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Convergent Numerical Method Using Transcendental Function of Exponential Type to Solve Continuous Dynamical Systems

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Abstract. This paper presents a numerical integration method recently proposed by means of an interpolating function involving a transcendental function of exponential type for the solution of continuous dynamical systems, that is, the initial value problems (IVPs) in ordinary differential equations (ODEs). The analysis of the local truncation error $\left(T_n\left(h\right)\right)$, order of convergence, consistency and the stability of the proposed method have been investigated in the present study. The principal term of $T_n\left(h\right)$ for the method has been derived via Taylor's series expansion. The standard test problem is taken into account to investigate the linear stability region and the corresponding stability interval of the method. It is observed that the newly proposed numerical integration method is second order convergent, consistent and conditionally stable. In order to test the performance measure of the proposed method, five IVPs of varying nature have been illustrated in the context of the maximum absolute global relative errors, the absolute relative errors computed at the final mesh point of the integration interval under consideration and the ℓ^2- error norm. Furthermore, the results are compared with two existing second order explicit numerical methods taken from the relevant literature. The so far obtained results have demonstrated that the proposed numerical integration method agrees with the second order convergence based upon the analysis conducted. Hence the proposed method is considered to be a good approach for finding the solution of different types of IVPs in ODEs.

AMS (MOS) Subject Classification Codes: 6502; 65L05; 65L12; 65L20; 65L70.
Key Words: Local error, Relative error, Absolute stability, Interpolation, ℓ^2- error norm.

1. INTRODUCTION

There are various natural and physical phenomena in which differential equations play a vital role. In as many as possible engineering and scientific fields, it is a known fact that several mathematical models emanating from the real and physical life situations cannot be solved explicitly in most of the cases such as nonlinear lotka volterra competition model and logistic equation in population dynamics, Lorenz system in meteorology, pendulum and duffing equations in mechanical engineering, Van der Pol equation in electrical engineering, Newton's law of cooling in thermodynamics, geodesic equation in geology, radioactive decay in nuclear physics, motion of a charged particle, Fermi–Ulam–Pasta Oscillator and many more. In such situations, numerical approximate methods of different characteristics and orders are needed mainly due to nonlinear terms involved in the practical problems.

Development of new numerical integration methods with varying characteristics for the solution of IVPs in ODEs has attracted the attention of many researchers in past and recent years. There are numerous numerical integration methods that produce approximations to the solution of IVPs such as the very fundamental Euler's method which is the oldest and simplest method proposed by Leonhard Euler in 1768 but later modified in the form of Improved Euler method, and then arrived the Runge Kutta (RK) methods which were described by Carl Runge and Martin Kutta in 1895 and 1905 respectively [5–7, 32]. In continuation of this effort, many researchers have derived new numerical integration methods of explicit and implicit nature in an attempt to obtain better approximate results than various of the available ones in the present literature such as [8,9,15,18,19,29,30,34], just to mention a few.

In order to tackle the computational complexity involved in numerical methods, the authors in [2, 11, 26–28] have attempted to reduce the number of slope evaluations in the incremental function of the methods. In addition to this, for dealing with the IVPs having singular solutions, the nonlinear numerical methods have been devised to handle the situation and the papers in [12, 13, 20] are the good start to get into such nonlinear numerical methods.

Besides many existing numerical methods, there are few more including variational iteration, optimal perturbation and collocation methods as described in [16, 17, 31] which play vital role in approximating practical IVPs in the field of ODEs. The methods are useful to solve various practical problems ubiquitous almost in all branches of sciences. Moreover, classical numerical methods are now being generalized to allow the underlying differential equations to take any order of differentiation. This is what we call fractional order systems whose numerous applications can be found in many recently published research works such as [1, 3, 4, 21–25, 35].

The fundamental aim of the present study is to analysis of the Local Truncation Error (LTE), order of convergence, consistency and stability of the numerical integration method derived in [10] for the solution of the IVPs in ODEs. In addition to this, the proposed numerical integration method is found to be a better performer in comparison to the non-linear numerical methods called the Wambecq's and the Ramos' method. However, these methods

$$\text{Wambecq [33]}: \quad y_{n+1} = y_n + h\left(\frac{k_1^2}{k_2}\right), \tag{1}$$

where

$$k_1 = f\left(x_n, y_n\right), \ k_2 = f\left(x_n - \frac{h}{2}, y_n - \frac{h}{2}k_1\right), \tag{2}$$

$$\text{Ramos [14]}: \quad y_{n+1} = y_n + \frac{2hf^2(x_n, y_n)}{2f(x_n, y_n) - hf'(x_n, y_n)}, \tag{3}$$

being nonlinear in nature are well suited for the IVPs having singularly perturbed solutions in ODEs.

The rest of the paper is organized as follows: the Section 2 consists of the methodology required to carry out the present study. Section 3 presents the analysis of the LTE and the order of convergence of the proposed integration method. In the Section 4, the consistency and the stability properties of the method are discussed whereas the Section 5 presents five IVPs for the testing and comparison of the proposed numerical method with two nonlinear numerical methods of explicit nature called the Wambecq's and the Ramos' methods followed by the Section 6 having some concluding remarks.

2. METHODOLOGY

Consider the IVP of the form

$$y'(x) = f(x, y(x)), \quad y(a) = y_0, \quad x_0 = a \leq x \leq b = x_n, \quad -\infty < y < \infty. \tag{4}$$

The existence and uniqueness for the solution of (4) has been guaranteed by means of the Lipschitz condition on the interval $I = [a, b]$. The uniform step size used for the proposed numerical integration method is given by $h = \frac{b-a}{N}$ where N is the number of integration steps. The mesh point is defined as $x_{n+1} = x_0 + (n+1)h, n = 0(1)N$.
At $x = x_n$, $y\left(x_n\right) \in C^3\left[a, b\right]$ is called the exact/theoretical solution of (4) whereas y_n has been reserved for denoting the numerical approximate solution of (4) for $x \in [a, b] \subset I$. In [10], a new numerical integration method of explicit nature was proposed by means of the interpolating function of exponential type which can be written in a simplified form as

$$y_{n+1} = y_n + h\left(f_n + f_n^{(1)}\right) + \left(e^{-h} - 1\right) f_n^{(1)}; \ \ n = 0, 1, \cdots, N-1. \tag{5}$$

This method is a good candidate to be included in the family of linear explicit numerical integration methods of RK type as its analysis carried out in the following sections agrees with those most of the standard numerical methods used for the purpose of solving IVPs in ODEs.

3. Local Truncation Error and Order of Convergence

The analysis of the LTE denoted by $T_n(h)$ indeed determines the order of convergence for any numerical integration method designed to solve the IVPs in ODEs. In order to check the order of the method, the formula of the numerical method is subtracted from the Taylor's series expansion for $y(x)$ in powers of h under the localizing assumptions.
The Taylor's series expansion for $y(x)$ in powers of h is given by

$$\begin{aligned} y(x_n+h) &= y(x_n) + hy'(x_n) + \tfrac{1}{2}h^2y''(x_n) + \tfrac{1}{6}h^3y'''(x_n) + O(h^4) \\ &= y(x_n) + hf(x_n, y(x_n)) + \tfrac{1}{2}h^2 f^{(1)}(x_n, y(x_n)) \\ &+ \tfrac{1}{6}h^3 f^{(2)}(x_n, y(x_n)) + O(h^4). \end{aligned} \tag{6}$$

The local truncation error is given by

$$T_n(h) = y(x_n+h) - y_{n+1}. \tag{7}$$

Using the Equations (5) and (6), the Equation (7) becomes

$$\begin{aligned} T_n(h) &= y(x_n) + hf(x_n, y(x_n)) + \tfrac{h^2}{2!} f^{(1)}(x_n, y(x_n)) + \tfrac{h^3}{3!} f^{(2)}(x_n, y(x_n)) + O\left(h^4\right) \\ &- \left[y_n + h\left(f(x_n, y_n) + f^{(1)}(x_n, y_n)\right) + \left(e^{-h} - 1\right) f^{(1)}(x_n, y_n)\right] \end{aligned}.$$

Using the Maclaurin's series expansion of e^{-h} and simplifying the above equation under the localization assumption, one obtains

$$T_n(h) = \tfrac{1}{3!}h^3\left[f^{(2)}(x_n, y_n) + f^{(1)}(x_n, y_n)\right] + O\left(h^4\right). \tag{8}$$

It is clearly seen from the Equation (8) that the principal term of $T_n(h)$ involves h^3 which confirms the second order accuracy of the method. Hence, the proposed numerical integration method given by (5) has the convergence of second order accuracy.

4. Consistency and Stability

This section presents the consistency and stability properties of the proposed numerical integration method (5) as follows.

4.1. **Consistency Property.**

A numerical integration method is said to be consistent if it has at least order $p = 1$. Additionally, for a numerical integration method to be consistent it is important for the truncation errors to be zero when the step size gets smaller and ultimately vanishes. Among many, one of the ways of analyzing the consistency of a numerical method is to check whether

$$\lim_{h\to 0}\left(\frac{T_n(h)}{h}\right) = \lim_{h\to 0}\left[\frac{y(x_n+h) - y_{n+1}}{h}\right] = 0.$$

Using the Equations (7) and (8) and the above criterion, it is easy to deduce that the proposed numerical integration method has consistency characteristics. It is a known fact that any numerical method having order of accuracy greater than or equal to 1 is considered to be consistent. On the basis of this fact, It can be deduced that the proposed numerical integration method (5) is consistent since it has second order accuracy.

4.2. **Numerical Stability.** A one step explicit numerical integration method is reserved to be stable if a small perturbation in the initial conditions of the IVP leads to a small perturbation in the following numerical approximation. To discuss the stability analysis of the proposed numerical integration method, consider the following Dahlquist's test equation:

$$y'(x) = \omega y(x), y(0) = 1 \qquad \text{with} \quad \omega = \text{constant} < 0.$$

In this test equation, the rate of change is proportional to the current value with the negative proportionality constant. This means that per time step, we are losing a specific percentage of the current value. Its exact solution is given by $y(x) = e^{\omega x}, \omega < 0$ which means that it would decay to 0 regardless of the value of $\omega < 0$. If ω is larger, this decay is faster and slower if ω is smaller. Indeed, it is required that the numerical solution should exhibit the same behavior. Technical term used for this discussion is the notion of stability which means how small the step size has to be for the numerical integration method to stay stable. For an integration interval $[x_n, x_{n+1}]$ where $h = x_{n+1} - x_n$; the exact solution at the point $x = x_{n+1}$ is obtained as

$$y(x_{n+1}) = e^{\omega x_{n+1}} = e^{\omega(x_n + h)} = e^{\omega x_n} e^{\omega h} = y(x_n) e^{\omega h}, \ \omega < 0. \tag{9}$$

When applied the proposed numerical integration method on this test problem; it yields

$$y_{n+1} = \Phi y_n \qquad \text{where} \qquad \Phi = 1 + \omega h + \frac{(\omega h)^2}{2!}. \tag{10}$$

Comparing the Equations (9) and (10), it is clearly seen that the Equation (10) is a three-term approximation for the function $e^{\omega h}$ in the exact solution. The error growth factor given by (10) can be controlled by $|\Phi| < 1$ so that the errors may not amplify. Thus, the stability function of the proposed numerical integration method (5) requires that

$$\left| 1 + \omega h + \frac{(\omega h)^2}{2!} \right| < 1.$$

Setting $z = \omega h$, then (10) yields $\left| 1 + z + \frac{z^2}{2!} \right| < 1$.
The region of absolute stability for the proposed numerical integration method (5) is defined by the region in the complex plane such that $\left| 1 + z + \frac{z^2}{2!} \right| < 1$. The stability region is plotted in the Figure 1.

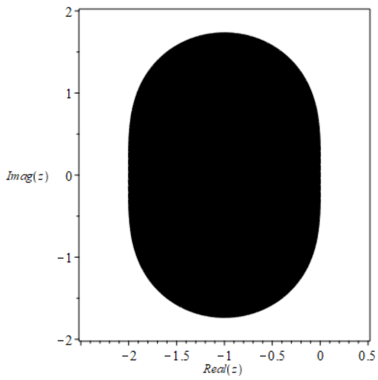

FIGURE 1. The stability region (black shaded) of the second order proposed numerical integration method with the stability interval of $(-2.01, 0.01)$.

5. NUMERICAL EXAMPLES

As many as five initial value problems of different types have been selected in the Table 1 to illustrate the performance of the second order proposed numerical integration method (5) in comparison to the two second order explicit nonlinear numerical methods called the Wambecq's and the Ramos' methods as found in [14, 33].

All of these three methods have been used to determine the maximum absolute relative global errors $\left(E_{max} = max_{a \leq n \leq b}|y(x_{n+1}) - y_{n+1}|\right)$, the absolute relative errors computed at the final mesh point of the given integration interval $\left(E(x = b) = |y(b) - y_N|\right)$ and the ℓ^2- error norm $\left(\ell^2 = \sqrt{\sum_{k=0}^{n} |y(x_{n+1}) - y_{n+1}|^2}\right)$ as shown in the Tables 2–6.

Over and above, the proposed numerical method follows the exact solution curve more elegantly as shown by the error curves in the Figures 2–4 for the IVP–1. In these Figures, one can see that the absolute relative errors are smaller than the errors produced by the Wambecq's and the Ramos' methods while taking varying values of the step size of h. In addition to this, the Table 3 represents the comparison of the proposed numerical integration method (5) with the Wambecq's and the Ramos' methods on the basis of maximum absolute global relative errors, absolute relative error at the final mesh point and the ℓ^2- error norm wherein it can easily be observed that performance of the method (5) is better than rest of the two methods since for every decreasing step size h the proposed method yields smaller errors in each case.

Similar sort of behavior was observed for rest of the test problems and hence the graphical illustrations have been omitted for the sake of brevity. Moreover, the second order accuracy of the proposed method has been confirmed from the experimental point of view, that is; how does it behave when it is applied on the selected test problems (IVPs) taking the step size h having a first order decrease in its magnitude. In connection to this, the absolute relative errors at the final mesh point of the associated integration interval are computed in the Table 7 to demonstrate that for every one-order decrease in h, there are two-order decrease in the magnitude of the computed errors.

IVP	$y'(x)$	Exact Solution	$y(0) = y_0$	$[a, b]$
1	$\frac{y(x)^2}{(1+x)^3}$	$\frac{2(1+x)^2}{x^2+2x+2}$	1	$[0, 10]$
2	$\frac{x\sin(x)}{y(x)}$	$\sqrt{-2x\cos(x) + 2\sin(x) + 1}$	1	$[0, 1]$
3	$-\frac{xy(x)}{(1+x^2)}$	$\frac{1}{\sqrt{x^2+1}}$	1	$[0, 1]$
4	$xy(x)^2$	$-\frac{2}{x^2-1}$	2	$[0, 0.5]$
5	$\frac{x^2\cos(x)}{y(x)^2}$	$\left(3x^2\sin x + 6x\cos x - 6\sin x + 1\right)^{\frac{1}{3}}$	1	$[0, 1]$

TABLE 1. Test Problems 1–5.

Method\NI	80	160	320	640	1280
Proposed	3.7547e-04	9.0379e-05	2.2002e-05	5.4170e-06	1.3434e-06
	3.1397e-04	7.1236e-05	1.6705e-05	4.0250e-06	9.8648e-07
	2.7003e-03	8.6808e-04	2.8840e-04	9.8394e-05	3.4129e-05
Wambecq	1.8000e-03	4.6611e-04	1.1872e-04	2.9965e-05	7.5276e-06
	1.8000e-03	4.6611e-04	1.1872e-04	2.9965e-05	7.5276e-06
	1.4721e-02	5.3859e-03	1.9392e-03	6.9206e-04	2.4584e-04
Ramos	1.7520e-03	4.6058e-04	1.1807e-04	2.9889e-05	7.5192e-06
	1.7520e-03	4.6058e-04	1.1807e-04	2.9889e-05	7.5192e-06
	1.4625e-02	5.4291e-03	1.9667e-03	7.0383e-04	2.5035e-04

TABLE 2. Maximum absolute global relative errors on [0, 10] (first row), absolute relative errors at $t = 10$ (second row), and ℓ^2- error norm (third row) for the IVP–1.

Method\NI	80	160	320	640	1280
Proposed	2.8962e-05	7.2392e-06	1.8097e-06	4.5240e-07	1.1310e-07
	2.3550e-05	5.8783e-06	1.4684e-06	3.6696e-07	9.1722e-08
	2.0039e-04	7.0733e-05	2.4988e-05	8.8310e-06	3.1216e-06
Wambecq	5.0829e-05	1.2340e-05	3.0325e-06	7.5073e-07	1.8665e-07
	5.0829e-05	1.2340e-05	3.0325e-06	7.5073e-07	1.8665e-07
	3.2374e-04	1.0933e-04	3.7605e-05	1.3085e-05	4.5849e-06
Ramos	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
	infinity	infinity	infinity	infinity	infinity
	infinity	infinity	infinity	infinity	infinity

TABLE 3. Maximum absolute global relative errors on [0, 1] (first row), absolute relative errors at $t = 1$ (second row), and ℓ^2- error norm (third row) for the IVP–2.

Method\NI	80	160	320	640	1280
Proposed	2.4762e-05 2.4762e-05 1.2342e-04	6.2046e-06 6.2046e-06 4.3608e-05	1.5529e-06 1.5529e-06 1.5413e-05	3.8845e-07 3.8845e-07 5.4484e-06	9.7140e-08 9.7140e-08 1.9261e-06
Wambecq	1.7770e-04 1.7770e-04 1.2816e-03	5.1255e-05 5.1255e-05 5.3612e-04	1.4513e-05 1.4513e-05 2.1913e-04	4.0524e-06 4.0524e-06 8.7983e-05	1.1190e-06 1.1190e-06 3.4835e-05
Ramos	1.5250e-04 1.5250e-04 1.1503e-03	4.5057e-05 4.5057e-05 4.9130e-04	1.2976e-05 1.2976e-05 2.0362e-04	3.6695e-06 3.6695e-06 8.2575e-05	1.0234e-06 1.0234e-06 3.2942e-05

TABLE 4. Maximum absolute global relative errors on [0, 1] (first row), absolute relative errors at $t = 1$ (second row), and ℓ^2- error norm (third row) for the IVP–3.

Method\NI	80	160	320	640	1280
Proposed	5.0710e-05 5.0710e-05 1.7987e-04	1.2786e-05 1.2786e-05 6.3441e-05	3.2102e-06 3.2102e-06 2.2402e-05	8.0426e-07 8.0426e-07 7.9151e-06	2.0128e-07 2.0128e-07 2.7975e-06
Wambecq	9.5379e-05 9.5379e-05 6.4106e-04	2.8411e-05 2.8411e-05 2.7309e-04	8.2376e-06 8.2376e-06 1.1302e-04	2.3423e-06 2.3423e-06 4.5796e-05	6.5618e-07 6.5618e-07 1.8259e-05
Ramos	1.1285e-04 1.1285e-04 6.9265e-04	3.2764e-05 3.2764e-05 2.9076e-04	9.3241e-06 9.3241e-06 1.1915e-04	2.6137e-06 2.6137e-06 4.7933e-05	7.2401e-07 7.2401e-07 1.9007e-05

TABLE 5. Maximum absolute global relative errors on [0, 0.5] (first row), absolute relative errors at $t = 0.5$ (second row), and ℓ^2- error norm (third row) for the IVP–4.

Method\NI	80	160	320	640	1280
Proposed	1.9648e-05 2.5928e-06 1.2318e-04	4.9034e-06 6.2637e-07 4.3441e-05	1.2248e-06 1.5386e-07 1.5340e-05	3.0605e-07 3.8125e-08 5.4199e-06	7.6496e-08 9.4885e-09 1.9156e-06
Wambecq	5.3949e-05 5.3949e-05 3.3100e-04	1.3165e-05 1.3165e-05 1.1218e-04	3.2446e-06 3.2446e-06 3.8670e-05	8.0455e-07 8.0455e-07 1.3473e-05	2.0022e-07 2.0022e-07 4.7243e-06
Ramos	0.0000e+00 infinity infinity	0.0000e+00 infinity infinity	0.0000e+00 infinity infinity	0.0000e+00 infinity infinity	0.0000e+00 infinity infinity

TABLE 6. Maximum absolute global relative errors on [0, 1] (first row), absolute relative errors at $t = 1$ (second row), and ℓ^2- error norm (third row) for the IVP–5.

IVP/h	10^{-1}	10^{-2}	10^{-3}	10^{-4}
1	1.9412e-04	1.6254e-06	1.5872e-08	1.5837e-10
2	1.5403e-03	1.5063e-05	1.5029e-07	1.5025e-09
3	1.5344e-03	1.5862e-05	1.5914e-07	1.5919e-09
4	9.9329e-03	1.2849e-04	1.3169e-06	1.3201e-08
5	2.4399e-04	1.6370e-06	1.5565e-08	1.5488e-10

TABLE 7. Behavior of the Absolute Relative Errors computed at $t = b$ for decreasing step size h values for the IVPs 1–5 using the proposed numerical integration method (5).

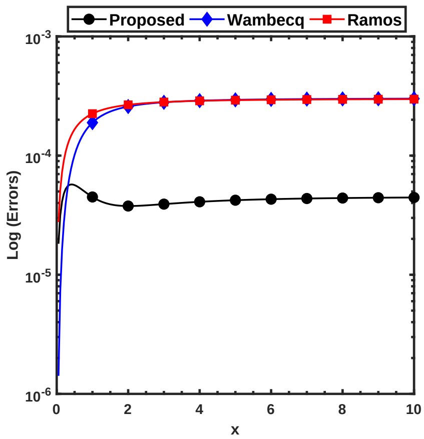

FIGURE 2. Comparison of the absolute relative errors for the IVP–1 taking $h = 0.05$.

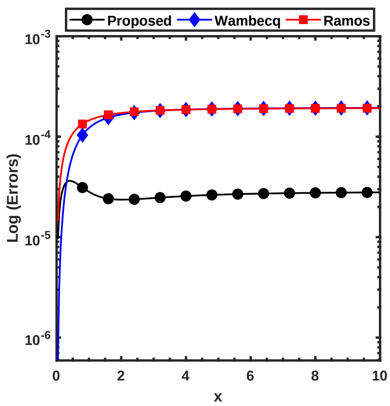

FIGURE 3. Comparison of the absolute relative errors for the IVP–1 taking $h = 0.04$.

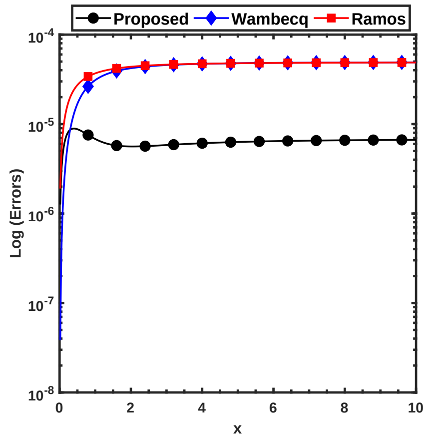

FIGURE 4. Comparison of the absolute relative errors for the IVP–1 taking $h = 0.02$.

6. Concluding Remarks

In this paper, we have investigated the truncation error analysis, convergence, consistency and the stability of the proposed numerical integration method obtained via a transcendental interpolating function of exponential type.

Five numerical examples have been solved to test the performance of the proposed method in terms of the maximum absolute global relative errors, the absolute relative errors computed at the final mesh point of the integration interval under consideration and the ℓ^2- error norms as demonstrated in the Tables 2–6.

When compared with the second order explicit nonlinear numerical integration methods (Wambecq and Ramos), the proposed method yielded smaller amount of errors in all cases as evident from the above tabular data; however, these two nonlinear methods perform well enough on stiff and singularly perturbed IVPs.

In addition to this, comparison of all these methods is shown graphically in the Figures 2–4 via absolute relative errors with the error curves produced by the proposed method lying always beneath errors curves of the later methods. It is also observed that the proposed numerical method is of second order convergence and consistent with conditional stability.

From the Figure 1, it can be seen that the proposed numerical method is conditionally stable with the region of linear stability and stability interval found to be $(-2.01, 0.01)$. Further, it may be noted that the second order convergence is clearly depicted in the Table 7 by every IVP considered in the numerical examples' section above. Hence, the proposed numerical integration method is a good approach for solving the IVPs of various nature and characteristics in diverse areas of ODEs.

It is believed that more sophisticated numerical methods can be proposed using transcendental function of exponential type in order to solve continuous dynamical systems which will certainly improve the stability characteristics and convergence order of the existing numerical methods. In this connection, the authors of the present paper will demonstrate such methods in near future.

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Authors' Contribution. First author of the present paper presented the fundamental idea of devising a new method which can efficiently solve the initial value problems whereas the second author worked on the error analysis, stability, convergence and consistency of the devised method. Write up of the paper is equally divided.

Conflict of Interest. The authors declare no conflict of interest.

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