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Solving Flexible Fuzzy Multi-Objective Linear Programming Problems: Feasibility and Efficiency Concept of Solutions

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Abstract. We propose a new algorithm to solve multi-objective fuzzy linear programming problem. Although various models in the literature have been introduced, but we concentrate a multi-objective linear programming model with flexible fuzzy constraints. For solving the introduced model, we propose a new approach ,where using different cuts, allows the original problem reduced to the crisp multi-parametric multi-objective linear programming problems. The mentioned approach first uses a one phase method just in one phase to an optimal solution can be achieved for every objective and then, the optimal pareto solution for the reduced multi parametric multi-objective linear programming can be found by goal programming model. This approach will be illustrated by a numerical example.

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1. Introduction

The issue of the allocation of the resources of consequently the volume of activity appointment are prudery doing some activities contingent upon common use of limited resource. We can formulate these issues by mathematical programming and have their corresponding models. One of these methods is operations research. If in the related model all mathematical relations are in the linear type, then we call it entitled linear mathematical programming. In these models, the objectives are stated in a combined goal function. For example, maximizing the total profit or minimizing the total cost. But one difficulty of linear mathematical programming is facts simplifying which consequently causes some limitations for planners interests and his problem solving. Lot of organizations are seeking non-economic goal such as social responsibilities and cooperation and appropriable public sociability. Actually Theas organization are pursuing some Multiple Criteria Decision Making(MCDM) which linear cannot afford it. As a matter of fact, the real world decision maker must fulfil several objectives which are sometimes conflicting of contradictory and even unpredictable. And this is the being divided into types; Multi- Objective Decision

Makers (MODM) and Multi Attribute Decision Makers (MADM)(Climaco,[5]) . A brief explanation about the development of MODM will be presented in this part. Kuhn and Tucker [11] for the first time published multiple objective by use of vector optimization concept, following that in (Yu [26]) presented the compromise solution method to cope with MODM problems which had outstanding influence on different application such as financial planning, econometric and development planning, transportation investment and planning, business management, investment portfolio selection, land use planning, water resource management, public policy and environmental issues. Also theoretically, the work develops of from simple multiple objective programming to multi-level, multi-objective programming to cope with very complex the problem real world. But we should consider another matter that the conventional MODM actually ignored the problem of subjective uncertainty if we want to with real word problem with fuzzy number and fuzzy variable we should in corporate then into MODM. Bellman and Zadeh [4] presented the idea of decision making in fuzzy environments. Following that many authors had some great and successful works such as Dudek et al. [7], Kondo et al. [10], Rahman et al. [16], Shakeel et al. ([23],[24]), Hwang and Yoon [9], Zimmerman [27], Sakawa([18],[19],[20]), Lee and Li [12] which resulted in studied of fuzzy multiple objective linear programming (FMOLP). Different types of FMOLP model have been suggested and also methods to solve these models have been progressed in literatures, such that, Wu et al. [25], Deep et al. [6], Hu et al. [8]. Various type of such models and solution are presented in [2]. And this question is proposed: how can we solve multi-objective problems By studying literature, we can see that most available method. Tried to accumulate multiple objective problem into a single objective by employing some real-valued functions and utility function [17], which real-valued function have various form such as weighted sum [15], max-min or weighted max-min ([14],[21],[13],[1]), the product form [1]. The utility function methods are placed on Decision Makers(DMs) selection. Which this selection is expressed in mathematical expressions by employing the utility function ([17],[22]).

The rest of this paper is organized as follows: in section 2, the relationships needed to transform the multi-objective linear programming problem with flexible fuzzy constraint (MOLPFFC) into a multi parametric multiple objective linear programming(MOMOLP) problem are presented, and at the end of the definitions and the basic theorems related to them are expressed; In section 3, the proposed a new algorithm for solving MOLPFFC problems and obtaining the optimal pareto solution is recommended for MPMOLP problems; Section 4 presents a numerical illustration and the finally the paper concludes in section 5.

2. Mathematical model and the fundamental theoretical results

One of the practical kind of fuzzy multi-objective linear programming model is a Multi-Objective Linear Programming with Flexible Fuzzy Constraints(MOLPFFC). The general model of the MOLPFFC problem with fuzzy flexible constraints can be written as

$$\begin{array}{lll} \max & Z_1(x) = \sum\limits_{j=1}^{n} c_{1j}x_j, \ldots, \max\ Z_K(x) = \sum\limits_{j=1}^{n} c_{Kj}x_j & \\ s.t. & \sum\limits_{j=1}^{n} a_{ij}x_j \precsim_{FF} b_i,\ \ i = 1, \ldots, m, & \\ & x_j \geq\ 0, \qquad\qquad j = 1, \ldots, n. & \end{array} \tag{2. 1}$$

In this study we have taken in to account the fuzziness in the aspiration level of the constraints set. Then for each constraint, the assumption can be shown by $\precsim_{FF}$ and modelled by use of any kind of membership function here, $\precsim_{FF}$ show that the inequalities are flexible and may be modified by a fuzzy set whose membership state that the decision maker can stand the violation of the constraints up to a definite level of tolerance. stand for the tolerance $p_i, i = 1, \ldots, m$, level of the ith constraint of the model (2.1).
?

2.1. Formulation using linear membership function

Take into account a decision maker encounters a MOLP with Fuzzy Flexible Constraint (MOLPFFC) problem in which he /she can endure violation in completion at the constraints, which is he allows the constraints to be hold as far as possible. For each constraint in the constraints set this presumption can be denoted by $a_i x \precsim_{FF} b_i, i = 1, \ldots, m$ and for each, modeled by use of the membership function,

$$\mu_i(x) = \begin{cases} 1, & \sum\limits_{j=1}^{n} a_{ij}x_j \leq b_i \\ F_i(\sum\limits_{j=1}^{n} a_{ij}x_j), & b_i \leq \sum\limits_{j=1}^{n} a_{ij}x_j \leq \quad b_i + p_i \\ 0, & \sum\limits_{j=1}^{n} a_{ij}x_j \geq b_i + p_i \end{cases} \tag{2. 2}$$

where $F_i\left(0\right)$ is completely reducing and continuous for $\sum\limits_{j=1}^{n} a_{ij}x_j, F_i\left(b_i\right) = 1$ and $F_i\left(b_i + p_i\right) = 0$. This membership function states that the decision maker tolerates violation in the accomplishment of the constraints i up the value $b_i + p_i$. The function $\mu_i\left(x\right)$ gives the degree of satisfaction of the i-th constrains for $x \in R^n$, but this value is achieved by use of function F_i which are defined over R. By considering these assumptions, the associated MOLPFFC problem will be presented as

$$\begin{array}{lll} \max & Z_1(x) = \sum\limits_{j=1}^{n} c_{1j}x_j, \ldots, \max\ Z_K(x) = \sum\limits_{j=1}^{n} c_{Kj}x_j & \\ s.t. & H_i\left(x, a_i\right) = \sum\limits_{j=1}^{n} a_{ij}x_j - b_i \precsim_{FF} 0, & i = 1, \ldots, m, \\ & x_j \geq 0, & j = 1, \ldots, n. \end{array} \tag{2. 3}$$

where $C_k = \left(c_{k1}, c_{k2}, \ldots, c_{kn}\right), k = 1, \ldots, K$ is an n-dimensional vector of parameters concerned in the objective function $Z_k, k = 1, \ldots, K$.

The crisp shape of MOLPFFC problem is obtained as follows:

$$\begin{array}{ll} \max & Z_1(x) = \sum\limits_{j=1}^{n} c_{1j}x_j, \ldots, \max\ Z_K(x) = \sum\limits_{j=1}^{n} c_{Kj}x_j \\ s.t. & \mu_i \left\{H_i\left(x, a_i\right) \precsim_{FF} 0\right\} \geq \alpha_i, \\ & x \geq 0,\ \alpha_i \geq \alpha_i^D,\ 0 \leq \alpha_i \leq 1, \quad i = 1, \ldots, m. \end{array} \tag{2. 4}$$

We consider α_i^D, as the lower bounded for α_i (minimum degree of membership ith constraints) based on the point view of the decision maker.

$$\mu_i \left\{H_i\left(x, a_i\right) \precsim 0\right\} = \begin{cases} 1\,, & H_i\left(x, a_i\right) \leq 0 \\ 1 - \frac{H_i(x, a_i)}{p_i}\,, & 0 \leq H_i\left(x, a_i\right) \leq p_i \\ 0\,, & H_i\left(x, a_i\right) \geq 0 \end{cases} \tag{2. 5}$$

This relation is equal to relation (2.6). The following membership function,

$$\mu_i \left\{H_i\left(x, a_i\right) \precsim 0\right\} = \begin{cases} 1\,, & \sum\limits_{j=1}^{n} a_{ij}x_j - b_i \leq 0 \\ 1 - \dfrac{\sum\limits_{j=1}^{n} a_{ij}x_j - b_i}{p_i}\,, & 0 \leq \sum\limits_{j=1}^{n} a_{ij}x_j - b_i \leq p_i \\ 0\,, & \sum\limits_{j=1}^{n} a_{ij}x_j - b_i \geq 0 \end{cases} \tag{2. 6}$$

And (2.4) gets

$$\begin{array}{ll} \max & Z_1(x) = \sum\limits_{j=1}^{n} c_{1j}x_j, \ldots, \max\ Z_K(x) = \sum\limits_{j=1}^{n} c_{Kj}x_j \\ s.t. & \sum\limits_{j=1}^{n} a_{ij}x_j \leq b_i + p_i\left(1 - \alpha_i\right), \\ & x_j \geq 0,\ \alpha_i \geq \alpha_i^D,\ 0 \leq \alpha_i \leq 1, \quad i = 1, \ldots, m, j = 1, \ldots, n. \end{array} \tag{2. 7}$$

We consider, α_i^D as the lower bounded for α_i according to the point view of the decision maker. We name the above problem as Multi-Parametric multi-objective Linear Programming problem and also we will show it in the abbreviated form as MPMOLP.
?

2.2. Notation fundamental definitions and main theatrical results

Now, we are going to give the concept of feasible solution to the (MPMOLP) problem and fuzzy-efficient solution to the MOLPFFC problem.

Definition 2.1. Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in (0, 1]^m$, and

$$X_\alpha = \left\{X \in R^n \left| x \geq 0,\ a_i x \leq b_i + p_i\left(1 - \alpha_i\right),\ \alpha_i^D \leq \alpha_i, i = 1, \ldots, m.\right.\right\} \tag{2. 8}$$

A vector $x \in X_\alpha$ is called the α-feasible solution to Problem (2.7).

Example 2.2. Consider the following Multi-parametric linear programming problem

$$\begin{array}{ll} \max & z = x_1 + 3x_2 \\ s.t. & x_1 + x_2 \leq 5 + (1 - \alpha_1), \\ & x_1 \leq 2 + 3(1 - \alpha_2), \\ & x_2 \leq 1 + 2(1 - \alpha_3), \\ & x_1 \geq 0, x_2 \geq 0, 0 \leq \alpha_1 \leq 1, 0 \leq \alpha_2 \leq 1, 0 \leq \alpha_3 \leq 1. \end{array}$$

An α-feasible solution can be obtained by $\alpha_1 = \alpha_2 = \alpha_3 = 0$ as follows: $X_\alpha = (x_1, x_2) = (5, 1)$.

Remark 2.3. If the problem (2.7) is not infeasible, then X_α is not empty.

Definition 2.4. Let $\prec^F$ be a fuzzy extension of binary relation $\leq$ and let $x = (x_1, \ldots, x_n)^T \in R^n$ be an α-feasibility solution to

$$\begin{array}{ll} \max & z_k(x) = \sum\limits_{j=1}^{n} c_{1j} x_j \\ s.t. & a_i x \leq b_i + p_i (1 - \alpha_i), \\ & x \geq 0, \ \alpha_i \geq \alpha_i^D, \ 0 \leq \alpha_i \leq 1, i = 1, \ldots, m. \end{array} \tag{2. 9}$$

where $\alpha = (\alpha_1, \ldots, \alpha_m) \in (0, 1]^m$ and let $Z_k(x)$, be k-th the objective $(k = 1, \ldots, K)$. The vector $x \in R^n$ is an $\bar{\alpha}-$efficient solution to with maximization of the objective function, if there is no any $x' \in X_{\bar{\alpha}}$ such that $Z_K(x) < Z_k(x')$.

Pay attention that any $\alpha-$efficient solution to the (2.9) problem is an αfeasible solution to the (2.9) problem with some extra properties. In the following theorem, we represent the necessary and sufficient condition for an αefficient solution to (2.9).

Example 2.5. Consider the following Multi-parametric linear programming problem which is given in Example 2.2. The optimal solution $X^T = (x_1 = 3, x_2 = 3)$ of the problem (in Example 2.2)

$$\begin{array}{ll} \max & z = x_1 + 3x_2 \\ s.t. & x_1 + x_2 \leq 5 + (1 - \alpha_1), \\ & x_1 \leq 2 + 3(1 - \alpha_2), \\ & x_2 \leq 1 + 2(1 - \alpha_3), \\ & x_1 \geq 0, x_2 \geq 0, 0 \leq \alpha_1 \leq 1, 0 \leq \alpha_2 \leq 1, 0 \leq \alpha_3 \leq 1. \end{array}$$

With $\alpha_1 = \alpha_2 = \alpha_3 = 0$ is an α-efficient solution for the main problem.

Definition 2.6. $X^* = (x^*, \alpha^*)$ is supposed to be a perfect optimal solution for MPMOLP problem if there is $(x^*, \alpha^*) \in X_\alpha$ such that $Z_k(x^*) \geq Z_k(x), k = 1, \ldots, K$ for all $(x, \alpha) \in X_\alpha$.

After all, generally, a complete optimal solution like this, which is at the same time maximizing all the multiple objective is not always available when the objective function differs each other, so rather than a complete optimal solution, a novel solution concept, which is called pareto optimality, is put forward in multi-objective programming.

Example 2.7. Consider the following Multi-objective linear programming problem

$$\begin{array}{ll} \max & z_1 = x_1 + 3x_2 \\ \max & z_2 = 2x_1 + 5x_2 \\ s.t. & x_1 + x_2 \leq 5 + (1-\alpha_1), \\ & x_1 \leq 2 + 3(1-\alpha_2), \\ & x_2 \leq 1 + 2(1-\alpha_3), \\ & x_1 \geq 0, x_2 \geq 0, 0 \leq \alpha_1 \leq 1, 0 \leq \alpha_2 \leq 1, 0 \leq \alpha_3 \leq 1. \end{array}$$

In the assumption $\alpha_1 = \alpha_2 = \alpha_3 = 0$ the α-efficient solution for each objective as $X^T = (x_1 = 3, x_2 = 3)$ is a prefect optimal solution for the above multi-objective linear programming problem.

Definition 2.8. $X^o = (x^o, \alpha^o)$ is mentioned to be a pareto optimal solution to the MP-MOLP problem if there is not exist another $X^o = (x^o, \alpha^o)$ such that $z_k(x^o) \leq z_k(y')$ for all $k = 1, \dots, K$ and $z_j(x^o) < z_j(y')$ for at least one $j \in \{1, \dots, K\}$.

Example 2.9. Consider the following Multi-objective linear programming problem

$$\begin{array}{ll} \max & z_1 = 5x_1 + x_2 \\ \max & z_2 = x_1 + 4x_2 \\ s.t. & x_1 + x_2 \leq 5 + (1-\alpha_1), \\ & x_1 \leq 2 + 3(1-\alpha_2), \\ & x_2 \leq 1 + 2(1-\alpha_3), \\ & x_1 \geq 0, x_2 \geq 0, 0 \leq \alpha_1 \leq 1, 0 \leq \alpha_2 \leq 1, 0 \leq \alpha_3 \leq 1. \end{array}$$

The α-efficient solutions $X^T = (x_1 = 3, x_2 = 3)$ and $X^T = (x_1 = 1, x_2 = 5)$ for the problem in the form of single objective are the pareto-optimal solution is a prefect optimal solution for the above multi-objective linear programming problem.

Theorem 2.10. Let $\bar{\alpha}_k = (\alpha_{k1}, \dots, \alpha_{km}) \in (0,1]^m$ and $x_k{}^* = (x^*_{k1}, \dots, x^*_{kn})^T$, $x^*_{kj} \geq 0$, $j = 1, \dots, n$, be an $\bar{\alpha}_k-$feasible solution to (2.9). Then a vector $x_k{}^* \in R^n$ is an $\bar{\alpha}_k-$efficient solution to Problem (2.9) if and only if x^*_k is an optimal to the following problem

$$\begin{array}{lll} \max & Z_k(x) = \sum\limits_{j=1}^{n} c_{kj} x_j & \\ s.t. & a_i x \leq b_i + p_i\,(1-\alpha_i)\,, & i = 1, \dots, m, \\ & x_j \geq 0,\ \ \alpha_i^D \leq \alpha_i, 0 \leq \alpha_i \leq 1, & j = 1, \dots, n. \end{array} \tag{2. 10}$$

where p_i is the predefined maximum tolerance.

We name the above problem (2.10) as the first Multi-Parametric Linear Programming (MPLP1) problem of k-th objective $(k = 1, \dots, K)$ MPMOLP(2.7) problem.

Proof: The proof is given in details in [3] and so we omit it here.

In Theorem 2.10., we have provided a computational method to solve multi-parametric linear programming problem (2.9). Thus by assigning a specific α_i^D by a decision maker, we may replace the α_i^D in the corresponding constraint of problem (2.9), and solve the resulted problem to compute the $\bar{\alpha}-$efficient solution to the problem (2.9). An $\bar{\alpha}-$efficient solution to (2.9) has two characteristics:

i. The solution has various satisfaction degrees corresponding to each constraint.

ii. The acquired solution is optimal.

This solution permits decision maker to obtain a more flexible and more compatibility by assigning desired preferences, especially, in online optimization in more noticeable.

We will call the Problem (2.9) as Phase I problem. Let $\bar{\alpha}^0 = \left(\alpha_1^0, \ldots, \alpha_m^0\right)$ and (x_k^*, z_k^*) be the optimal solution of Phase I with $\bar{\alpha}^0$ degree of efficiency $\alpha_{ki}^* = \mu_i \left\{H_i\left(x^*, a_i\right) \prec^F 0\right\} \geq \alpha_i^0$. Now, in In Phase 2, we solve the following problem,

$$
\begin{array}{lll}
\max & W_k(\alpha) = \sum\limits_{i=1}^{m} \alpha_i & \\
s.t. & Z_k(x) \geq Z_k(x_k^*), & \\
& a_i x \leq b_i + p_i\left(1 - \alpha_i\right), & i = 1, \ldots, m, \\
& \alpha_{ki}^* \leq \alpha_i \leq 1, & \\
& x = (x_1, \ldots, x_n), & \\
& x_j \geq 0, & j = 1, \ldots, n.
\end{array} \tag{2. 11}
$$

We name the above problem (2.11) as the second Multi-Parametric Linear Programming **(MPLP2)** problem of objective $(k = 1, \ldots, K)$ MPMOLP(2.7) problem.

Let $(x_k^{**}, \alpha_{k1}^{**}, \ldots, \alpha_{km}^{**})$ be an optimal solution to MPLP2 problem. Then, we have the following theorem.

Theorem 2.11. In the optimal solution x_k^{**} to the problem (2.11) (MPLP 2), x_k^{**} is a maximum $\bar{\alpha}-$ efficient solution to the Problem (2.9).

Proof: the proof is given in details in [3] and so we omit it here.

3. Proposed new approach for improving the solving process of MOLPFFC

According to the discussion, which is given in the last section, in phase1, we consider, α_i^D as the lower bounded for α_i based on the point view of the decision maker. Now, solving Problem (2.10) by using Attari and Nasseri method [3] , gives the optimal solution including the optimal values of x and α as x^*, and α^*

$$
\begin{array}{lll}
\max & Z_k(x) = \sum\limits_{j=1}^{n} c_{kj} x_j & \\
s.t. & a_i x \leq b_i + p_i\left(1 - \alpha_i\right), & i = 1, \ldots, m, \\
& x_j \geq 0,\ \ \alpha_i^D \leq \alpha_i, 0 \leq \alpha_i \leq 1, & j = 1, \ldots, n.
\end{array} \tag{3. 12}
$$

Based on the optimal solution of above problem (MPLP1), make MPLP2 problem as follows:

$$
\begin{array}{lll}
\max & W_k(\alpha) = \sum\limits_{i=1}^{m} \alpha_i & \\
s.t. & Z_k(x) \geq Z_k(x_k^*), & \\
& a_i x \leq b_i + p_i\left(1 - \alpha_i\right), & i = 1, \ldots, m, \\
& \alpha_{ki}^* \leq \alpha_i \leq 1, & \\
& x = (x_1, \ldots, x_n),\ x_j \geq 0, & j = 1, \ldots, n.
\end{array} \tag{3. 13}
$$

Attari and Nasseri [3] advised that the optimal solution of kth objective function of MP-MOLP problem (2.7) can be obtained by solving MPLP2 (3.13) problem. In fact, Theorem 2.11. supports their claim.
Now, we are in a place to introduce our new approach for solving of k-th objective function MPMOLP (2.7) problem. Our main contribution of the suggested approach defines an auxiliary problem as follows:

$$\begin{array}{lll} \max & \hat{Z}_k(x,\alpha) = Z_k(x) + \sum\limits_{i=1}^{m} \alpha_i & \\ s.t. & a_i x \leq b_i + p_i(1-\alpha_i), & i = 1,\dots,m, \\ & 0 < \alpha_i \leq 1, \alpha_i^D \leq \alpha_i, & k = 1,\dots,K, \\ & x \geq 0, x = (x_1,\dots,x_n), \alpha = (\alpha_1,\dots,\alpha_m). & \end{array} \tag{3. 14}$$

We name the above problem (3.14) as the Third Multi-Parametric Linear Programming **(MPLP3)** problem of k-th objective $(k = 1,\dots,K)$ MPMOLP(2.7) problem.
Remark 3.1 The suggested approach can solve the original k-th objective of MPMOLP(2.7) problem directly, while the proposed approach proposed by Attari and et al. [3] needs to solve the mentioned problem in two phases.
The following theorem shows that the optimal solution of MPLP 2 is actually a feasible solution for MPLP 3 problem.
Theorem 3.2. The optimal solution of MPLP 2 problem (3.13), which is defined for k-th objective $(k = 1,\dots,K)$ MPMOLP (2.7) problem is a feasible solution for MPLP3 problem that is model (3.14).
Proof: Suppose x^{**} and α^{**} be the optimal value of the decision variables in the optimal solution of MPLP2 problem. Since the optimal solution is indeed a feasible solution, and on the other hand, the set of feasible solution of MPLP 2 problem is a subset of the MPLP 3 problem, and thus the optimal solution of MPLP 2 problem is a feasible solution for MPLP 3 problem.
Remark 3.3. Theorem 3.2. shows that the optimal solution of k-th objective $(k = 1,\dots,K)$ MPMOLP(2.7) problem which is obtained from problem MPLP3, is not worse than the optimal.
solution of k-th objective $(k = 1,\dots,K)$ MPMOLP(2.7) problem , which is obtained from the proposed approach by Attari and et al. [3]
Algorithm1 (Main steps of the improved proposed algorithm for MOLPFFC (2.1) problem) Assumption1: A Multi-Objective Linear Programming with Fuzzy Flexible Constraint (MOLP FFC) problem (2.1) is given to solve. (The parameters of the model including a_i, b_i, p_i and α_i^D for all $i = 1,\dots,m$ are given).
Step 1: Using Equation (2.2) to (2.6), convert problem (2.1) to the form of Problem (2.7).
Step 2: For every objective function in problem (2.7), Make an associated Multi-Parametric Linear programming (MPLP3) Problem based on the model (3.14).
Step 3: For every objective function in problem (2.7) Solve the associated MPLP3 problem and obtain the optimal value of $\hat{x}^* = (\hat{x}_1^*,\dots,\hat{x}_n^*)$ and $\hat{\alpha}^* = (\hat{\alpha}_1^*,\dots,\hat{\alpha}_m^*)$, and finally the optimal value of the objective function

$$\hat{Z}_k^*(x,\alpha) = \hat{Z}(\hat{x}^*,\hat{\alpha}^*), \hat{Z}_k^*(x) = Z_k(\hat{x}^*), \;\; \hat{W}_k^*(\alpha) = \sum_{i=1}^{m} \hat{\alpha}_i^*. \tag{3. 15}$$

Step 4: Determaine the minimum and the maximum values of $W_k^*(\alpha)$ which are obtained in the last step:

$$W_{\max}^*(\alpha) = \max\left\{W_k^*(\alpha)\ |k = 1,\ldots,K\right\}, W_{\min}^*(\alpha) = \min\left\{W_k^*(\alpha)\ |k = 1,\ldots,K\right\}. \tag{3. 16}$$

Step 5: Make and solve the following Goal Programming (GP) model to obtain a pareto-optimal solution for problem (2.7):

$$\begin{array}{ll}
\min & \hat{D} = \sum\limits_{k=1}^{K}(D_{k1} + D_{k2}) \\
s.t. & Z_k(x) + \sum\limits_{i=1}^{m}\alpha_i - D_{k1} + D_{k2} = \hat{Z}_k^*(x,\alpha), \\
& W_{\min}^*(\alpha) \le \sum\limits_{i=1}^{m}\alpha_i \le W_{\max}^*(\alpha), \\
& a_i x \le b_i + p_i(1-\alpha_i), \\
& 0 \le \alpha_i^D \le \alpha_i \le 1, i = 1,\ldots,m, \\
& x_j \ge 0, D_{k1}, D_{k2} \ge 0, k = 1,\ldots,K, \\
& x = (x_1,\ldots,x_n), \alpha = (\alpha_1,\ldots,\alpha_m).
\end{array} \tag{3. 17}$$

Algorithm 1 will be finished after 5 steps.
In the above algorithm, we name problem (3.17) as Multi-Parametric Goal Programming (MPGP) problem.

4. Numerical study on proposed method for solving MOLPFFC problems

In this section, we are going to explain the suggested algorithm which are introduced in the last part by solving a numerical example.
Example 4.1. Consider the following MOLPFFC problem:

$$\begin{array}{ll}
\max & z_1(x) = x_1 + 2x_2 \\
\max & z_2(x) = x_1 - x_2 \\
s.t. & -5x_1 + 2x_2 \precsim_{FF} 7, \\
& -x_1 + 3x_2 \precsim_{FF} 30, \\
& x_1 + x_2 \precsim_{FF} 90, \\
& 5x_1 - x_2 \le_{FF} 390, \\
& -4x_1 + 79x_2 \ge_{FF} 79, \\
& x_1 \ge 0, x_2 \ge 0.
\end{array} \tag{4. 18}$$

where $p_1 = 5$, $p_2 = 10$, $p_3 = 30$, $p_4 = 60$ and $p_5 = 20$ are predefined the maximum tolerance for $b_i, (i = 1,2,3,4,5)$, also $\alpha_1^D = 0$, $\alpha_2^D = 0, \alpha_3^D = 0, \alpha_4^D = 0$ and $\alpha_5^D = 0$ are the lower bound of the satisfaction degree which is desired for the i-th constraint $(i = 1,2,3,4,5)$ by the decision maker.
Solution process: The optimal solution of the above MOLPFFC problem can be obtained by using Algorithm 1
?

?4.1. ?Optimal solution using Algorithm 1

Step 1: Obtain the corresponding MPMOLP for the MOLPFFC problem as follows:

$$\begin{array}{ll} \max & Z_1(x) = x_1 + 2x_2 \\ \max & Z_2(x) = x_1 - x_2 \\ s.t. & -5\,x_1 + 2x_2 \leq 7 + 5(1-\alpha_1), \\ & -x_1 + 3x_2 \leq 30 + 10(1-\alpha_2), \\ & x_1 + x_2 \leq 90 + 30(1-\alpha_3), \\ & 5x_1 - x_2 \leq 390 + 60(1-\alpha_4), \\ & -4x_1 + 79x_2 \geq 79 - 20(1-\alpha_5), \\ & 0 \leq \alpha_1 \leq 1,\ 0 \leq \alpha_2 \leq 1, \\ & 0 \leq \alpha_3 \leq 1, 0 \leq \alpha_4 \leq 1, \\ & 0 \leq \alpha_5 \leq 1, x_1 \geq 0, x_2 \geq 0. \end{array} \tag{4.19}$$

Step 2: Make the following multi-parametric linear programming which is the corresponding to each objective in Problem (4.19) as follows:

Multi-parameter linear programming problem corresponding to the first objective function

$$\begin{array}{ll} \max & \hat{Z}_1(x,\alpha) = x_1 + 2x_2 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \\ s.t. & -5\,x_1 + 2x_2 \leq 7 + 5(1-\alpha_1), \\ & -x_1 + 3x_2 \leq 30 + 10(1-\alpha_2), \\ & x_1 + x_2 \leq 90 + 30(1-\alpha_3), \\ & 5x_1 - x_2 \leq 390 + 60(1-\alpha_4), \\ & -4x_1 + 79x_2 \geq 79 - 20(1-\alpha_5), \\ & 0 \leq \alpha_1 \leq 1,\ 0 \leq \alpha_2 \leq 1,\ 0 \leq \alpha_3 \leq 1, \\ & 0 \leq \alpha_4 \leq 1,\ 0 \leq \alpha_5 \leq 1,\ x_1 \geq 0, x_2 \geq 0, \\ & x = (x_1, x_2), \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5). \end{array} \tag{4.20}$$

Multi-parameter linear programming problem corresponding to the second objective function

$$\begin{array}{ll} \max & \hat{Z}_2(x,\alpha) = x_1 - x_2 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \\ s.t. & -5\,x_1 + 2x_2 \leq 7 + 5(1-\alpha_1), \\ & -x_1 + 3x_2 \leq 30 + 10(1-\alpha_2), \\ & x_1 + x_2 \leq 90 + 30(1-\alpha_3), \\ & 5x_1 - x_2 \leq 390 + 60(1-\alpha_4), \\ & -4x_1 + 79x_2 \geq 79 - 20(1-\alpha_5), \\ & 0 \leq \alpha_1 \leq 1,\ 0 \leq \alpha_2 \leq 1,\ 0 \leq \alpha_3 \leq 1, \\ & 0 \leq \alpha_4 \leq 1,\ 0 \leq \alpha_5 \leq 1,\ x_1 \geq 0, x_2 \geq 0, \\ & x = (x_1, x_2), \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5). \end{array} \tag{4.21}$$

Step 3: The optimal solution to the Problem (4.20) (related to the first objective function) as follows:

$$\begin{array}{l} \hat{x}^* = (\hat{x}_1^*, x_2^*) = (80, 40), \hat{\alpha}^* = (\hat{\alpha}_1^*, \ldots, \hat{\alpha}_5^*) = (1, 0, 0, 1, 1), \hat{Z}_1^*(x,\alpha) = 163, \\ \hat{Z}_1^*(x) = 160,\ \hat{W}_1^*(\alpha) = 3. \end{array} \tag{4.22}$$

The optimal solution to the Problem (4.21) (related to the second objective function) as follows:

$$\hat{x}^* = (\hat{x}_1^*, x_2^*) = (91.12276, 5.613811), \hat{\alpha}^* = (\hat{\alpha}_1^*, \ldots, \hat{\alpha}_5^*) = (1, 1, 0.7754476, 0, 1),$$
$$\hat{Z}_2^*(x,\alpha) = 89.28440\,, \; \hat{Z}_2^*(x) = 85.5089524, \; \hat{W}_2^*(\alpha) = 3.7754476. \tag{4. 23}$$

Step 4: Make and solve the following Goal Programming (GP) model to obtain a pareto-optimal solution for problem (4.19):

$$\begin{array}{ll}
\min & \hat{D} = D_{11} + D_{12} + D_{21} + D_{22} \\
s.t. & x_1 - x_2 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 - D_{11} + D_{12} = 89.28440, \\
& x_1 + 2x_2 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 - D_{21} + D_{22} = 163, \\
& -5\,x_1 + 2x_2 \leq 7 + 5(1 - \alpha_1), \\
& -x_1 + 3x_2 \leq 30 + 10(1 - \alpha_2), \\
& x_1 + x_2 \leq 9\,0\; + 30(1 - \alpha_3), \\
& 5x_1 - x_2 \leq 390 + 60(1 - \alpha_4), \\
& -4x_1 + 79x_2 \geq 79 - 20(1 - \alpha_5), \\
& 0 \leq \alpha_1 \leq 1,\; 0 \leq \alpha_2 \leq 1,\; 0 \leq \alpha_3 \leq 1,\; 0 \leq \alpha_4 \leq 1,\, 0 \leq \alpha_5 \leq 1, \\
& x_1 \geq 0, x_2 \geq 0, D_{11} \geq 0, D_{12} \geq 0, D_{21} \geq 0, D_{22} \geq 0.
\end{array} \tag{4. 24}$$

Finally, the pareto-optimal solution of the MPMOLP problem as follows:

$$\hat{D}^* = 31.28440, \hat{D}_{11}^* = 0, \hat{D}_{12}^* = 16.28440, \hat{D}_{21}^* = 0, \hat{D}_{22}^* = 15, \hat{x}^* = (\hat{x}_1^*, x_2^*)$$
$$= (95, 25), \hat{\alpha}^* = (\hat{\alpha}_1^*, \ldots, \hat{\alpha}_5^*) = (1, 1, 0, 0, 1), \hat{Z}_2^*(x,\alpha) = 73\,, \; \hat{Z}_2^*(x) = 70,$$
$$\hat{Z}_1^*(x,\alpha) = 148\,, \; \hat{Z}_1^*(x) = 145. \tag{4. 25}$$

5. SENSITIVITY ANALYSIS

It is seen that solving MOLPFFC problem has led to solving MPMOLP and MPGP problems. The most valuable features of These problems is that we can obtain different optimal solution for the given problem by considering different value for $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$,in different situation. In the real word, the decision maker faces a different situation in the solution of a problem. For this purpose, we can use the sensitivity analysis of the problem. In this section, we give some of the sensitivity analysis of the obtained model.

5-1 For considering some limitation assumption on the total value of the parameters α_i based on the point of view of the decision maker, we add the following constraint, $(3 \leq \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \leq 3.7754476\,)$ to the problem 24, the Optimal solution is no changed.

5-2 While we would like to improve the optimal value of the aspreation level, we replace the following objective function $(Min\,\hat{D} = -D_{11} + D_{12} - D_{21} + D_{22})$ as the objective function, the optimal solution to problem 26 is no changed.

5-3 For the minimum membership degree of every constraint based on the point of view the decision maker we consider the minimum value 0.4 for all constraint $(\alpha_1 \geq 0.4, \alpha_2 \geq 0.4, \alpha_3 \geq 0.4, \alpha_4 \geq 0.4, \alpha_5 \geq 0.4)$ to the (4.24) problem, the optimal solution is changed as it is.

$$\hat{D}^* = 28.45064, \hat{D}^*_{11} = 0, \hat{D}^*_{12} = 11.45064, \hat{D}^*_{21} = 0, \hat{D}^*_{22} = 17, \hat{x}^* = (\hat{x}^*_1, x^*_2) = (89, 19), \hat{\alpha}^* = (\hat{\alpha}^*_1, \ldots, \hat{\alpha}^*_5) = (1, 1, 0.4, 0.4, 1), \hat{Z}^*_2(x, \alpha) = 73.8\,,\ \hat{Z}^*_2(x) = 70, \hat{Z}^*_1(x, \alpha) = 130.8\,,\ \hat{Z}^*_1(x) = 127. \tag{5. 26}$$

6. Conclusion

In this paper, a Multi-Objective Linear Programming with Flexible Fuzzy Constraint set is considered. We saw that the proposed model is a generalized form of the fuzzy multi-objective linear programming model which is appeared in the literature. The solving process is established based on using $\alpha-$cut concept of fuzzy sets and by considering a minimum desirable level of the feasibility of the solution for the Flexible Fuzzy Constraints a multi- parameters multi objective Linear Programming obtained. Finally, the achieved model by a new algorithm which is constructed based on a Multi-Parametric Linear Goal Programming model, the pareto optimal solution is obtained. We emphasize that the proposed approach is also important for the post optimality discussion.

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