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On a Study of (s,t)-Generalized Pell Sequence and its Matrix Sequence

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Abstract. First of all in this article, we consider (s,t)-type sequences such as (s,t)-Pell sequence $\langle \mathcal{P}_n(s,t)\rangle$, (s,t)-Pell-Lucas sequence $\langle \mathcal{Q}_n(s,t)\rangle$ and (s,t)-Modified Pell sequence $\langle \mathcal{R}_n(s,t)\rangle$. Also we consider (s,t)-type matrix sequences such as (s,t)-Pell matrix sequence $\langle \mathcal{U}_n(s,t)\rangle$, (s,t)-Pell-Lucas matrix sequence $\langle \mathcal{V}_n(s,t)\rangle$ and (s,t)-Modified Pell matrix sequence $\langle \mathcal{W}_n(s,t)\rangle$. Then we introduce (s,t)-generalized Pell sequence $\langle \mathcal{T}_n(s,t)\rangle$ and its matrix sequence named (s,t)-generalized Pell matrix sequence $\langle \mathcal{X}_n(s,t)\rangle$. But the main aim here to present many new results for (s,t)-generalized Pell sequence and (s,t)-generalized Pell matrix sequence and study the relations for (s,t)-generalized Pell sequence and (s,t)-generalized Pell matrix sequence with other (s,t)-type sequences and (s,t)-type matrix sequences. In addition to this we also define matrix sequences to (s,t)-type matrix sequences.

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Key Words: Pell Sequences, (s,t)-Type Sequences and (s,t)-Type Matrix Sequences.

1. INTRODUCTION

Many authors worked on Fibonacci, Lucas, Pell, Jacobsthal sequences etc by various aspects (see[15]-[2]). The well-known Fibonacci sequence is given by the following equation

$$F_n = F_{n-1} + F_{n-2}, \;\; n \geq 2$$

From several past years many authors investigated the generalizations of Fibonacci, Lucas, Pell sequences etc by adding parameters s and t to the recurrence relations of these sequences then named the resulted sequences as (s,t)-type sequences like (s,t)-Fibonacci sequence, (s,t)-Lucas sequence, (s,t)-Pell sequence etc. In addition to this they also defined the matrix sequences for (s,t)-type sequences and called the matrix sequences as (s,t)-type matrix sequences like (s,t)-Fibonacci matrix sequence, (s,t)-Lucas matrix sequence, (s,t)-Pell sequence matrix etc. A matrix sequence is the sequence in which the terms of the sequences are in the form of matrices and the elements of these matrices are the terms of simple (s,t)-type sequences.

In 2008 Civciv and Turkmen in [3] and [4] defined (s,t)-Fibonacci sequence $\langle F_n(s,t)\rangle_{n\in\mathbb{N}}$ and (s,t)-Lucas sequence $\langle L_n(s,t)\rangle_{n\in\mathbb{N}}$ and their matrix sequences (s,t)-Fibonacci matrix sequence $\langle \mathcal{F}_n(s,t)\rangle_{n\in\mathbb{N}}$ and (s,t)-Lucas matrix sequence $\langle \mathcal{L}_n(s,t)\rangle_{n\in\mathbb{N}}$ respectively.

$$F_{n+1}(s,t) = sF_n(s,t) + tF_{n-1}(s,t), \;\; n \geq 1 \;\text{ and }\; F_0(s,t) = 0, \; F_1(s,t) = 1$$

$$\mathcal{F}_{n+1}(s,t) = s\mathcal{F}_n(s,t) + t\mathcal{F}_{n-1}(s,t), \; n \geq 1$$

with $\mathcal{F}_0(s,t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathcal{F}_1(s,t) = \begin{bmatrix} s & 1 \\ t & 0 \end{bmatrix}$ for $s,t \in \mathbb{R}^+$.

and

$$L_{n+1}(s,t) = sL_n(s,t) + tL_{n-1}(s,t), \;\; n \geq 1 \;\text{ and }\; L_0(s,t) = 2, \; L_1(s,t) = s$$

$$\mathcal{L}_{n+1}(s,t) = s\mathcal{L}_n(s,t) + t\mathcal{L}_{n-1}(s,t), \; n \geq 1$$

with $\mathcal{L}_0(s,t) = \begin{bmatrix} s & 2 \\ 2t & -s \end{bmatrix}$, $\mathcal{L}_1(s,t) = \begin{bmatrix} s^2+2t & s \\ st & 2t \end{bmatrix}$ for $s,t \in \mathbb{R}^+$.

In 2011 Yazlik et al.[13] introduced the generalizations of (s,t)-Fibonacci sequence and (s,t)-Fibonacci matrix sequence by defining the sequences $\langle G_n(s,t)\rangle_{n\in\mathbb{N}}$ called the generalized (s,t)-Fibonacci sequence and $\langle \mathfrak{R}_n(s,t)\rangle_{n\in\mathbb{N}}$ called the generalized (s,t)-Fibonacci matrix sequence. In 2015 Gulec and Taskara [6] studied (s,t)-Pell sequence $\langle P_n(s,t)\rangle_{n\in\mathbb{N}}$ and Pell-Lucas sequence $\langle Q_n(s,t)\rangle_{n\in\mathbb{N}}$ and their matrix representations $\langle \mathcal{P}_n(s,t)\rangle_{n\in\mathbb{N}}$ and $\langle \mathcal{Q}_n(s,t)\rangle_{n\in\mathbb{N}}$. Srisawat and Sripad [9] investigated (s,t)-Pell sequence and (s,t)-Pell-Lucas sequence by some matrix methods. Then in 2015 Ipek et al.[7] delineated the another generalized (s,t)-Fibonacci sequence $\langle G_n(s,t)\rangle_{n\in\mathbb{N}}$ and its matrix sequence $\langle \mathfrak{R}_n(s,t)\rangle_{n\in\mathbb{N}}$. Yazlik et al.[14] applied binomial transforms to the (s,t)-Fibonacci matrix sequence and generalized (s,t)-Fibonacci matrix sequence. In 2016 Uygun and Uslu [11] studied the generalizations of (s,t)-Jacobsthal and (s,t)-Jacobsthla-Lucas sequences as well as generalizations of their matrix ones. In [12] Wani et al. studied the matrix sequences for generalized Fibonacci sequence and k-Pell sequence. The main aim of this article to obtain the relations between the (s, t)-generalized Pell sequence and other (s, t)-type sequences also obtain the relations between (s, t)-generalized Pell matrix sequence and other (s, t)-type matrix sequences.

2. Definitions of (s, t)-Type Sequences

For $g, h, k \in \mathbb{Z}^+$ and $s, t \in \mathbb{R}^+$, we have the following definitions

Definition 2.1. [6] *The (s, t)-Pell sequence $\langle \mathcal{P}_n(s, t) \rangle$ is defined by the following recurrence relation*

$$\mathcal{P}_n(s, t) = 2s\mathcal{P}_{n-1}(s, t) + t\mathcal{P}_{n-2}(s, t), \quad n \geq 2 \tag{2. 1}$$

with $\mathcal{P}_0(s, t) = 0$, $\mathcal{P}_1(s, t) = 1$

Definition 2.2. [6] *The (s, t)-Pell-Lucas sequence $\langle \mathcal{Q}_n(s, t) \rangle$ is defined by the following recurrence relation*

$$\mathcal{Q}_n(s, t) = 2s\mathcal{Q}_{n-1}(s, t) + t\mathcal{Q}_{n-2}(s, t), \quad n \geq 2 \tag{2. 2}$$

with $\mathcal{Q}_0(s, t) = 2$, $\mathcal{Q}_1(s, t) = 2s$

Definition 2.3. *The (s, t)-Modified Pell sequence $\langle \mathcal{R}_r(s, t) \rangle$ is given by the following equation*

$$\mathcal{R}_n(s, t) = 2s\mathcal{R}_{n-1}(s, t) + t\mathcal{R}_{n-2}(s, t), \quad n \geq 2 \tag{2. 3}$$

with $\mathcal{R}_{-1} = \dfrac{-1}{t}$, $\mathcal{R}_0(s, t) = \dfrac{1}{s}$, $\mathcal{R}_1(s, t) = 1$

Definition 2.4. *The (s, t)-Generalized Pell sequence $\langle \mathcal{T}_n(s, t) \rangle$ is given by the following equation*

$$\mathcal{T}_n(s, t) = 2s\mathcal{T}_{n-1}(s, t) + t\mathcal{T}_{n-2}(s, t), \quad n \geq 2 \tag{2. 4}$$

with $\mathcal{T}_{-1} = \dfrac{gt + 2ht - k}{t}$, $\mathcal{T}_0(s, t) = 2gs + 2hs + \dfrac{k}{s}$, $\mathcal{T}_1(s, t) = g\big(4s^2 + t\big) + h\big(4s^2 + t\big) + k$

3. Definitions of (s, t)-Type Matrix Sequences

In this section we introduce matrix sequences of above (s, t)-type sequences and are called (s, t)-type matrix sequences and the elements of these (s, t)-type matrix sequences are the terms of simple (s, t)-type sequences. For $g, h, k \in \mathbb{Z}^+$ and $s, t \in \mathbb{R}^+$, the following definitions are hold

Definition 3.1. [6] *The (s, t)-Pell matrix sequence $\langle \mathcal{U}_n(s, t) \rangle$ is recurrently defined by*

$$\mathcal{U}_n(s, t) = 2s\mathcal{U}_{n-1}(s, t) + t\mathcal{U}_{n-2}(s, t), \quad n \geq 2 \tag{3. 1}$$

with $\mathcal{U}_0(s, t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathcal{U}_1(s, t) = \begin{bmatrix} 2s & 1 \\ t & 0 \end{bmatrix}$

Definition 3.2. [6] *The (s,t)-Pell-Lucas matrix sequence $\langle \mathcal{V}_n(s,t)\rangle$ is recurrently defined by*

$$\mathcal{V}_n(s,t) = 2s\mathcal{V}_{N-1}(s,t) + t\mathcal{V}_{n-2}(s,t)\,, \quad n \geq 2 \tag{3. 2}$$

with $\mathcal{V}_0(s,t) = \begin{bmatrix} 2s & 2 \\ 2t & -2s \end{bmatrix}, \mathcal{V}_1(s,t) = \begin{bmatrix} 4s^2+2t & 2s \\ 2st & 2t \end{bmatrix}$

Definition 3.3. *The (s,t)-Modified Pell matrix sequence $\langle \mathcal{W}_n(s,t)\rangle$ is recurrently defined by*

$$\mathcal{W}_n(s,t) = 2s\mathcal{W}_{n-1}(s,t) + t\mathcal{W}_{n-2}(s,t)\,, \quad n \geq 2 \tag{3. 3}$$

with $\mathcal{W}_0(s,t) = \begin{bmatrix} \mathcal{R}_1 & \mathcal{R}_0 \\ t\mathcal{R}_0 & t\mathcal{R}_{-1} \end{bmatrix}, \mathcal{W}_1(s,t) = \begin{bmatrix} \mathcal{R}_2 & \mathcal{R}_1 \\ t\mathcal{R}_1 & t\mathcal{R}_0 \end{bmatrix}$

Definition 3.4. *The (s,t)-generalized Pell matrix sequence $\langle \mathcal{X}_n(s,t)\rangle$ is recurrently defined by*

$$\mathcal{X}_n(s,t) = 2s\mathcal{X}_{n-1}(s,t) + t\mathcal{X}_{n-2}(s,t)\,, \quad n \geq 2 \tag{3. 4}$$

with $\mathcal{X}_0(s,t) = \begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_0 \\ t\mathcal{T}_0 & t\mathcal{T}_{-1} \end{bmatrix}, \mathcal{X}_1(s,t) = \begin{bmatrix} \mathcal{T}_2 & \mathcal{T}_1 \\ t\mathcal{T}_1 & t\mathcal{T}_0 \end{bmatrix}$

4. Matrix Sequences of the (s,t)-Type Matrix Sequences

In this section we introduce matrix sequences for (s,t)-type matrix sequences which are mentioned in the section (3). In other words here we defined sequences whose elements are the terms of (s,t)-type matrix sequences. For $g,h,k \in \mathbb{Z}^+$ and $s,t \in \mathbb{R}^+$, we have

Definition 4.1. *The matrix sequence $\left\langle \widehat{\mathcal{U}}_n(s,t)\right\rangle$ of the (s,t)-Pell matrix sequence $\langle \mathcal{P}_n(s,t)\rangle$ is recurrently defined by*

$$\widehat{\mathcal{U}}_n(s,t) = 2s\widehat{\mathcal{U}}_{n-1}(s,t) + t\widehat{\mathcal{U}}_{n-2}(s,t)\,, \quad n \geq 2 \tag{4. 1}$$

with $\widehat{\mathcal{U}}_0(s,t) = \begin{bmatrix} \mathcal{U}_1 & \mathcal{U}_0 \\ t\mathcal{U}_0 & t\mathcal{U}_{-1} \end{bmatrix}, \widehat{\mathcal{U}}_1(s,t) = \begin{bmatrix} \mathcal{U}_2 & \mathcal{U}_1 \\ t\mathcal{U}_1 & t\mathcal{U}_0 \end{bmatrix}$

Definition 4.2. *The matrix sequence $\left\langle \widehat{\mathcal{V}}_n(s,t)\right\rangle$ of the (s,t)-Pell-Lucas matrix sequence $\langle \mathcal{V}_n(s,t)\rangle$ is given by the following equation*

$$\widehat{\mathcal{V}}_n(s,t) = 2s\widehat{\mathcal{V}}_{n-1}(s,t) + t\widehat{\mathcal{V}}_{n-2}(s,t)\,, \quad n \geq 2 \tag{4. 2}$$

with $\widehat{\mathcal{V}}_0(s,t) = \begin{bmatrix} \mathcal{V}_1 & \mathcal{V}_0 \\ t\mathcal{V}_0 & t\mathcal{V}_{-1} \end{bmatrix}, \widehat{\mathcal{V}}_1(s,t) = \begin{bmatrix} \mathcal{V}_2 & \mathcal{V}_1 \\ t\mathcal{V}_1 & t\mathcal{V}_0 \end{bmatrix}$

Definition 4.3. *The matrix sequence $\left\langle \widehat{\mathcal{W}}_n(s,t)\right\rangle$ of the (s,t)-Modified Pell matrix sequence $\langle \mathcal{W}_n(s,t)\rangle$ is given by the following equation*

$$\widehat{\mathcal{W}}_n(s,t) = 2s\widehat{\mathcal{W}}_{n-1}(s,t) + t\widehat{\mathcal{W}}_{n-2}(s,t)\,, \quad n \geq 2 \tag{4. 3}$$

with $\widehat{\mathcal{W}}_0(s,t)=\begin{bmatrix}\mathcal{W}_1 & \mathcal{W}_0\\ t\mathcal{W}_0 & t\mathcal{W}_{-1}\end{bmatrix}, \widehat{\mathcal{W}}_1(s,t)=\begin{bmatrix}\mathcal{W}_2 & \mathcal{W}_1\\ t\mathcal{W}_1 & t\mathcal{W}_0\end{bmatrix}$

Definition 4.4. *The matrix sequence* $\left\langle\widehat{\mathcal{T}}_n(s,t)\right\rangle$ *of the* (s,t)*-Generalized Pell matrix sequence* $\langle\mathcal{T}_n(s,t)\rangle$ *is delineated as*

$$\widehat{\mathcal{T}}_n(s,t)=2s\widehat{\mathcal{T}}_{n-1}(s,t)+t\widehat{\mathcal{T}}_{n-2}(s,t), \quad n\geq 2 \tag{4. 4}$$

with $\widehat{\mathcal{T}}_0(s,t)=\begin{bmatrix}\mathcal{T}_1 & \mathcal{T}_0\\ t\mathcal{T}_0 & t\mathcal{T}_{-1}\end{bmatrix}, \widehat{\mathcal{T}}_1(s,t)=\begin{bmatrix}\mathcal{T}_2 & \mathcal{T}_1\\ t\mathcal{T}_1 & t\mathcal{T}_0\end{bmatrix}$

In the rest of this paper, for convenience we will use the symbol $\langle\mathcal{Y}_n\rangle$ instead of $\langle\mathcal{Y}_n(s,t)\rangle$ actually $\langle\mathcal{Y}_n(s,t)\rangle$ is any sequence from the equation (2. 1) to (4. 4).

5. Characteristic Equation

Since the recurrence relation is similar for all the sequences that will be the general as well as matrix sequences. So these sequences have the same characteristic equation.

$$z^2-2sz-t=0 \tag{5. 1}$$

Let λ and μ be its roots which are given by

$$\lambda=s+\sqrt{s^2+t} \text{ and } \mu=s-\sqrt{s^2+t} \tag{5. 2}$$

Certainly λ and μ holds the following properties

i. $\lambda+\mu=2s$, $\lambda\mu=-t$ and $\lambda-\mu=2\sqrt{s^2+t}$

ii. $2s\lambda+t=\lambda^2$ and $2s\mu+t=\mu^2$

iii. $s\lambda+t=\dfrac{\lambda(\lambda-\mu)}{2}$ and $s\mu+t=-\dfrac{\mu(\lambda-\mu)}{2}$

iv. $\dfrac{\lambda}{s}-1=\dfrac{\lambda-\mu}{2s}$ and $\dfrac{\mu}{s}-1=-\dfrac{\lambda-\mu}{2s}$

Lemma 5.1. **(General Result)** *For* $\mathcal{U}_1$ *and* $n,p\in\mathbb{Z}_0$*, we have*

$$\begin{bmatrix}\frac{\mathcal{Y}_{n+p+1}}{t}\\ \mathcal{Y}_{n+p}\end{bmatrix}=\mathcal{U}_1^n\begin{bmatrix}\frac{\mathcal{Y}_{p+1}}{t}\\ \mathcal{Y}_p\end{bmatrix} \tag{5. 3}$$

$$\mathcal{Y}_{n+p}=\frac{\mathcal{Y}_{p+1}\lambda^n-\mathcal{Y}_{p+1}\mu^n-\mathcal{Y}_p\mu\lambda^n+\mathcal{Y}_p\lambda\mu^n}{\lambda-\mu} \tag{5. 4}$$

$$=\mathcal{A}\lambda^{n+p}-\mathcal{B}\mu^{n+p}, \quad \mathcal{A}=\frac{\mathcal{Y}_{p+1}-\mathcal{Y}_p\mu}{\lambda^p(\lambda-\mu)}, \quad \mathcal{B}=\frac{\mathcal{Y}_{p+1}-\mathcal{Y}_p\lambda}{\mu^p(\lambda-\mu)} \tag{5. 5}$$

where $\langle\mathcal{Y}_n\rangle$ *is any sequence from the equation* (2. 1) *to* (4. 4).

Proof. The proof of the equation (5. 3) can be easily obtained by using induction on n.

To prove the equation (5. 4) we use the concept of diagonalization of a square matrix.

Since $\mathcal{U}_1$ is a square matrix. Suppose that z be the eigenvalue of $\mathcal{U}_1$ and then by the concept of Cayley Hamilton theorem the characteristic equation of $\mathcal{U}_1$ is defined as

$$det\left(\mathcal{U}_1 - zI\right) = 0 \Rightarrow \begin{vmatrix} 2s-z & 1 \\ t & -z \end{vmatrix} = z^2 - 2sz - t = 0$$

λ and μ be the characteristic roots as well as eigen values of $\mathcal{U}_1$. Now the eigen vectors corresponding to eigen values can be found by evaluating the equation

$$\left(\mathcal{U}_1 - zI\right)\mathcal{C} = 0$$

where $\mathcal{C}$ is the column vector of order 2×1. Then the eigen vector corresponding to λ is delineated by

$$\left(\mathcal{U}_1 - \lambda I\right)\mathcal{C} = 0 \Rightarrow \begin{bmatrix} 2s-\lambda & 1 \\ t & -\lambda \end{bmatrix} \begin{bmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{bmatrix} = \begin{bmatrix} \left(2s-\lambda\right)\mathcal{C}_1 + \mathcal{C}_2 \\ t\mathcal{C}_1 - \lambda\mathcal{C}_2 \end{bmatrix} = 0$$

Consider the equation

$$t\mathcal{C}_1 - \lambda\mathcal{C}_2 = 0 \Rightarrow \mathcal{C}_2 = -\mu\mathcal{C}_1 \tag{5. 6}$$

Put $\mathcal{C}_1 = x$ in the equation (5. 6), we get $\mathcal{C}_2 = -\mu x$. Thus the eigenvectors corresponding to λ are of kind $\begin{bmatrix} x \\ -\mu x \end{bmatrix}$. For $x = 1$, the eigen vector assigning to λ is $\begin{bmatrix} 1 \\ -\mu \end{bmatrix}$. Similarly the eigenvector assigning to μ is $\begin{bmatrix} 1 \\ -\lambda \end{bmatrix}$. Let $\overline{\mathcal{C}} = \begin{bmatrix} 1 & 1 \\ -\mu & -\lambda \end{bmatrix}$ be the matrix of eigenvectors and $\mathcal{D} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ be the diagonal matrix. Then by the process of diagonalization, we have

$$\begin{aligned} \mathcal{U}_1^n &= \overline{\mathcal{C}}\mathcal{D}^n\left(\overline{\mathcal{C}}\right)^{-1} \\ &= \left(\lambda-\mu\right)^{-1} \begin{bmatrix} \lambda^n & \mu^n \\ -\mu\lambda^n & -\lambda\mu^n \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ -\mu & -1 \end{bmatrix} \\ &= \left(\lambda-\mu\right)^{-1} \begin{bmatrix} \lambda^{n+1} - \mu^{n+1} & \lambda^n - \mu^n \\ -\mu\lambda^{n+1} + \lambda\mu^{n+1} & -\mu\lambda^n + \lambda\mu^n \end{bmatrix} \end{aligned}$$

After using the equation (5. 3), we get

$$\begin{bmatrix} \frac{\mathcal{Y}_{n+p+1}}{t} \\ \mathcal{Y}_{n+p} \end{bmatrix} = \left(\lambda-\mu\right)^{-1} \begin{bmatrix} \lambda^{n+1} - \mu^{N+1} & \lambda^n - \mu^N \\ -\mu\lambda^{n+1} + \lambda\mu^{n+1} & -\mu\lambda^n + \lambda\mu^n \end{bmatrix} \begin{bmatrix} \frac{\mathcal{Y}_{p+1}}{t} \\ \mathcal{Y}_p \end{bmatrix}$$

$$\begin{bmatrix} \frac{\mathcal{Y}_{n+p+1}}{t} \\ \mathcal{Y}_{n+p} \end{bmatrix} = \left(\lambda-\mu\right)^{-1} \begin{bmatrix} \mathcal{E} \\ -\frac{\mathcal{Y}_{p+1}\mu\lambda^{n+1}}{t} + \frac{\mathcal{Y}_{p+1}\lambda\mu^{n+1}}{t} - \mathcal{Y}_p\mu\lambda^n + \mathcal{Y}_p\lambda\mu^n \end{bmatrix}$$

where $\mathcal{E}$ is the corresponding term of the matrix. Therefore, we have

$$\mathcal{Y}_{n+p} = (\lambda-\mu)^{-1}\left(-\frac{\mathcal{Y}_{p+1}\mu\lambda^{n+1}}{t}+\frac{\mathcal{Y}_{p+1}\lambda\mu^{n+1}}{t}-\mathcal{Y}_p\mu\lambda^n+\mathcal{Y}_p\lambda\mu^n\right)$$

$$= \frac{\mathcal{Y}_{p+1}\lambda^n-\mathcal{Y}_{p+1}\mu^n-\mathcal{Y}_p\mu\lambda^n+\mathcal{Y}_p\lambda\mu^N}{\lambda-\mu}$$

$$= \frac{\mathcal{Y}_{p+1}-\mathcal{Y}_p\mu}{\lambda-\mu}\lambda^n-\frac{\mathcal{Y}_{p+1}-\mathcal{Y}_p\lambda}{\lambda-\mu}\mu^n$$

$$= \frac{\mathcal{Y}_{p+1}-\mathcal{Y}_p\mu}{\lambda^p(\lambda-\mu)}\lambda^{n+p}-\frac{\mathcal{Y}_{p+1}-\mathcal{Y}_p\lambda}{\mu^n(\lambda-\mu)}\mu^{n+p}$$

This completes the proof of the lemma. □

6. RESULTS FOR (s,t)-TYPE SEQUENCES

In the present section we delineate some new results for (s,t)-type sequences as well some some relations among them.

Theorem 6.1. [9] *For $n\in\mathbb{Z}_0$, the n^{th} terms of (s,t)-Pell sequence $\langle\mathcal{P}_n\rangle$ and (s,t)-Pell-Lucas sequence $\langle\mathcal{Q}_n\rangle$ are given respectively by*

$$\mathcal{P}_n=\frac{\lambda^n-\mu^n}{\lambda-\mu} \tag{6. 1}$$

$$\mathcal{Q}_n=\lambda^n+\mu^n \tag{6. 2}$$

Theorem 6.2. *If $n,p\geq 1$, the following results hold*

$$\mathcal{P}_n\mathcal{Q}_p+t\mathcal{P}_{n-1}\mathcal{Q}_{p-1}=\mathcal{Q}_{n+p-1} \tag{6. 3}$$

$$\mathcal{P}_n\mathcal{R}_p+t\mathcal{P}_{n-1}\mathcal{R}_{p-1}=\mathcal{R}_{n+p-1} \tag{6. 4}$$

Theorem 6.3. *For $n\geq 0$, we have*

$$\mathcal{R}_n=\frac{\lambda^n+\mu^n}{\lambda+\mu} \tag{6. 5}$$

Theorem 6.4. *For $n,p\geq 0$, we get*

$$\mathcal{T}_{n+p}=\mathcal{A}_1\lambda^{n+p}-\mathcal{B}_1\mu^{n+p},\ \ \mathcal{A}_1=\frac{\mathcal{T}_{p+1}-\mathcal{T}_p\mu}{\lambda^p(\lambda-\mu)},\ \ \mathcal{B}_1=\frac{\mathcal{T}_{p+1}-\mathcal{T}_p\lambda}{\mu^p(\lambda-\mu)} \tag{6. 6}$$

$$=\mathcal{T}_p\mathcal{P}_{n+1}+t\mathcal{T}_{p-1}\mathcal{P}_n \tag{6. 7}$$

$$=\frac{\mathcal{T}_p}{2}\mathcal{Q}_n+(\mathcal{T}_{p+1}-s\mathcal{T}_p)\mathcal{P}_n \tag{6. 8}$$

$$=s\mathcal{T}_p\mathcal{R}_n+(\mathcal{T}_{p+1}-s\mathcal{T}_p)\mathcal{P}_n \tag{6. 9}$$

Proof. First of all replace $\mathcal{Y}$ by $\mathcal{T}$ in the equations (5. 4) and (5. 5). Then the proof of the equation (6. 6) is obvious. Also

$$\mathcal{T}_{n+p}=\frac{\mathcal{T}_{p+1}\lambda^n-\mathcal{T}_{p+1}\mu^n-\mathcal{T}_p\mu\lambda^n+\mathcal{T}_p\lambda\mu^n}{\lambda-\mu}$$

$$= \frac{1}{\lambda-\mu}\left\{\left[\left(\mathcal{T}_{p+1}-\mathcal{T}_p\left(2s-\lambda\right)\lambda^n\right]-\left[\mathcal{T}_{p+1}-\mathcal{T}_p\left(2s-\mu\right)\mu^n\right]\right\}$$
$$= \frac{1}{\lambda-\mu}\left[\mathcal{T}_p\lambda^{n+1}+\left(\mathcal{T}_{p+1}-2s\mathcal{T}_p\right)\lambda^n-\mathcal{T}_p\mu^{n+1}-\left(\mathcal{T}_{p+1}-2s\mathcal{T}_p\right)\mu^n\right]$$
$$= \mathcal{T}_p\frac{\lambda^{n+1}-\mu^{n+1}}{\lambda-\mu}+\left(\mathcal{T}_{p+1}-2s\mathcal{T}_p\right)\frac{\lambda^n-\mu^n}{\lambda-\mu}$$
$$= \mathcal{T}_p\mathcal{P}_{n+1}+t\mathcal{T}_{p-1}\mathcal{P}_n$$

Again

$$\mathcal{T}_{n+p} = \frac{\mathcal{T}_{p+1}\lambda^n-\mathcal{T}_{p+1}\mu^n-\mathcal{T}_p\mu\lambda^n+\mathcal{T}_p\lambda\mu^n}{\lambda-\mu}$$
$$= \frac{1}{\lambda-\mu}\left(\mathcal{T}_{p+1}\lambda^n-\frac{\mathcal{T}_p\mu\lambda^n}{2}-\mathcal{T}_{p+1}\mu^n+\frac{\mathcal{T}_p\lambda\mu^n}{2}-\frac{\mathcal{T}_p\mu\lambda^n}{2}+\frac{\mathcal{T}_p\lambda\mu^n}{2}\right)$$
$$= \frac{1}{\lambda-\mu}\left[\frac{2\mathcal{T}_{p+1}-\mathcal{T}_p\left(2s-\lambda\right)}{2}\lambda^n-\frac{2\mathcal{T}_{p+1}-\mathcal{T}_p\left(2s-\mu\right)}{2}\mu^n-\frac{\mathcal{T}_p\mu\lambda^n}{2}+\frac{\mathcal{T}_p\lambda\mu^n}{2}\right]$$
$$= \frac{1}{\lambda-\mu}\left[\frac{\mathcal{T}_p\lambda^{n+1}}{2}-\frac{\mathcal{T}_p\mu\lambda^n}{2}+\frac{\mathcal{T}_p\lambda\mu^n}{2}-\frac{\mathcal{T}_p\mu\lambda^{n+1}}{2}+\left(\mathcal{T}_{p+1}-s\mathcal{T}_p\right)\lambda^n\right.$$
$$\left.-\left(\mathcal{T}_{p+1}-s\mathcal{T}_p\right)\mu^n\right]$$
$$= \frac{1}{\lambda-\mu}\left[\frac{\mathcal{T}_p}{2}\lambda^n\left(\lambda-\mu\right)+\frac{\mathcal{T}_p}{2}\mu^n\left(\lambda-\mu\right)+\left(\mathcal{T}_{p+1}-s\mathcal{T}_p\right)\left(\lambda^n-\mu^n\right)\right]$$
$$= \frac{\mathcal{T}_p}{2}\mathcal{Q}_n+\left(\mathcal{T}_{p+1}-s\mathcal{T}_p\right)\mathcal{P}_n \qquad \text{By the Eqns. (6. 1) and (6. 2)}$$

Since $\mathcal{R}_n = \dfrac{\mathcal{Q}_n}{2s}$, we get

$$\mathcal{T}_{n+p} = s\mathcal{T}_p\mathcal{R}_n+\left(\mathcal{T}_{p+1}-s\mathcal{T}_p\right)\mathcal{P}_n$$

Hence the proof of the theorem. □

Corollary 6.5. *If $n \geq 0$, the n^{th} term of (s,t)-generalized Pell sequence $\langle\mathcal{T}_n(s,t)\rangle$ is given by the following equation*

$$\left.\begin{aligned}\mathcal{T}_n &= \mathcal{A}_2\lambda^n-\mathcal{B}_2\mu^n,\ \ \mathcal{A}_2=\frac{\mathcal{T}_1-\mathcal{T}_0\mu}{\lambda-\mu},\ \ \mathcal{B}_2=\frac{\mathcal{T}_1-\mathcal{T}_0\lambda}{\lambda-\mu}\\ &= g\frac{\lambda^{n+2}-\mu^{n+2}}{\lambda-\mu}+h\left(\lambda^{n+1}+\mu^{n+1}\right)+k\frac{\lambda^n-\mu^n}{\lambda+\mu}\\ &= g\mathcal{P}_{n+2}+h\mathcal{Q}_{n+1}+k\mathcal{R}_n\end{aligned}\right\} \qquad (6.\ 10)$$

Proof. if we put $p = 0$ in the equation (6. 6), we get

$$\mathcal{T}_n = \mathcal{A}_2\lambda^n-\mathcal{B}_2\mu^n,\ \ \mathcal{A}_2=\frac{\mathcal{T}_1-\mathcal{T}_0\mu}{\lambda-\mu},\ \ \mathcal{B}_2=\frac{\mathcal{T}_1-\mathcal{T}_0\lambda}{\lambda-\mu}$$

Now we use the values of $\mathcal{T}_0$ and $\mathcal{T}_1$ to find the required result. Therefore, we have

$$\begin{aligned}
\mathcal{A}_2\lambda^n &= \frac{\mathcal{T}_1 - \mathcal{T}_0\mu}{\lambda-\mu}\lambda^n \\
&= \big(\lambda-\mu\big)^{-1}\Big(4gs^2+4hs^2+gt+2ht+k-2gs\mu-2hs\mu-\frac{k}{s}\mu\Big)\lambda^n \\
&= \big(\lambda-\mu\big)^{-1}\Big[4gs^2+4hs^2+gt+2ht+k-2gs\big(2s-\lambda\big)-2hs\big(2s-\lambda\big) \\
&\qquad -\frac{k}{s}\big(2s-\lambda\big)\Big]\lambda^n \\
&= \big(\lambda-\mu\big)^{-1}\Big[g\big(2s\lambda+t\big)+2h\big(s\lambda+t\big)+k\Big(\frac{\lambda}{s}-1\Big)\Big]\lambda^n \\
&= \big(\lambda-\mu\big)^{-1}\left[g\lambda^{n+2}+h\big(\lambda-\mu\big)\lambda^{n+1}+k\Big(\frac{\lambda-\mu}{2s}\Big)\lambda^n\right] \qquad \text{By (iii.) and (iv.)}
\end{aligned}$$

Similarly

$$\mathcal{B}_2\mu^n = \big(\lambda-\mu\big)^{-1}\left[g\mu^{n+2}-h\big(\lambda-\mu\big)\mu^{n+1}-k\Big(\frac{\lambda-\mu}{2s}\Big)\mu^n\right]$$

Therefore, we have

$$\begin{aligned}
\mathcal{T}_n &= \frac{1}{\lambda-\mu}\left[g\lambda^{n+2}-g\mu^{n+2}+h\big(\lambda-\mu\big)\lambda^{n+1}+h\big(\lambda-\mu\big)\mu^{n+1}+k\Big(\frac{\lambda-\mu}{2s}\Big)\lambda^n\right. \\
&\qquad \left.+\,k\Big(\frac{\lambda-\mu}{2s}\Big)\mu^n\right] \\
&= g\frac{\lambda^{n+2}-\mu^{n+2}}{\lambda-\mu}+h\Big(\lambda^{n+1}+\mu^{n+1}\Big)+k\frac{\lambda^n-\mu^n}{\lambda+\mu} \\
&= g\mathcal{P}_{n+2}+h\mathcal{Q}_{n+1}+k\mathcal{R}_n \qquad \text{By the Eqns. (6. 1), (6. 2) and(6. 5)}
\end{aligned}$$

This completes the proof of corollary. □

Lemma 6.6. *Let $0 \le p \le n$, the following result holds*

$$\mathcal{P}_{n-p} = \frac{\widehat{\mathcal{Y}}_p\widehat{\mathcal{Y}}_{n+1}-\widehat{\mathcal{Y}}_{p+1}\widehat{\mathcal{Y}}_n}{\widehat{\mathcal{Y}}_p\widehat{\mathcal{Y}}_{p+2}-\widehat{\mathcal{Y}}_{p+1}^2} \tag{6. 11}$$

where $\left\langle\widehat{\mathcal{Y}}_n\right\rangle$ is any sequence from the equation (2. 2) *to* (2. 4).

Proof. From the equation (5. 5), we get

$$\begin{aligned}
&\frac{\widehat{\mathcal{Y}}_p\widehat{\mathcal{Y}}_{n+1}-\widehat{\mathcal{Y}}_{p+1}\widehat{\mathcal{Y}}_n}{\widehat{\mathcal{Y}}_p\widehat{\mathcal{Y}}_{p+2}-\widehat{\mathcal{Y}}_{p+1}^2} \\
&= \frac{\big(\mathcal{A}\lambda^p-\mathcal{B}\mu^p\big)\big(\mathcal{A}\lambda^{n+1}-\mathcal{B}\mu^{N+1}\big)-\big(\mathcal{A}\lambda^{p+1}-\mathcal{B}\mu^{p+1}\big)\big(\mathcal{A}\lambda^n-\mathcal{B}\mu^n\big)}{\mathcal{Y}_p\mathcal{Y}_{p+2}-\mathcal{Y}_{p+1}^2}
\end{aligned}$$

$$= \frac{-\mathcal{AB}\Big(\lambda^{n+1}\mu^{p} - \lambda^{n}\mu^{p+1} - \lambda^{p+1}\mu^{n} + \lambda^{p}\mu^{n+1}\Big)}{\mathcal{Y}_p\mathcal{Y}_{p+2} - \mathcal{Y}_{p+1}^2}$$
$$= \frac{-\mathcal{AB}\big(\lambda-\mu\big)\big(\lambda\mu\big)^p\big(\lambda^{n-p} - \mu^{n-p}\big)}{\mathcal{Y}_p\mathcal{Y}_{p+2} - \mathcal{Y}_{p+1}^2}$$

Since $\mathcal{AB} = -\dfrac{\widehat{\mathcal{Y}}_p\widehat{\mathcal{Y}}_{p+2} - \widehat{\mathcal{Y}}_{p+1}^2}{\big(\lambda\mu\big)^p\big(\lambda-\mu\big)^2}$. Then we get

$$\frac{\widehat{\mathcal{Y}}_p\widehat{\mathcal{Y}}_{n+1} - \widehat{\mathcal{Y}}_{p+1}\widehat{\mathcal{Y}}_n}{\widehat{\mathcal{Y}}_p\widehat{\mathcal{Y}}_{p+2} - \widehat{\mathcal{Y}}_{p+1}^2} = \mathcal{P}_{n-p}$$

Hence the result. □

Theorem 6.7. *For $n \in \mathbb{Z}_0$, we get*

$$\mathcal{P}_n = \frac{\mathcal{Q}_{n+1} - s\mathcal{Q}_n}{2\big(s^2+t\big)} = \frac{s\big(\mathcal{R}_{n+1} - s\mathcal{R}_n\big)}{s^2+t} = \frac{\mathcal{T}_0\mathcal{T}_{n+1} - \mathcal{T}_1\mathcal{T}_n}{\mathcal{T}_0\mathcal{T}_2 - \mathcal{T}_1^2} \tag{6. 12}$$

Proof. The proof can be clearly seen by using the equation (6. 11). □

7. RESULTS FOR (s,t)-TYPE MATRIX SEQUENCES

Here we introduce some new results for (s,t)-type matrix sequences. In addition to this some relations among (s,t)-type matrix sequences are obtained.

Theorem 7.1. [6] *The n^{th} terms of the $\big(s,t\big)$-Pell matrix sequence $\langle\mathcal{U}_n\rangle$ and $\big(s,t\big)$-Pell-Lucas matrix sequence $\langle\mathcal{V}_n\rangle$ are given by*

$$\mathcal{U}_n = \begin{bmatrix} \mathcal{P}_{n+1} & \mathcal{P}_n \\ t\mathcal{P}_n & t\mathcal{P}_{n-1} \end{bmatrix}, \quad n \geq 1 \tag{7. 1}$$

$$\mathcal{V}_n = \begin{bmatrix} \mathcal{Q}_{n+1} & \mathcal{Q}_n \\ t\mathcal{Q}_n & t\mathcal{Q}_{n-1} \end{bmatrix}, \quad n \geq 1 \tag{7. 2}$$

Theorem 7.2. *For $n \in \mathbb{N}$, the following relation holds*

$$\mathcal{U}_n = \begin{bmatrix} \mathcal{Q}_{n+2} - s\mathcal{Q}_{n+1} & \mathcal{Q}_{n+1} - s\mathcal{Q}_n \\ t\big(\mathcal{Q}_{n+1} - s\mathcal{Q}_n\big) & t\big(\mathcal{Q}_n - s\mathcal{Q}_{n-1}\big) \end{bmatrix}$$

$$= \frac{s}{s^2+t} \begin{bmatrix} \mathcal{R}_{n+2} - s\mathcal{R}_{n+1} & \mathcal{R}_{n+1} - s\mathcal{R}_n \\ t\big(\mathcal{R}_{n+1} - s\mathcal{R}_n\big) & t\big(\mathcal{R}_n - s\mathcal{R}_{n-1}\big) \end{bmatrix} \tag{7. 3}$$

$$= \frac{1}{\mathcal{T}_0\mathcal{T}_2 - \mathcal{T}_1^2} \begin{bmatrix} \mathcal{T}_0\mathcal{T}_{n+2} - \mathcal{T}_1\mathcal{T}_{n+1} & \mathcal{T}_0\mathcal{T}_{n+1} - \mathcal{T}_1\mathcal{T}_n \\ t\big(\mathcal{T}_0\mathcal{T}_{n+1} - \mathcal{T}_1\mathcal{T}_n\big) & t\big(\mathcal{T}_0\mathcal{T}_n - \mathcal{T}_1\mathcal{T}_{n-1}\big) \end{bmatrix}$$

Proof. The proof of this theorem can be obtained by using the equations (6. 12) and (7. 1). □

Corollary 7.3. *If $n \geq 1$, we get*

$$\mathcal{Q}_n = \frac{\mathcal{T}_0\big(\mathcal{T}_{n+2} + t\mathcal{T}_n\big) - \mathcal{T}_1\big(\mathcal{T}_{n+1} + t\mathcal{T}_{n-1}\big)}{\mathcal{T}_0\mathcal{T}_2 - \mathcal{T}_1^2}$$

Theorem 7.4. *For $n \in \mathbb{N}$, the n^{th} terms of the sequences (s,t)-Modified Pell matrix sequence $\langle \mathcal{W}_n \rangle$ and (s,t)-generalized Pell matrix sequence $\langle \mathcal{X}_n \rangle$ are delineated by*

$$\mathcal{W}_n = \begin{bmatrix} \mathcal{R}_{n+1} & \mathcal{R}_n \\ t\mathcal{R}_n & t\mathcal{R}_{n-1} \end{bmatrix} \tag{7. 4}$$

$$\mathcal{X}_n = \begin{bmatrix} \mathcal{T}_{n+1} & \mathcal{T}_n \\ t\mathcal{T}_n & t\mathcal{T}_{n-1} \end{bmatrix} \tag{7. 5}$$

Proof. By using the equation (5. 4), we get

$$\mathcal{W}_n = \frac{\mathcal{W}_1\lambda^n - \mathcal{W}_1\mu^n - \mathcal{W}_0\mu\lambda^n + \mathcal{W}_0\lambda\mu^n}{\lambda - \mu}$$

$$= \frac{1}{\lambda-\mu}\Bigg[\begin{pmatrix} \mathcal{R}_2 & \mathcal{R}_1 \\ t\mathcal{R}_1 & \mathcal{R}_0 \end{pmatrix}\lambda^n - \begin{pmatrix} \mathcal{R}_2 & \mathcal{R}_1 \\ t\mathcal{R}_1 & \mathcal{R}_0 \end{pmatrix}\mu^n - \begin{pmatrix} \mathcal{R}_1 & \mathcal{R}_0 \\ t\mathcal{R}_0 & \mathcal{R}_{-1} \end{pmatrix}\mu\lambda^n + \begin{pmatrix} \mathcal{R}_1 & \mathcal{R}_0 \\ t\mathcal{R}_0 & \mathcal{R}_{-1} \end{pmatrix}\lambda\mu^n\Bigg]$$

$$= \frac{1}{\lambda-\mu}\begin{bmatrix} \mathcal{R}_2\lambda^n - \mathcal{R}_2\mu^n - \mathcal{R}_1\mu\lambda^n + \mathcal{R}_2\lambda\mu^n & \mathcal{R}_1\lambda^n - \mathcal{R}_1\mu^n - \mathcal{R}_0\mu\lambda^n + \mathcal{R}_0\lambda\mu^n \\ t\Big(\mathcal{R}_1\lambda^n - \mathcal{R}_1\mu^n - \mathcal{R}_0\mu\lambda^n + \mathcal{R}_0\lambda\mu^n\Big) & t\Big(\mathcal{R}_0\lambda^n - \mathcal{R}_0\mu^n - \mathcal{R}_{-1}\mu\lambda^n + \mathcal{R}_{-1}\lambda\mu^n\Big) \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{R}_{n+1} & \mathcal{R}_n \\ t\mathcal{R}_n & t\mathcal{R}_{n-1} \end{bmatrix} \qquad \text{By the Eqn. (5. 4)}$$

Similarly

$$\mathcal{X}_n = \begin{bmatrix} \mathcal{T}_{n+1} & \mathcal{T}_n \\ t\mathcal{T}_n & t\mathcal{T}_{n-1} \end{bmatrix}$$

□

Theorem 7.5. *Let $n \geq 0$, the following result holds for the matrix sequences $\langle \mathcal{U}_n \rangle$, $\langle \mathcal{V}_n \rangle$, $\langle \mathcal{W}_n \rangle$ and $\langle \mathcal{X}_n \rangle$, we have*

$$\mathcal{X}_n = g\mathcal{U}_{n+2} + h\mathcal{V}_{n+1} + k\mathcal{W}_n$$

Proof. From the equations (7. 1), (7. 2) and (7. 4), we get

$$\begin{aligned} g\mathcal{U}_{n+2} + h\mathcal{V}_{n+1} + k\mathcal{W}_n &= g\begin{bmatrix} \mathcal{P}_{n+3} & \mathcal{P}_{n+2} \\ t\mathcal{P}_{n+2} & t\mathcal{P}_n \end{bmatrix} + h\begin{bmatrix} \mathcal{Q}_{n+2} & \mathcal{Q}_{n+1} \\ t\mathcal{Q}_{n+1} & t\mathcal{Q}_n \end{bmatrix} \\ &\quad + k\begin{bmatrix} \mathcal{R}_{n+1} & \mathcal{R}_n \\ t\mathcal{R}_n & t\mathcal{R}_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{T}_{n+1} & \mathcal{T}_n \\ t\mathcal{T}_n & t\mathcal{T}_{n-1} \end{bmatrix} && \text{By the Eqn. (6. 10)} \\ &= \mathcal{X}_n && \text{By the Eqn. (7. 5)} \end{aligned}$$

as required. □

Theorem 7.6. *For $n \geq 0$, we get*

$$\begin{aligned} \mathcal{X}_n &= \mathcal{T}_0\mathcal{U}_{n+1} + t\mathcal{T}_{-1}\mathcal{U}_n \\ &= \mathcal{X}_0\mathcal{P}_{n+1} + t\mathcal{X}_{-1}\mathcal{P}_n \end{aligned} \tag{7. 6}$$

$$\begin{aligned} \mathcal{X}_n &= \frac{\mathcal{T}_0}{2}\mathcal{V}_n + \big(\mathcal{T}_1 - s\mathcal{T}_0\big)\mathcal{U}_n \\ &= \frac{\mathcal{X}_0}{2}\mathcal{Q}_n + \big(\mathcal{X}_1 - s\mathcal{X}_0\big)\mathcal{P}_n \end{aligned} \tag{7. 7}$$

$$\begin{aligned} \mathcal{X}_n &= s\mathcal{T}_0\mathcal{W}_n + \big(\mathcal{T}_1 - s\mathcal{T}_0\big)\mathcal{U}_n \\ &= s\mathcal{X}_0\mathcal{R}_n + \big(\mathcal{X}_1 - s\mathcal{X}_0\big)\mathcal{P}_n \end{aligned} \tag{7. 8}$$

Proof. By using the equation (7. 1), we obtain

$$\begin{aligned} \mathcal{T}_0\mathcal{U}_{n+1} + t\mathcal{T}_{-1}\mathcal{U}_n &= \mathcal{T}_0\begin{bmatrix} \mathcal{P}_{n+2} & \mathcal{P}_{n+1} \\ t\mathcal{P}_{n+1} & t\mathcal{P}_n \end{bmatrix} + t\mathcal{T}_{-1}\begin{bmatrix} \mathcal{P}_{n+1} & \mathcal{P}_n \\ t\mathcal{P}_n & t\mathcal{P}_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{T}_0\mathcal{P}_{n+2} + t\mathcal{T}_{-1}\mathcal{P}_{n+1} & \mathcal{T}_0\mathcal{P}_{n+1} + t\mathcal{T}_{-1}\mathcal{P}_n \\ t\big(\mathcal{T}_0\mathcal{P}_{n+1} + t\mathcal{T}_{-1}\mathcal{P}_n\big) & t\big(\mathcal{T}_0\mathcal{P}_n + t\mathcal{T}_{-1}\mathcal{P}_{n-1}\big) \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{T}_{n+1} & \mathcal{T}_n \\ t\mathcal{T}_n & t\mathcal{T}_{n-1} \end{bmatrix} && \text{By the Eqn. (6. 7)} \\ &= \mathcal{X}_n \end{aligned}$$

Again

$$\begin{aligned}\mathcal{X}_0\mathcal{P}_{n+1}+t\mathcal{X}_{-1}\mathcal{P}_n &= \begin{bmatrix}\mathcal{T}_1 & \mathcal{T}_0\\ t\mathcal{T}_0 & t\mathcal{T}_{-1}\end{bmatrix}\mathcal{P}_{n+1}+\begin{bmatrix}t\mathcal{T}_0 & t\mathcal{T}_{-1}\\ t^2\mathcal{T}_{-1} & t^2\mathcal{T}_{-2}\end{bmatrix}\mathcal{P}_n\\ &= \begin{bmatrix}\mathcal{T}_1\mathcal{P}_{n+1}+t\mathcal{T}_0\mathcal{P}_n & \mathcal{T}_0\mathcal{P}_{n+1}+t\mathcal{T}_{-1}\mathcal{P}_n\\ t\big(\mathcal{T}_0\mathcal{P}_{n+1}+t\mathcal{T}_{-1}\mathcal{P}_n\big) & t\big(\mathcal{T}_{-1}\mathcal{P}_{n+1}+t\mathcal{T}_{-2}\mathcal{P}_n\big)\end{bmatrix}\\ &= \begin{bmatrix}\mathcal{T}_{n+1} & \mathcal{T}_n\\ t\mathcal{T}_n & t\mathcal{T}_{n-1}\end{bmatrix} \qquad \text{By the Eqn. (6. 7)}\\ &= \mathcal{X}_n\end{aligned}$$

Hence the proof of the equation (7. 6).
The proof of the equations (7. 7) and (7. 8) is same as the proof of the equation (7. 6). □

Theorem 7.7. *For $n \geq 0$, the following results are obvious*

$$\begin{aligned}\mathcal{U}_n &= \frac{\mathcal{V}_{n+1}-s\mathcal{V}_n}{2\left(s^2+t\right)}\\ &= \frac{s\big(\mathcal{W}_{n+1}-s\mathcal{W}_n\big)}{s^2+t} \qquad (7.\ 9)\\ &= \frac{\mathcal{T}_0\mathcal{X}_{n+1}-\mathcal{T}_1\mathcal{X}_n}{\mathcal{T}_0\mathcal{T}_2-\mathcal{T}_1^2}\end{aligned}$$

Proof. The proof can be clearly seen by using the equations (6. 12), (7. 2), (7. 4) and (7. 5). □

Theorem 7.8. **(Commutative property)** *If $n,p \in \mathbb{Z}_0$, we get*

$$\mathcal{U}_n\mathcal{W}_p = \mathcal{W}_p\mathcal{U}_n = \mathcal{W}_{n+p} \qquad (7.\ 10)$$

$$\mathcal{X}_p\mathcal{U}_n = \mathcal{U}_n\mathcal{X}_p = \mathcal{X}_{n+p} \qquad (7.\ 11)$$

Proof.

$$\begin{aligned}\mathcal{U}_n\mathcal{W}_p &= \begin{bmatrix}\mathcal{R}_{p+1}\mathcal{P}_{n+1}+t\mathcal{R}_p\mathcal{P}_n & \mathcal{R}_p\mathcal{P}_{n+1}+t\mathcal{R}_{p-1}\mathcal{P}_n\\ t\big(\mathcal{R}_p\mathcal{P}_{n+1}+t\mathcal{R}_{p-1}\mathcal{P}_n\big) & t\big(\mathcal{R}_p\mathcal{P}_n+t\mathcal{R}_{p-1}\mathcal{P}_{n-1}\big)\end{bmatrix}\\ &= \begin{bmatrix}\mathcal{R}_{n+p+1} & \mathcal{R}_{n+p}\\ t\mathcal{R}_{n+p} & t\mathcal{R}_{n+p-1}\end{bmatrix}\\ &= \mathcal{W}_{n+p}\end{aligned}$$

Similarly

$$\mathcal{W}_p\mathcal{U}_n = \mathcal{W}_{n+p}$$

Thus, we get

$$\mathcal{U}_n\mathcal{W}_p = \mathcal{W}_p\mathcal{U}_n = \mathcal{W}_{n+p}$$

The proof of equation (7. 11) is similar to the proof of equation (7. 10). □

Lemma 7.9.

$$\mathcal{U}_n^p = \mathcal{U}_{np}, \quad n, p \geq 1 \tag{7. 12}$$

Proof. To prove the result we use induction on p. Let $p = 1$, we have

$$\mathcal{U}_n = \mathcal{U}_n$$

Suppose that the result is true for all values i less than or equal to p. Then, we have

$$\begin{aligned} \mathcal{U}_n^{p+1} &= \mathcal{U}_n^p \mathcal{U}_n \\ &= \mathcal{U}_{np} \mathcal{U}_n \\ &= \mathcal{U}_{np+n} \\ &= \mathcal{U}_{n(p+1)} \end{aligned}$$

Since $\mathcal{U}_{np}\mathcal{U}_n = \mathcal{U}_{np+n}$ [6], we get

$$\begin{aligned} \mathcal{U}_n^{p+1} &= \mathcal{U}_{np+n} \\ &= \mathcal{U}_{n(p+1)} \end{aligned}$$

as required. □

Theorem 7.10. *For $n, p \geq 0$ and $r \geq 1$, we have*

$$\mathcal{W}_{n+r}^p = (\mathcal{W}_r^p)\mathcal{U}_{np} \tag{7. 13}$$

$$\mathcal{X}_{n+r}^p = (\mathcal{X}_r^p)\mathcal{U}_{np} \tag{7. 14}$$

Proof. Since

$$\begin{aligned} (\mathcal{W}_r^p)\mathcal{U}_{np} &= \frac{\mathcal{U}_n^p(\mathcal{W}_r^p)\mathcal{U}_{np}}{\mathcal{U}_n^p} \\ &= \frac{(\mathcal{U}_n \mathcal{W}_r)^p\, \mathcal{U}_{np}}{\mathcal{U}_n^p} \\ &= \mathcal{W}_{n+r}^p \qquad \text{By the Eqns. (7. 10) and (7. 12).} \end{aligned}$$

Similarly □

$$\mathcal{X}_{n+r}^p = (\mathcal{X}_r^p)\mathcal{U}_{np}$$

8. Results for Matrix Sequences of the (s, t)-Type Matrix Sequences

Theorem 8.1. *For $n \geq 0$, the n^{th} terms of all matrix sequences of (s, t)-type matrix sequences are given by the following equations*

$$\widehat{\mathcal{U}}_n = \begin{bmatrix} \mathcal{U}_{n+1} & \mathcal{U}_n \\ t\mathcal{U}_n & t\mathcal{U}_{n-1} \end{bmatrix} \tag{8. 1}$$

$$\widehat{\mathcal{V}}_n = \begin{bmatrix} \mathcal{V}_{n+1} & \mathcal{V}_n \\ t\mathcal{V}_n & t\mathcal{V}_{n-1} \end{bmatrix} \tag{8. 2}$$

$$\widehat{\mathcal{W}}_n = \begin{bmatrix} \mathcal{W}_{n+1} & \mathcal{W}_n \\ t\mathcal{W}_n & t\mathcal{W}_{n-1} \end{bmatrix} \tag{8. 3}$$

$$\widehat{\mathcal{X}}_n = \begin{bmatrix} \mathcal{X}_{n+1} & \mathcal{X}_n \\ t\mathcal{X}_n & t\mathcal{X}_{n-1} \end{bmatrix} \tag{8. 4}$$

Proof. Replace $\mathcal{Y}$ by $\widehat{\mathcal{U}}$ in the equation (5. 4), we get

$$\widehat{\mathcal{U}}_n = \frac{\widehat{\mathcal{U}}_1\lambda^n - \widehat{\mathcal{U}}_1\mu^n - \widehat{\mathcal{U}}_0\mu\lambda^n + \widehat{\mathcal{U}}_0\lambda\mu^n}{\lambda-\mu}$$

$$= \frac{1}{\lambda-\mu}\begin{bmatrix} \mathcal{U}_2\lambda^n - \mathcal{U}_2\mu^n - \mathcal{U}_1\mu\lambda^n + \mathcal{U}_1\lambda\mu^n & \mathcal{U}_1\lambda^n - \mathcal{U}_1\mu^n - \mathcal{U}_0\mu\lambda^n + \mathcal{U}_0\lambda\mu^n \\ t\Big(\mathcal{U}_1\lambda^n - \mathcal{U}_1\mu^n - \mathcal{U}_0\mu\lambda^n + \mathcal{U}_0\lambda\mu^n\Big) & t\Big(\mathcal{U}_0\lambda^n - \mathcal{U}_0\mu^n - \mathcal{U}_{-1}\mu\lambda^n + \mathcal{U}_{-1}\lambda\mu^n\Big) \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{U}_{n+1} & \mathcal{U}_n \\ t\mathcal{U}_n & t\mathcal{U}_{n-1} \end{bmatrix} \qquad \text{By the Eqn. (5. 4)}$$

This proves the equation (8. 1). □

The other equations can be proved same as the equation (8. 1).

Theorem 8.2. *If* $n \geq 1$, *we get*

$$\begin{aligned} \widehat{\mathcal{U}}_n &= \frac{1}{2(s^2+t)}\begin{bmatrix} \mathcal{V}_{n+2} - s\mathcal{V}_{n+1} & \mathcal{V}_{n+1} - s\mathcal{V}_n \\ t(\mathcal{V}_{n+1} - s\mathcal{V}_n) & t(\mathcal{V}_n - s\mathcal{W}_{n-1}) \end{bmatrix} \\ &= \frac{s}{s^2+t}\begin{bmatrix} \mathcal{W}_{n+2} - s\mathcal{W}_{n+1} & \mathcal{W}_{n+1} - s\mathcal{W}_n \\ t(\mathcal{W}_{n+1} - s\mathcal{W}_n) & t(\mathcal{W}_n - s\mathcal{W}_{n-1}) \end{bmatrix} \\ &= \frac{1}{\mathcal{T}_0\mathcal{T}_2 - \mathcal{T}_1^2}\begin{bmatrix} \mathcal{T}_0\mathcal{X}_{n+2} - \mathcal{T}_1\mathcal{X}_{n+1} & \mathcal{T}_0\mathcal{X}_{n+1} - \mathcal{T}_1\mathcal{X}_n \\ t(\mathcal{T}_0\mathcal{X}_{n+1} - \mathcal{T}_1\mathcal{X}_n) & t(\mathcal{T}_0\mathcal{X}_n - \mathcal{T}_1\mathcal{X}_{n-1}) \end{bmatrix} \end{aligned} \tag{8. 5}$$

Proof. The proof can be easily established by using the equation (7. 9). □

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