c^+}f(a,c+\xi_2(d,c)) \\
&+\ J^{\alpha,\beta}_{a^+,c^+}f(a+\xi_1(b,a),c+\xi_2(d,c))\Big)\Bigg| \\
\leq\ &\tfrac{\xi_1(b,a)\xi_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\left(\int_0^1\int_0^1\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q\right)^{(1-t)(1-\lambda)}\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right)^{t\lambda}\right. \\
&\left.\times\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q\right)^{(1-t)\lambda}\left(\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right)^{t(1-\lambda)}d\lambda dt\right)^{\frac{1}{q}} \\
\leq\ &\tfrac{\xi_1(b,a)\xi_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|\left(\int_0^1\int_0^1 l^{t\lambda}m^{1-t\lambda}d\lambda dt\right)^{\frac{1}{q}},
\end{aligned} \tag{3. 23}$$

where l and m are defined as in (3. 20) and (3. 21) respectively.

Now, applying the weighted arithmetic-geometric mean inequality for (3. 23), and then integrating the result, we get

$$\begin{aligned}
&\Bigg|\tfrac{f(a,c)+f(a,c+\xi_2(d,c))+f(a+\xi_1(b,a),c)+f(a+\xi_1(b,a),c+\xi_2(d,c))}{4} - A \\
&+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\xi_1(b,a))^{\alpha}(\xi_2(d,c))^{\beta}}\Big(J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,(c+\xi_2(d,c))^-}f(a,c) \\
&+J^{\alpha,\beta}_{a^+,(c+\xi_2(d,c))^-}f(a+\xi_1(b,a),c)+J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,c^+}f(a,c+\xi_2(d,c)) \\
&+\ J^{\alpha,\beta}_{a^+,c^+}f(a+\xi_1(b,a),c+\xi_2(d,c))\Big)\Bigg| \\
\leq\ &\tfrac{\xi_1(b,a)\xi_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|\left(\int_0^1\int_0^1(t\lambda l+(1-t\lambda)m)\,d\lambda dt\right)^{\frac{1}{q}} \\
=\ &\tfrac{\xi_1(b,a)\xi_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|\left(\tfrac{l+3m}{4}\right)^{\frac{1}{q}},
\end{aligned}$$

which is the desired result. □

Corollary 3.7. *In Theorem 3.6 if we choose $\xi_1(b,a)=\xi_2(b,a)=b-a$, we obtain the following fractional inequality*

$$\begin{aligned}
&\Bigg|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - A + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \\
&\times\Big(J^{\alpha,\beta}_{b^-,d^-}f(a,c)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{a^+,c^+}f(b,d)\Big)\Bigg| \\
\leq\ &\tfrac{(b-a)(d-c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|\left(\tfrac{l+3m}{4}\right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 3.8. *Let $f : K \to \mathbb{R}$ be a partially differentiable function on K. If $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q$ is co-ordinated* $\log$*-preinvex function on K with respect to ξ_1 and ξ_2 where $q > 1$, then the following inequality holds*

$$\begin{aligned}
&\Big| \tfrac{f(a,c)+f(a,c+\xi_2(d,c))+f(a+\xi_1(b,a),c)+f(a+\xi_1(b,a),c+\xi_2(d,c))}{4} - A \\
&+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\xi_1(b,a))^{\alpha}(\xi_2(d,c))^{\beta}} \Big(J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,(c+\xi_2(d,c))^-} f(a,c) \\
&+J^{\alpha,\beta}_{a^+,(c+\xi_2(d,c))^-} f(a+\xi_1(b,a),c) + J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,c^+} f(a,c+\xi_2(d,c)) \\
&+ J^{\alpha,\beta}_{a^+,c^+} f(a+\xi_1(b,a),c+\xi_2(d,c)) \Big) \Big| \\
\leq\ & \tfrac{\xi_1(b,a)\xi_2(d,c)}{4(1+\alpha)(1+\beta)} \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a,c)\right| \Bigg(\left(\tfrac{(\alpha+1)(\beta+1)l+(\beta+\alpha+3)m}{(\alpha+2)(\beta+2)}\right)^{\frac{1}{q}} + \left(\tfrac{l+(\beta\alpha+2\beta+2\alpha+3)m}{(\alpha+2)(\beta+2)}\right)^{\frac{1}{q}} \\
&+ \left(\tfrac{(\alpha+1)l+(\beta(\alpha+2)+\alpha+3)m}{(\alpha+2)(\beta+2)}\right)^{\frac{1}{q}} + \left(\tfrac{(\beta+1)l+((\beta+2)\alpha+\beta+3)m}{(\alpha+2)(\beta+2)}\right)^{\frac{1}{q}} \Bigg),
\end{aligned}$$

where A, l and m are defined as in (3. 12)*,* (3. 20) *and* (3. 21) *respectively.*

Proof. From Lemma 3.3, properties of modulus, and power mean inequality, we have

$$\begin{aligned}
&\Big| \tfrac{f(a,c)+f(a,c+\xi_2(d,c))+f(a+\xi_1(b,a),c)+f(a+\xi_1(b,a),c+\xi_2(d,c))}{4} - A \\
&+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\xi_1(b,a))^{\alpha}(\xi_2(d,c))^{\beta}} \Big(J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,(c+\xi_2(d,c))^-} f(a,c) \\
&+J^{\alpha,\beta}_{a^+,(c+\xi_2(d,c))^-} f(a+\xi_1(b,a),c) + J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,c^+} f(a,c+\xi_2(d,c)) \\
&+ J^{\alpha,\beta}_{a^+,c^+} f(a+\xi_1(b,a),c+\xi_2(d,c)) \Big) \Big| \\
\leq\ & \tfrac{\xi_1(b,a)\xi_2(d,c)}{4} \Bigg(\left(\int_0^1 \int_0^1 t^{\alpha}\lambda^{\beta} d\lambda dt \right)^{1-\frac{1}{q}} \\
&\times \left(\int_0^1 \int_0^1 t^{\alpha}\lambda^{\beta} \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a+t\xi_1(b,a), c+\lambda\xi_2(d,c))\right|^q d\lambda dt \right)^{\frac{1}{q}} \\
&+ \left(\int_0^1 \int_0^1 t^{\alpha}(1-\lambda)^{\beta} d\lambda dt \right)^{1-\frac{1}{q}} \\
&\times \left(\int_0^1 \int_0^1 t^{\alpha}(1-\lambda)^{\beta} \left|\tfrac{\partial^2 f}{\partial t \partial \lambda}(a+t\xi_1(b,a), c+\lambda\xi_2(d,c))\right|^q d\lambda dt \right)^{\frac{1}{q}} \\
&+ \left(\int_0^1 \int_0^1 (1-t)^{\alpha}\lambda^{\beta} d\lambda dt \right)^{1-\frac{1}{q}}
\end{aligned}$$

$$\times\left(\int_0^1\int_0^1 (1-t)^{\alpha}\lambda^{\beta}\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\xi_1(b,a),c+\lambda\xi_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\int_0^1 (1-t)^{\alpha}(1-\lambda)^{\beta}d\lambda dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\int_0^1 (1-t)^{\alpha}(1-\lambda)^{\beta}\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\xi_1(b,a),c+\lambda\xi_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}\Bigg)$$

$$= \frac{\xi_1(b,a)\xi_2(d,c)}{4(1+\alpha)^{1-\frac{1}{q}}(1+\beta)^{1-\frac{1}{q}}}\left(\left(\int_0^1\int_0^1 t^{\alpha}\lambda^{\beta}\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\xi_1(b,a),c+\lambda\xi_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1\int_0^1 t^{\alpha}(1-\lambda)^{\beta}\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\xi_1(b,a),c+\lambda\xi_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\int_0^1 (1-t)^{\alpha}\lambda^{\beta}\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\xi_1(b,a),c+\lambda\xi_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1\int_0^1 (1-t)^{\alpha}(1-\lambda)^{\beta}\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\xi_1(b,a),c+\lambda\xi_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}\right).$$

Using log-preinvexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$, and then Applying the A-G inequality for the result we get

$$\left|\frac{f(a,c)+f(a,c+\xi_2(d,c))+f(a+\xi_1(b,a),c)+f(a+\xi_1(b,a),c+\xi_2(d,c))}{4}-A\right.$$

$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\xi_1(b,a))^{\alpha}(\xi_2(d,c))^{\beta}}\left(J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,(c+\xi_2(d,c))^-}f(a,c)\right.$$

$$+J^{\alpha,\beta}_{a^+,(c+\xi_2(d,c))^-}f\left(a+\xi_1(b,a),c\right)+J^{\alpha,\beta}_{(a+\xi_1(b,a))^-,c^+}f\left(a,c+\xi_2(d,c)\right)$$

$$\left.\left.+\ J^{\alpha,\beta}_{a^+,c^+}f\left(a+\xi_1(b,a),c+\xi_2(d,c)\right)\right)\right|$$

$$\leq \frac{\xi_1(b,a)\xi_2(d,c)}{4(1+\alpha)^{1-\frac{1}{q}}(1+\beta)^{1-\frac{1}{q}}}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|$$

$$\times\left(\left((l-m)\int_0^1\int_0^1 t^{\alpha+1}\lambda^{\beta+1}d\lambda dt+m\int_0^1\int_0^1 t^{\alpha}\lambda^{\beta}d\lambda dt\right)^{\frac{1}{q}}\right.$$

$$+\left((l-m)\int_0^1\int_0^1 t^{\alpha+1}\lambda\left(1-\lambda\right)^{\beta}d\lambda dt+m\int_0^1\int_0^1 t^{\alpha}\left(1-\lambda\right)^{\beta}d\lambda dt\right)^{\frac{1}{q}}$$

$$+\left((l-m)\int_0^1\int_0^1 t\left(1-t\right)^{\alpha}\lambda\left(1-\lambda\right)^{\beta}d\lambda dt+m\int_0^1\int_0^1\left(1-t\right)^{\alpha}\left(1-\lambda\right)^{\beta}d\lambda dt\right)^{\frac{1}{q}}$$

$$+\left((l-m)\int_0^1\int_0^1 t\left(1-t\right)^{\alpha}\lambda^{\beta+1}d\lambda dt+m\int_0^1\int_0^1\left(1-t\right)^{\alpha}\lambda^{\beta}d\lambda dt\right)^{\frac{1}{q}}\Bigg)$$

$$=\frac{\xi_1(b,a)\xi_2(d,c)}{4(1+\alpha)(1+\beta)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,c\right)\right|\left(\left(\frac{(\alpha+1)(\beta+1)l+(\beta+\alpha+3)m}{(\alpha+2)(\beta+2)}\right)^{\frac{1}{q}}+\left(\frac{l+(\beta\alpha+2\beta+2\alpha+3)m}{(\alpha+2)(\beta+2)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(\alpha+1)l+(\beta(\alpha+2)+\alpha+3)m}{(\alpha+2)(\beta+2)}\right)^{\frac{1}{q}}+\left(\frac{(\beta+1)l+((\beta+2)\alpha+\beta+3)m}{(\alpha+2)(\beta+2)}\right)^{\frac{1}{q}}\right),$$

which is the desired result. □

Corollary 3.9. *In Theorem 3.8 if we choose $\xi_1(b,a)=\xi_2(b,a)=b-a$, we obtain the following fractional inequality*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\right.$$
$$\left.\times\left(J^{\alpha,\beta}_{b^-,d^-}f\left(a,c\right)+J^{\alpha,\beta}_{a^+,d^-}f\left(b,c\right)+J^{\alpha,\beta}_{b^-,c^+}f\left(a,d\right)+J^{\alpha,\beta}_{a^+,c^+}f\left(b,d\right)\right)\right|$$
$$\leq\frac{(b-a)(d-c)}{4(1+\alpha)(1+\beta)}\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,c\right)\right|\left(\left(\frac{(\alpha+1)(\beta+1)l+(\beta+\alpha+3)m}{(\alpha+2)(\beta+2)}\right)^{\frac{1}{q}}+\left(\frac{l+(\beta\alpha+2\beta+2\alpha+3)m}{(\alpha+2)(\beta+2)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(\alpha+1)l+(\beta(\alpha+2)+\alpha+3)m}{(\alpha+2)(\beta+2)}\right)^{\frac{1}{q}}+\left(\frac{(\beta+1)l+((\beta+2)\alpha+\beta+3)m}{(\alpha+2)(\beta+2)}\right)^{\frac{1}{q}}\right).$$

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