Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 51(3)(2019) pp. 61-69

Ridge Regression Based Subdivision Schemes for Noisy Data

Muhammad Asghar
Department of Mathematics,
The Islamia University of Bahawalpur, Pakistan.
Email: m.asgharmaths@gmail.com

Ghulam Mustafa
Department of Mathematics,
The Islamia University of Bahawalpur, Pakistan.
Email: Corresponding authors: ghulam.mustafa@iub.edu.pk

Received: 04 May, 2018 / Accepted: 23 September, 2018 / Published online: 27 December, 2018

Abstract. In this article, a generalized algorithm for curve and surface design has been presented. This algorithm is based on Ridge regression. The subdivision schemes generated by the proposed algorithm give less response to the outliers. The quality of proposed subdivision schemes are also better than the least squares based subdivision schemes. Moreover, least squares based subdivision schemes are the special case of our schemes.

AMS (MOS) Subject Classification Codes: 65D17, 65D15, 65D10, 62J05.
Key Words: Univariate; bivariate; stationary subdivision schemes; ridge regression; noisy data.

1. INTRODUCTION

Subdivision schemes are an efficient tool for generating smooth curves and surfaces. The advantage of subdivision scheme is that it can be applied to any data. But the subdivision schemes constructed by different methods other than the statistical method does not deal with noisy data and outliers.

In statistics, the least squares method is one of the oldest and most papular method for the data fitting when data is irregularly-spaced. In 2015, first time Dyn et al. [1] had introduced and analyzed a family of stationary subdivision schemes for refining noisy data by fitting local least squares polynomials (i.e. by using l_2 regression). In the same year, Mustafa et al. [2] presented l_1-regression for the construction of subdivision schemes to deal with noisy data with impulsive noises and outliers. The generalization of this work was presented in [3].

Ridge regression belongs to a class of regression tools that use l_2 regularization. Ridge regression is a technique for analyzing multiple regression. If the data contain outliers, the least squares estimates are unbiased; ridge regression is used for such type of data. Lu et al. [5] proposed a fast algorithm for ridge regression when the number of features is much larger than the number of observations. Khalaf et al. [6] used ridge regression for the estimator of predictors collinearity. Cule and Iorio [7] presented a method to determine the ridge parameter based on the data. Hang et al. [8] used ridge regression for remote sensing data analysis, including hyper-spectral image classification and atmospheric aerosol retrieval.

1.1. **Motivation and contributions.** Ridge regression is used in different fields for data analysis. We use the ridge regression in the field of subdivision schemes. The purpose of this article is to construct the univariate and bivariate subdivision schemes which deal with noisy data and outliers.

The contributions of our proposed work are:

- A generalized algorithm for univariate and bivariate subdivision schemes.
- Subdivision schemes generated by our proposed algorithm give less response to the outliers as compared to the least squares based subdivision schemes [1] (see Figure 1).
- The quality of the curve generated by the proposed schemes is also better than the curve generated by the least squares based subdivision schemes [1] (see Figure 1).

The paper is structured as follows: In Section 2, the generalized algorithm of univariate subdivision schemes is presented. Section 3 is for the bivariate case. Applications and comparison are presented in Section 4. Conclusions are drawn in Section 5.

2. UNIVARIATE CASE

In this section, we present a generalized algorithm for curve fitting. We present a family of α-ary subdivision schemes for curve designing. For the derivation of a family of α-ary subdivision schemes, we follow the following steps.

Step 1: Consider the pth degree polynomial

$$f(x_r) = a_0 + a_1 x_r + a_2 x_r^2 + a_3 x_r^3 + \cdots + a_p x_r^p. \tag{2. 1}$$

In ridge regression shrinks the regression coefficients by imposing a penalty on their size [9]. The ridge coefficients minimize a penalized residual sum of squares of (2. 1) with observations $(x_r = r, f(x_r) = f_r)$ for $r = -n, -n+1, \cdots, n$, where $n > 0$ is defined as

$$J(a_0, a_1, a_2, \cdots, a_p) = \sum_{r=-n}^{n} \left(f_r - \sum_{i=0}^{p} a_i r^i \right)^2 + \lambda \sum_{i=1}^{p} a_i^2. \tag{2. 2}$$

Here λ is an arbitrary parameter that controls the amount of shrinkage (i.e. The distance between control polygon and limit curve): as the value of λ increases, the amount of shrinkage also increases. See [10] for more details about shrinkage. For minimizing the sum of squares of residual, we adopt the same procedure as that of ordinary least squares. By taking the partial derivatives of (2.2) with respect to a_0, a_1, $a_2 \cdots$, a_p and setting them equal to zero. We get the values of a_0, a_1, $\cdots$, a_p. By substituting the values of a_p's in (2. 1), we get the best fit polynomial of degree p to the data.

Step 2: Once the best fit polynomial of any degree is obtained, the family of α-ary subdivision schemes is derived by putting $r = \pm\frac{2t+1}{2\alpha}$, where $\alpha \geq 2$ with $t = 0, 1, 2, \cdots, \alpha - 1$ and replacing f_r by f_{i+r}^k.

2.1. **Family of (2n+1)-point, α-ary schemes based on a polynomial of degree 1.** Here, we will present the family of $(2n+1)$-point, α-ary subdivision scheme based on a polynomial of degree 1. After substituting $p = 1$ in (2.1) and (2.2), we get the linear polynomial and penalized residual sum. By taking the partial derivatives of the penalized residual sum and setting them equal to zero, we get the values of a_0 and a_1.

$$a_0 = \frac{1}{2n+1}\sum_{r=-n}^{n} f_r \quad \text{and} \quad a_1 = \frac{3}{2n^3+3n^2+n+3\lambda}\sum_{r=-n}^{n} r f_r.$$

By putting the values of a_0 and a_1 in linear form, we get the best fit polynomial

$$f(r) = \frac{1}{2n+1}\sum_{r=-n}^{n} f_r + \left(\frac{3}{2n^3+3n^2+n+3\lambda}\sum_{r=-n}^{n} r f_r\right) r. \qquad (2.\ 3)$$

The family of $(2n+1)$-point α-ary subdivision schemes is obtained by substituting $r = \pm\frac{2t+1}{2\alpha}$ in (2.3) and replacing f_r by f_{i+r}^k

$$f_{\alpha i+\theta}^{k+1} = f\left(\pm\frac{2t+1}{2\alpha}\right) = \frac{1}{2n+1}\sum_{r=-n}^{n} f_{i+r}^k + \left(\frac{3}{2n^3+3n^2+n+3\lambda}\sum_{r=-n}^{n} r f_{i+r}^k\right) \times\left(\pm\frac{2t+1}{2\alpha}\right), \quad \theta = 0, 1 \cdots, \alpha - 1. \qquad (2.\ 4)$$

Which is the general form of a family of $(2n+1)$-point α-ary subdivision schemes based on linear polynomial. By substituting the different values of n and α, we get the family members of $(2n+1)$-point α-ary subdivision schemes.
Similarly, if $-n+1 \leq r, s \leq n$, where $n \geq 1$, then we get family of $2n$-point α-ary approximating subdivision schemes for curve design.
The least squares based subdivision schemes are also reproduced by our proposed algorithm.

Remark 2.2. *By taking $\lambda = 0$, we get the families of schemes proposed by Dyn et al.* [1].

Remark 2.3. *By taking $\lambda = 0$ and $p = 3$, we get the families of schemes proposed by Mustafa and Bari* [4].

Family members of binary approximating subdivision schemes: For a family of binary approximating subdivision schemes, put $\alpha = 2$ in (2. 4), we get

$$f_{2i+\theta}^{k+1} = \frac{1}{2n+1}\sum_{r=-n}^{n} f_{i+r}^k + \left(\frac{3}{2n^3+3n^2+n+3\lambda}\sum_{r=-n}^{n} r f_{i+r}^k\right)\left(\pm\frac{2t+1}{4}\right). \qquad (2.\ 5)$$

By substituting $n = 1$, we get a 3-point binary approximating subdivision scheme

$$\begin{aligned} f_{2i}^{k+1} &= \left(\frac{1}{3} + \frac{1}{8+4\lambda}\right) f_{i-1}^k + \frac{1}{3} f_i^k + \left(\frac{1}{3} - \frac{1}{8+4\lambda}\right) f_{i+1}^k, \\ f_{2i+1}^{k+1} &= \left(\frac{1}{3} - \frac{1}{8+4\lambda}\right) f_{i-1}^k + \frac{1}{3} f_i^k + \left(\frac{1}{3} + \frac{1}{8+4\lambda}\right) f_{i+1}^k. \end{aligned} \qquad (2.\ 6)$$

For $n = 2$, we have a 5-point binary approximating subdivision scheme

$$\begin{aligned} f^{k+1}_{2i+1} &= \Big(\frac{1}{5}+\frac{1}{20+2\lambda}\Big)f^k_{i-2}+\Big(\frac{1}{5}+\frac{1}{40+4\lambda}\Big)f^k_{i-1}+\frac{1}{5}f^k_i+\Big(\frac{1}{5}-\frac{1}{40+4\lambda}\Big)f^k_{i+1} \\ &\quad +\Big(\frac{1}{5}-\frac{1}{20+2\lambda}\Big)f^k_{i+2}, \\ f^{k+1}_{2i+1} &= \Big(\frac{1}{5}-\frac{1}{20+2\lambda}\Big)f^k_{i-2}+\Big(\frac{1}{5}-\frac{1}{40+4\lambda}\Big)f^k_{i-1}+\frac{1}{5}f^k_i+\Big(\frac{1}{5}+\frac{1}{40+4\lambda}\Big)f^k_{i+1} \\ &\quad +\Big(\frac{1}{5}+\frac{1}{20+2\lambda}\Big)f^k_{i+2}. \end{aligned} \tag{2. 7}$$

Similarly, for each value of n, we get a binary scheme.

2.4. **Family of (2n+1)-point, α-ary schemes based on polynomial of degree 2.** Here, we will present the family of $(2n+1)$-point, α-ary subdivision scheme based on a polynomial of degree 2. For this, put $p = 2$ in (2.1), we get the second degree polynomial. After the step 1 of the algorithm, we get a best fit polynomial

$$f(r) = \frac{1}{\beta_0}\Big(\eta_0 - \frac{\eta_0\beta_1^2-\eta_2\beta_0\beta_1}{\beta_1^2-\beta_0\beta_3}\Big)+\Big(\frac{\eta_1}{\beta_2}\Big)r+\Big(\frac{\eta_0\beta_1-\eta_2\beta_0}{\beta_1^2-\beta_0\beta_3}\Big)r^2, \tag{2. 8}$$

where

$$\beta_0 = \sum_{r=-n}^{n} 1, \quad \beta_1 = \sum_{r=-n}^{n} r^2, \quad \beta_2 = \sum_{r=-n}^{n} r^2-\lambda, \quad \beta_3 = \sum_{r=-n}^{n} r^4-\lambda,$$
$$\eta_0 = \sum_{r=-n}^{n} f_r, \quad \eta_1 = \sum_{r=-n}^{n} rf_r, \quad \text{and} \quad \eta_2 = \sum_{r=-n}^{n} r^2 f_r.$$

The family of $(2n+1)$-point α-ary subdivision schemes is obtained by substituting $r = \pm\frac{2t+1}{2\alpha}$ in (2.8) and replacing f_r by f^k_{i+r}

$$\begin{aligned} f^{k+1}_{\alpha i+\theta} &= f\left(\pm\frac{2t+1}{2\alpha}\right) = \frac{1}{\beta_0}\Big(\eta_0-\frac{\eta_0\beta_1^2-\eta_2\beta_0\beta_1}{\beta_1^2-\beta_0\beta_3}\Big)+\Big(\frac{\eta_1}{\beta_2}\Big)\Big(\pm\frac{2t+1}{2\alpha}\Big) \\ &\quad +\Big(\frac{\eta_0\beta_1-\eta_2\beta_0}{\beta_1^2-\beta_0\beta_3}\Big)\Big(\pm\frac{2t+1}{2\alpha}\Big)^2, \end{aligned} \tag{2. 9}$$

where

$$\beta_0 = \sum_{r=-n}^{n} 1, \quad \beta_1 = \sum_{r=-n}^{n} r^2, \quad \beta_2 = \sum_{r=-n}^{n} r^2-\lambda, \quad \beta_3 = \sum_{r=-n}^{n} r^4-\lambda,$$
$$\eta_0 = \sum_{r=-n}^{n} f^k_{i+r}, \quad \eta_1 = \sum_{r=-n}^{n} rf^k_{i+r}, \quad \text{and} \quad \eta_2 = \sum_{r=-n}^{n} r^2 f^k_{i+r}.$$

Which is the general form of a family of $(2n+1)$-point α-ary subdivision schemes. By substituting the different values of α and n, we get the family members. The schemes can be analyzed by using the techniques given in [1].

3. BIVARIATE CASE

This section is for non-tensor product surface subdivision schemes. For the derivation of a family of α-ary subdivision schemes, we follow the following steps.
Step 1: Consider the bivariate pth degree polynomial with respect to the observations $(x_r = r, y_s = s, f(x_r, y_s) = f_{r,s})$

$$\begin{aligned} f(r,s) &= a_0 + \sum_{i=0}^{1} a_{i+1} r^{1-i} s^i + \sum_{i=0}^{2} a_{i+3} r^{2-i} s^i + \sum_{i=0}^{3} a_{i+6} r^{3-i} s^i + \cdots \\ &+ \sum_{i=0}^{p-1} a_{i+\alpha} r^{p-1-i} s^i + \sum_{i=0}^{p} a_{i+\alpha+p} r^{p-i} s^i. \end{aligned} \quad (3.\ 10)$$

By ridge regression approximation, the penalized residual sum of squares is defined as:

$$\begin{aligned} J(a_0, a_1, \cdots, a_{2p+\alpha}) &= \sum_{r=-n}^{n} \sum_{s=-n}^{n} \Big(f_{r,s} - a_0 - \sum_{i=0}^{1} a_{i+1} r^{1-i} s^i - \cdots \\ &- \sum_{i=0}^{p} a_{i+\alpha+p} r^{p-i} s^i \Big)^2 + \lambda \sum_{i=0}^{p} a_{i+\alpha+p}^2. \end{aligned} \quad (3.\ 11)$$

By taking the partial derivatives with respect to a_0, $a_1 \cdots a_{2p+\alpha}$ and setting them equal to zero, we will get a system of linear equations. The solution of these equations gives the values of $a_{2p+\alpha}$'s. By substituting these values in (3. 10), we get the best fit bivariate polynomial.
Step 2: Once the best fit polynomial of any degree is obtained, the family of $(2n+1)^2$-point α-ary non-tensor product surface subdivision schemes is derived by putting $(r, s) = (\pm\frac{2t+1}{2\alpha}, \pm\frac{2t+1}{2\alpha})$, where $\alpha \geq 2$ with $t = 0, 1, 2, \cdots, \alpha - 1$ and replacing $f_{r,s}$ by $f^k_{i+r,i+s}$. Similarly, if $-n+1 \leq r, s \leq n$, where $n \geq 1$, then we get family of $4n^2$-point α-ary approximating subdivision schemes for surface design.

3.1. Family of $(2n+1)^2$-point, α-ary bivariate schemes based on a polynomial of degree 1.

The family of $(2n+1)^2$-point, α-ary subdivision scheme based on the polynomial of degree one is presented in this section. By substituting $p = 1$ in (3.10), we get the linear bivariate polynomial

$$f(r,s) = a_0 + a_1 r + a_2 s. \quad (3.\ 12)$$

After applying step 1 of the above algorithm, we get the values of a_0, a_1 and a_2:

$$a_0 = \frac{1}{(2n+1)^2} \sum_{r=-n}^{n} \sum_{s=-n}^{n} f_{r,s}, \quad (3.\ 13)$$

$$a_1 = \frac{1}{3(4n^4 + 8n^3 + 5n^2 + n - 3\lambda)} \sum_{r=-n}^{n} \sum_{s=-n}^{n} r f_{r,s}, \quad (3.\ 14)$$

$$a_2 = \frac{1}{3(4n^4 + 8n^3 + 5n^2 + n - 3\lambda)} \sum_{r=-n}^{n} \sum_{s=-n}^{n} s f_{r,s}. \quad (3.\ 15)$$

By substituting the values of a_0, a_1 and a_2 in (3.12), we get the best fit bivariate polynomial of degree one. The family of $(2n+1)^2$-point α-ary non-tensor product subdivision schemes is obtained by putting $(r,s) = (\pm\frac{2t+1}{2\alpha}, \pm\frac{2t+1}{2\alpha})$ and replacing $f_{r,s}$ by $f^k_{r,s}$ in the best fit polynomial. The general form of the scheme is

$$f^{k+1}_{\alpha i+\theta,\alpha j+\theta} \quad = \quad a_0 + a_1\left(\pm\frac{2t+1}{2\alpha}\right) + a_2\left(\pm\frac{2t+1}{2\alpha}\right), \qquad (3.\ 16)$$

where a_0, a_1 and a_2 are defined in (3.4), (3.5) and (3.6) respectively. Corresponding to the different values of n and α, we get the family members of $(2n+1)^2$-point α-ary non-tensor product surface subdivision schemes.

Remark 3.2. *By taking $\lambda = 0$, we get the families of bivariate schemes proposed by Dyn et al.* [1].

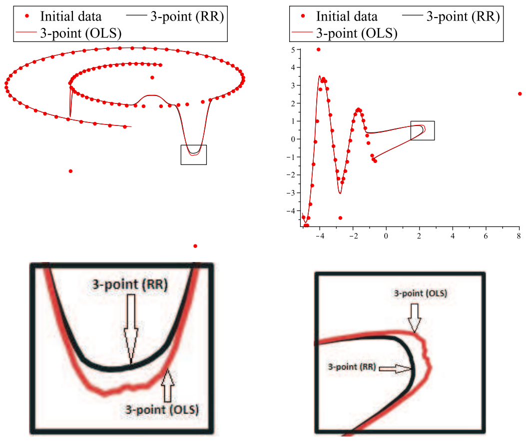
Remark 3.3. *By taking $\lambda = 0$ and $p = 3$, we get the families of bivariate schemes proposed by Mustafa and Bari* [4].

4. Applications and comparison

This section aims to present the visual performance and comparison of subdivision schemes generated by Ridge Regression (RR) algorithm and ordinary least squares (OLS) algorithm [1]. In Figure 1, we consider the data with three outliers and apply the proposed 3-point scheme (2. 6) when $\lambda = 2.5$ and ordinary least squares based 3-point schemes presented in [1]. We see that the proposed 3-point scheme (2. 6) gives less response to the outliers comparative to the 3-point scheme presented in [1]. The second row of Figure 1 is the mirror images of the selected region of images presented in the first row. From this figure, we can easily observe that the quality of curves generated by the proposed scheme is better than the quality of curves produced by [1].

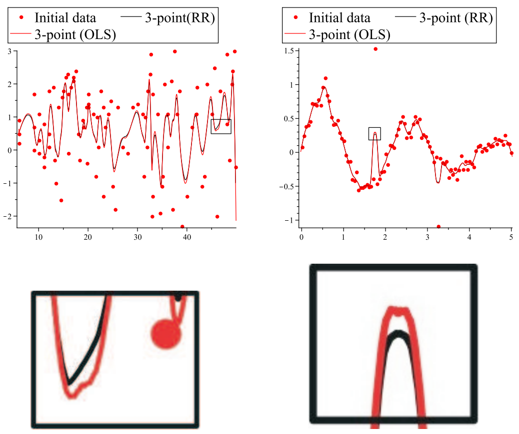
In Figure 2, we consider the noisy data and apply the proposed 3-point scheme (2. 6) when $\lambda = 2.5$ and 3-point scheme presented in [1]. The second row of Figure 2 is the mirror images of the selected region of the images presented in the first row. From this figure, we can again observe the quality of the curves generated by the proposed scheme is better than the quality of curves produced by [1].

In Figure 3, we present the comparison between the scheme (3. 16) with $\lambda = 2.5$ and the schemes presented in [1]. We observe that as the complexity of scheme increases the quality of surfaces increases. In surface case, it is difficult to see the differences among these surfaces by naked eyes. But as in the curve case by magnifying some parts of images, one can see the differences among these images.



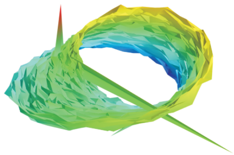
(**a**) (**b**)

FIGURE 1. (**a**) and (**b**) are the comparison of proposed 3-point schemes when $\lambda = 2.5$ and 3-point scheme of [1] in terms of outliers. The second row of (**a**) and (**b**) is the mirror images of the selected region, shows the quality of curves.



(**a**) (**b**)

FIGURE 2. (**a**) and (**b**) are the comparison of proposed 3-point schemes when $\lambda = 2.5$ and 3-point scheme of [1] in terms of noisy data. The second row of (**a**) and (**b**) is the mirror images of the selected region, shows the quality of curves.



$(\mathbf{a})$ Initial noisy surface

$(\mathbf{b})$ 3-point (OLS)

$(\mathbf{c})$ 3-point (RR)

$(\mathbf{d})$ 5-point (OLS)

$(\mathbf{e})$ 5-point (RR)

$(\mathbf{f})$ 7-point (OLS)

$(\mathbf{g})$ 7-point (RR)

FIGURE 3. $(\mathbf{a})$ Shows the initial noisy surface whereas $(\mathbf{b})$ - $(\mathbf{g})$ shows the comparison of proposed surface subdivision schemes when $\lambda = 2.5$ with surface schemes presented in [1].

5. Conclusions

In this article, the generalized algorithms for univariate and bivariate approximating subdivision schemes have been presented. A comparison between least squares based and ridge regression based subdivision schemes is presented. The subdivision schemes generated by our algorithm give less response to the outliers and also give good quality of the curves and surfaces. Moreover, the least squares based subdivision schemes are the special cases of our proposed algorithm.

Conflict of interests

The authors declare that there is no conflict of interest regarding the publication of this article and regarding the funding that they have received.

Acknowledgement

This work is supported by National Research Programme for Universities (NRPU) (P. No. 3183) HEC, Pakistan. We thanks the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

Authors Contribution

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[11] M. Asghar and G. Mustafa, *Stationary and Non-Stationary Univariate Subdivision Schemes*, Punjab Univ. j. math. **50**, No. 3 (2018) 25-42.

[10] P. Ashraf, G. Mustafa and J. Deng, *A six-point variant on the Lane-Riesenfeld algorithm*, Journal of Applied Mathematics, **2014**, Article ID 628285, 7 pages.

[7] E. Cule and M. De Iorio, *Ridge regression in prediction problems: automatic choice of the ridge parameter*, Genetic Epidemiology **37**, No. 7 (2013) 704-714.

[1] N. Dyn, A. Heard, K. Hormann and N. Sharon, *Univariate subdivision schemes for noisy data with geometric applications*, Computer Aided Geometric Design, **37**, (2015) 85-104.

[9] J. Friedman, T. Hastie and R. Tibshirani, *The elements of statistical learning*, Springer, **1**, (2001).

[8] R. Hang, Q. Liu, H. Song, Y. Sun, F. Zhu and H. Pei, *Graph regularized nonlinear ridge regression for remote sensing data analysis*, IEEE Journal of Selected Topics in Applied Earth Observations and Remote Sensing **10**, No. 1 (2017) 277-285.

[6] G. Khalaf, K. Mansson and G. Shukur, *Modified ridge regression estimators*, Communications in Statistics Theory and Methods **42**, No. 8 (2013) 1476-1487.

[5] Y. Lu, P. S. Dhillon, D. Foster and L. Ungar, *Faster ridge regression via the subsampled randomized hadamard transform*, In Advances in Neural Information Processing Systems, (2013) 369-377.

[2] G. Mustafa, H. Li, J. Zhang and J. Deng, l_1*-regression based subdivision schemes for noisy data*, Computer-Aided Design **58**, (2015) 189-199.

[3] G. Mustafa and R. Hameed, *Families of non-linear subdivision schemes for scattered data fitting and their non-tensor product extensions*, arXiv preprint arXiv:1709.04282 (2017).

[4] G. Mustafa and M. Bari, *Wide-ranging families of subdivision schemes for fitting data*, Punjab Univ. j. math. **48**, No. 2 (2016) 125-134.

[12] G. Mustafa, M. Bari and A. Sadiq, *The (2n)2 - Point Scheme Based on Bivariate Cubic Polynomial*, Punjab Univ. j. math. **49**, No. 3 (2017) 27-35.